# Numerical Solution for Solving fuzzy Fredholm Integro-Differential Equations by Euler Method

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**Abstract**

Numerical algorithms for solving Fuzzy Integro-Differential Equations (FIDEs) are considered. A scheme based on the classical Euler method is discussed in detail and this is followed by a complete error analysis. The algorithm is illustrated by solving linear first-order fuzzy integro-differential equations.

**Keywords :** Fuzzy integro-differential equations; Strongly Generalized H-differentiability; Numerical solution; Euler method..

## 1 Introduction

The idea of fuzzy set was first introduced by Latfi Zadah in 1960 as a means of handling uncertainty that is due to impression or vagueness rather than to randomness. Further Zadeh [12, 33] studied the concept of fuzzy numbers and arithmetic operations on it, which was further improved by Mizumoto and Tanaka [24]. Dubios and Prade [14] introduced the concept of LR fuzzy numbers and gave a computational formula for operations on fuzzy functions. They [15] also studied the concept of integration of fuzzy functions.

Recently due to industrial interest in fuzzy control, the applications of the theory of fuzzy differential and integral equation have been increased. Fuzzy integro-differential equations play a major role in modeling uncertainty of dynamical systems. Balachandran and Kanagarajan [5, 6]

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studied the existing results for fuzzy delay integro-differential equations and general fuzzy volterra fredholm integral equations.

Several numerical methods were developed to solve integro-differential equations. Abbasbandy and Hashemi [2] used the variation iteration method to solve fuzzy integro-differential equations. Further, they [3] used the homotopy analysis method to find the numerical solution for fuzzy integro-differential equations. A new method was introduced by Allahviranloo et all [4] for solving fuzzy integro differential equations under generalized differentiability. In this paper, we have used Euler method to find the numerical solution of fuzzy integro differential equations under generalized H-differentiability.

The structure of this paper is organized in the following way. In section 2, the basic definitions and notations that are used in our work are given. In section 3, the fuzzy integro-differential equations have been introduced and the proposed method has been discussed in detail. In section 4, some numerical examples have been given to illustrate the proposed technique. Finally in section 5, summary and conclusion of the study has been given.

## 2 Preliminaries

In this section, the basic notations used in fuzzy operations are introduced. We start by defining the fuzzy number.

**Definition 2.1** *[23] fuzzy number is a fuzzy set* $u : \Re \to I = [0, 1]$ *such that*

*i.* $u$ *is upper semi continuous;*

*ii.* $u(x) = 0$ *outside some interval* $[a, d]$*;*

*iii. There are real numbers* $b$ *and* $c$, $a \leq b \leq c \leq d$, *for which*

- $u(x)$ *is monotonically increasing on* $[a, b]$,
- $u(x)$ *is monotonically decreasing on* $[c, d]$,
- $u(x) = 1, b \leq x \leq c.$

*The set of all the fuzzy numbers (as given in Definition 2.1) is denoted by* $E$.

**Definition 2.2** *([16], [18]) Let* $f : [a, b] \to E$, *for each partition* $P = \{t_0, t_1, ...., t_n\}$ *of* $[a, b]$ *and for arbitrary* $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$ *suppose*

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$
$$\Delta := max\{|t_i - t_{i-1}|, i = 1, 2, ..., n\}.$$

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*The definite integral of $f(t)$ over $[a,b]$ is*

$$\int_a^b f(t)dt = lim_{\Delta\to 0} R_p.$$

*provided, that this limit exists in the metric $D$.*

*If the fuzzy function $f(t)$ is continuous in the metric $D$, its definite integral exists [18] and also,*

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t;r)dt,$$

**Definition 2.3** *An arbitrary fuzzy number is represented by an ordered pair of functions $[\underline{u}(r), \overline{u}(r)]$ for all $r \in [0,1]$, which satisfies the following requirements [17, 20].*

- $\underline{u}(r)$ is a bounded left continuous non decreasing function over $[0,1]$;
- $\overline{u}(r)$ is a bounded left continuous non increasing functions over $[0,1]$;
- $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

Let $E$ be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $r$-level intervals. This means that if $\tilde{v} \in E$, then the $r$-level set $[v]_r = \{s|v(s) \ge r\}$, is a closed bounded interval which is denoted by $[v]_r = [\underline{v}(r), \overline{v}(r)]$ for $r \in (0,1]$, and $[v]_0 = \overline{\bigcup_{r\in(0,1]}[v]_r}$.

Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are called equal, $\tilde{u} = \tilde{v}$, if $u(s) = v(s)$ for all $s \in \Re$ or $[u]_r = [v]_r$ for all $r \in [0,1]$.

**Lemma 2.4** *[28] if $\tilde{u}, \tilde{v} \in E$, then for $r \in (0,1]$,*

$[u+v]_r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$,

$[u,v]_r = [min\ k_r, max\ k_r]$,

*where $k_r = \{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$.*

*The Hausdorff distance between fuzzy numbers is given by $D : E \times E \to R_+ \bigcup \{0\}$,*

$$D(u,v) = sup_{r\in[0,1]}\ max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

*Where $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \subset E$ is utilized (see [9]). Then, it is easy to see that $D$ is a metric in $E$ and has the following properties (see [27]).*

$$\begin{aligned} D(u\oplus w, v\oplus w) &= D(u,v), \forall u,v,w \in E, \\ D(k\odot u, k\odot v) &= |k|\, D(u,v),\ \forall k \in \Re, u,v \in E, \\ D(u\oplus v, w\oplus e) &\le D(u,w) + D(v,e),\ \forall u,v,w,e \in E. \end{aligned}$$

*$(D,E)$ is a complete metric space.*

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**Definition 2.5** *[16] Let $f : R \to E$ be a fuzzy-valued function. If, for arbitrary fixed $t_0 \in R$ and $\epsilon > 0, \delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon.$$

*$f$ is said to be continuous.*

**Definition 2.6** *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then $z$ is called the H-difference of $x$ and $y$ and it is denoted by $x \ominus y$. In this paper, we consider the following definition of differentiability for fuzzy-valued functions that was introduced by Bede et al. [9] and investigated by Chalco-Cano et al. [11].*

**Definition 2.7** *Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that $f$ is strongly generalized H-differentiable at $x_0$. If there exists an element $f^{'}(x_0) \in E$ such that,*

(1) for all $h > 0$ sufficiently close to 0, there exist $f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric $D$)

$$\lim_{h\to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f\,'(x_0),$$

(2) for all $h < 0$ sufficiently close to 0, there exists $f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric $D$)

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{h} = \lim_{h\to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{h} = f\,'(x_0).$$

In the special case when $f$ is a fuzzy-valued function, we have the following results.

**Theorem 2.8** *[11] Let $f : R \to E$ be a function and denote $f(t) = (\underline{f}(t; r), \overline{f}(t; r))$, for each $r \in [0, 1]$. Then,*

*(1) if $f$ is differentiable in the first form (1) in Definition 2.7, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and $f^{'}(t) = (\underline{f}^{'}(t; r), \overline{f}^{'}(t; r))$*

*(2) if $f$ is differentiable in the second form (2) in Definition 2.7, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and $f^{'}(t) = (\overline{f}^{'}(t; r), \underline{f}^{'}(t; r))$*

The principal properties of the H-derivatives in the first form (1), some of which still holds for the second form (2), are well known and can be found in [22], and some properties for the second form (2) can be found in [11]. Notice that we say fuzzy-valued function $f$ is I - differentiable if it satisfies the first form (1) in Definition 2.7, and we say $f$ is II - differentiable if it satisfies the second form (2) in Definition 2.7.

# 3 Fuzzy integro-differential equations

The linear Fredholm integro-differential equations [21, 25]

$$\begin{cases} X^{'}(s) = y(s) + \lambda \int_a^b k(s,t)X(t)dt, \\ X(s_0) = X_0, \end{cases} \tag{1}$$

where $\lambda > 0$, $k$ is an arbitrary given kernel function over the square $a \le s, t \le b$ and $y(s)$ is a given function of $s \in [a, b]$. If $X$ is a fuzzy function, $y(s)$ is a given fuzzy function of $s \in [a, b]$ and $X^{'}$ is the fuzzy derivative (according to Definition 2.7 of $X$, this equation may only possess fuzzy solution. Sufficient for the existing equation of the second kind, are given in [7].

Let $X(s) = [\underline{X}(s;r), \overline{X}(s;r)]$ is a fuzzy solution of equation (1), therefore by Lemma 2.4 and Definition 2.7, we have the equivalent system

$$\begin{cases} \underline{X}^{'}(s) = \underline{y}(s) + \lambda \int_a^b \underline{k(s,t)X(t)}dt, \quad \underline{X}(s_0) = \underline{X}_0, \\ \\ \overline{X}^{'}(s) = \overline{y}(s) + \lambda \int_a^b \overline{k(s,t)X(t)}dt, \quad \overline{X}(s_0) = \overline{X}_0. \end{cases} \tag{2}$$

which possesses a unique solution $(\underline{X}, \overline{X}) \in B$ which is a fuzzy function, i.e. for each $s$, the pair $[\underline{X}(s;r), \overline{X}(s;r)]$ is a fuzzy number, therefore each solution of equations (1) is a solution of system (2) and conversely also, equations (1) and system (2) are equivalent.

The parametric form of equations (2) is given by

$$\begin{cases} \underline{X}^{'}(s,r) = \underline{y}(s,r) + \lambda \int_a^b \underline{k(s,t)X(t,r)}dt, \quad \underline{X}(s_0) = \underline{X}_0(r), \\ \\ \overline{X}^{'}(s,r) = \overline{y}(s,r) + \lambda \int_a^b \overline{k(s,t)X(t,r)}dt, \quad \overline{X}(s_0) = \overline{X}_0(r). \end{cases} \tag{3}$$

for $r \in [0, 1]$. Suppose, $k(s, t)$ be continuous in $a \le s \le b$ and for fix $t$, $k(s, t)$ changes its sign in finite points as $s_i$ where $x_i \in [a, s_1]$. For example, let $k(s, t)$ be non-negative over $[a, s_1]$ and negative over $[s_1, b]$, therefore we have

$$\begin{cases} \underline{X}^{'}(s,r) = \underline{y}(s,r) + \lambda \int_a^{s_1} k(s,t)\underline{X(t,r)}dt + \lambda \int_{s_1}^b k(s,t)\overline{X(t,r)}dt, \\ \underline{X}(s_0) = \underline{X}_0(r), \\ \\ \overline{X}^{'}(s,r) = \overline{y}(s,r) + \lambda \int_a^{s_1} k(s,t)\overline{X(t,r)}dt + \lambda \int_{s_1}^b k(s,t)\underline{X(t,r)}dt, \\ \overline{X}(s_0) = \overline{X}_0(r). \end{cases} \tag{4}$$

In most cases, however, analytical solution to equation (3) may not be found and a numerical approach must be considered.

In the following, we generalize the Euler method and give its error analysis [26]. Any other suitable known numerical methods can be generalized similarly. In the interval $I = [0, A]$, we consider a set of discrete equally spaced grid points $0 < t_0 < t_1 < t_2 < ..... < t_N = A$ at which two exact solutions $Y(t,\alpha) = \underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ are approximated by some $y(t,\alpha) = \underline{y}(t,\alpha), \overline{y}(t,\alpha)/$, respectively. The grid points at which the solutions are calculated are $t_n = t_0 + nh, \quad h = \frac{A}{N}$. The exact and approximate solutions at $t_n, 0 < n < N$ are denoted by $Y_n(\alpha)$ and $y_n(\alpha)$, respectively. The Euler method based on the first-order approximation of $\underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ and $\underline{y}(t,\alpha), \overline{y}(t,\alpha)$ is obtained as follows:

$$\begin{cases} \underline{y}_{n+1}(\alpha) = \underline{y}_n(\alpha) + hF[t_n, \underline{y}_n(\alpha), \overline{y}_n(\alpha)], \\ \overline{y}_{n+1}(\alpha) = \overline{y}_n(\alpha) + hG[t_n, \underline{y}_n(\alpha), \overline{y}_n(\alpha)], \\ \underline{y}_0^\alpha = \underline{y}_0, \\ \overline{y}_0^\alpha = \overline{y}_0, \end{cases} \tag{5}$$

# 4 Numerical Examples

In this section, some examples are given to illustrate the technique.

**Example 4.1** *Let us consider the following FIDE*

$$\begin{cases} y'(t) = \int_0^t y(s)ds, \\ y(0;r) = [\alpha - 1, 1 - \alpha], \quad 0 \le \alpha \le 1 \end{cases} \tag{6}$$

*The exact solution is given by*

$$Y(t;r) = [(\alpha - 1)cosht, \ (1 - \alpha)cosht] \tag{7}$$

*The approximate solution by using Euler method is given by*

$$\begin{cases} \underline{y}_{n+1}^\alpha = \underline{y}_n^\alpha[1 + h(sinht)], \\ \overline{y}_{n+1}^\alpha = \overline{y}_n^\alpha[1 + h(sinht)], \\ \underline{y}_0^\alpha = \underline{y}_0, \\ \overline{y}_0^\alpha = \overline{y}_0, \end{cases} \tag{8}$$

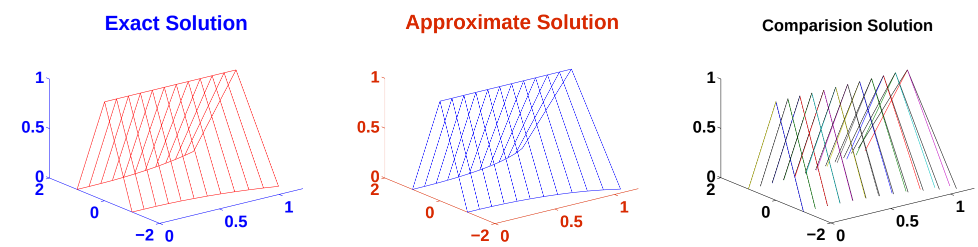


Figure 1: At t=0.1

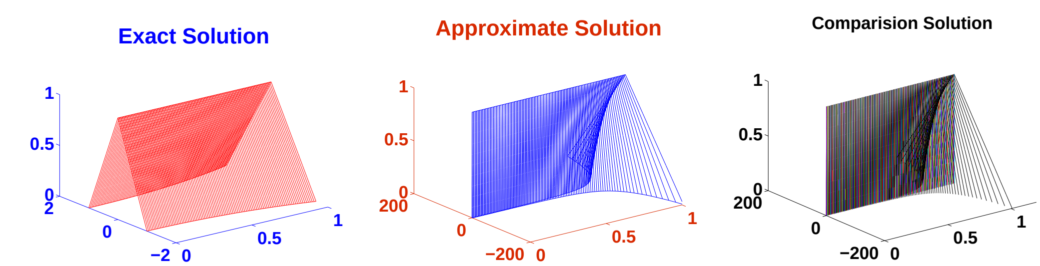


Figure 2: At t=0.01

*A comparison between the exact and the approximate solutions at $t$ = 0.1 and $t$ = 0.01 are shown in the above figures (1) and (2).*

**Example 4.2** *Let us consider the following FIDE*

$$\begin{cases} y'(t) = [0.5 + 0.5\alpha, 2 - \alpha] + \displaystyle\int_0^t y(s)ds, \\ y(0; r) = [0.5 + 0.5\alpha, 2 - \alpha], \quad 0 \leq \alpha \leq 1 \end{cases} \tag{9}$$

*The exact solution is given by*

$$Y(t; r) = [(0.5 + 0.5\alpha)e^t,\ (2 - \alpha)e^t] \tag{10}$$

*The approximate solution by using Euler method is given by*

$$\begin{cases} \underline{y}^{\alpha}_{n+1} = \underline{y}^{\alpha}_{n}[1 + h(e^{t})], \\ \overline{y}^{\alpha}_{n+1} = \overline{y}^{\alpha}_{n}[1 + h(e^{t})], \\ \underline{y}^{\alpha}_{0} = \underline{y}_{0}, \\ \overline{y}^{\alpha}_{0} = \overline{y}_{0}, \end{cases} \tag{11}$$

*A comparison between the exact and the approximate solutions at $t = 0.1$ and $t = 0.01$ are shown in the following figures (3) and (4).*

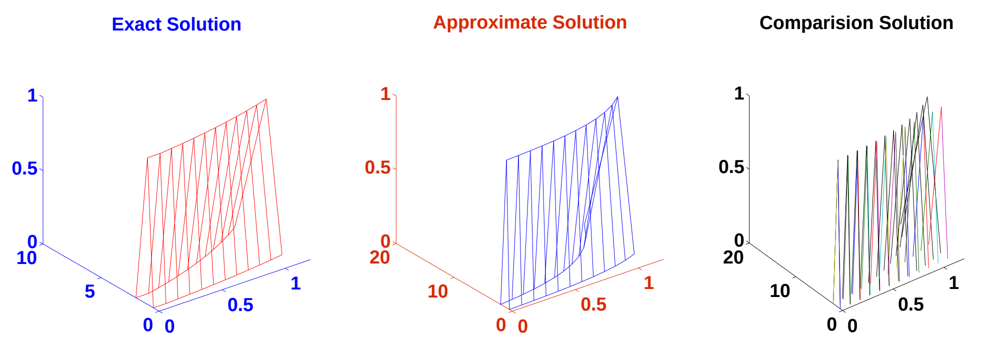


Figure 3: At t=0.1

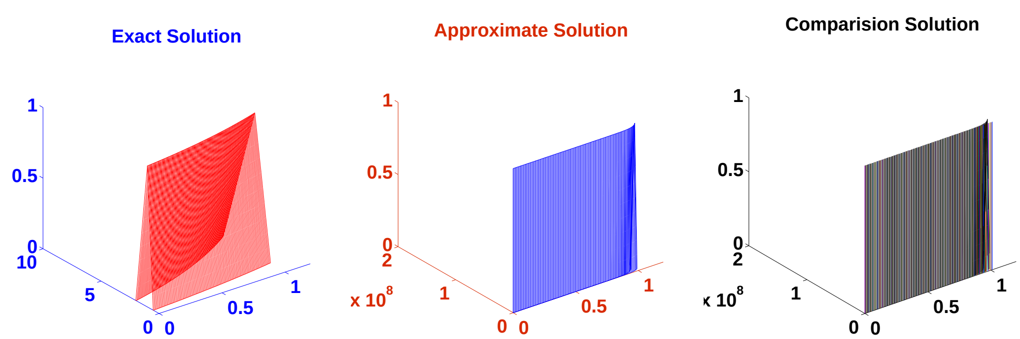


Figure 4: At t=0.01

# 5 Conclusion

This paper infers the solution of fuzzy integro-differential equations by using strongly generalized H-differentiability. We translated the fuzzy differential equations into two systems of ordinary differential equations and then solved numerically by Euler method. The obtained results concluded that the approximate solutions coincides with the corresponding exact solutions, which means that the method is suitable and effective to solve fuzzy integro-differential equations. Moreover, the solutions of the higher order fuzzy integro-differential equations can be calculated as future prospects, in a similar manner.

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