Application of Complex Analysis on Solving Some Definite Integrals

Chii-Huei Yu*[1]

[1]Department of Information Technology, Nan Jeon University of Science and Technology

October 4, 2014

Abstract

This paper studies two types of definite integrals and uses Maple for verification. The closed forms of the two types of definite integrals can be obtained mainly using Cauchy integral theorem and Cauchy integral formula. In addition, some examples are used to demonstrate the calculations.

Keywords : Definite integrals, closed forms, Cauchy integral theorem, Cauchy integral formula, Maple.

1 Introduction

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This article considers the following two types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{A}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}\, d\theta, \tag{1}$$

and

$$\int_0^{2\pi} \frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}\, d\theta. \tag{2}$$

*chiihuei@mail.nju.edu.tw

Where $A = e^{r\cos\theta}[-r^2\sin(r\sin\theta-\theta)-(b+d)r\cos(r\sin\theta)+(a+c)r\sin(r\sin\theta)+(ad+bc)\cos(r\sin\theta+\theta)-(ac-bd)\sin(r\sin\theta+\theta)]$, $B = e^{r\cos\theta}[r^2\cos(r\sin\theta-\theta)-(b+d)r\sin(r\sin\theta)-(a+c)r\cos(r\sin\theta)+(ad+bc)\sin(r\sin\theta+\theta)+(ac-bd)\cos(r\sin\theta+\theta)]$, r, θ, a, b, c, d are real numbers, $0 < a^2+b^2 < c^2+d^2$, and $|r| \neq \sqrt{a^2+b^2}, \sqrt{c^2+d^2}$. The closed forms of the two types of definite integrals can be obtained using Cauchy integral theorem and Cauchy integral formula; these are the major results of this paper (i.e., Theorems 1 and 2). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-27], Yu and B. -H. Chen [28], and Yu and Sheu [29-30] used some methods, for example, complex power series, integration term by term theorem, differentiation with respect to a parameter, Parsevals theorem, and area mean value theorem to evaluate some types of integrals. In this paper, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2 Preliminaries and Results

Some notations, formulas and theorems used in this paper are introduced below.

2.1 Notations

Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a , the real part of z , denoted as $\mathrm{Re}(z)$; b , the imaginary part of z , denoted as $\mathrm{Im}(z)$.

2.2 Formulas and theorems

2.2.1 Euler's formula:

$e^{ix} = \cos x + i\sin x$, where x is a real number.

Two important methods in complex analysis are used in this study, which can be found in [31, p 109] and [31, p121] respectively.

2.2.2 Cauchy integral theorem:

Let f be analytic on a simply connected region D and C be a simple closed curve in D, then $\int_C f(z)dz = 0$.

2.2.3 Cauchy integral formula:

If f is analytic on a region G, C is a simple closed curve in G, and a is inside C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$.

Before the major results can be derived, a lemma is needed.

Lemma 1 *Assume that z, λ, μ are complex numbers, $0 < |\lambda| < |\mu|$, r is a real number, n is a positive integer, and C is a circle centered at 0 with radius $|r|$. Then the contour integral*

$$\int_C \frac{e^z}{(z-\lambda)(z-\mu)} dz = \begin{cases} 0 & if 0 < |r| < |\lambda|, & (3) \\ \frac{2\pi i \cdot e^\lambda}{\lambda-\mu} & if |\lambda| < |r| < |\mu|, & (4) \\ \frac{2\pi i \cdot (e^\lambda - e^\mu)}{\lambda-\mu} & if |r| > |\mu|. & (5) \end{cases}$$

Proof. Because

$$\frac{e^z}{(z-\lambda)(z-\mu)} = \frac{1}{\lambda-\mu}\Big(\frac{e^z}{z-\lambda} - \frac{e^z}{z-\mu}\Big) \ . \tag{6}$$

Case 1. If $0 < |r| < |\lambda|$, it follows from Cauchy integral theorem that

$$\int_C \frac{e^z}{(z-\lambda)(z-\mu)} dz = 0.$$

Case 2. If $|\lambda| < |r| < |\mu|$, we have

$$\begin{aligned} & \int_C \frac{e^z}{(z-\lambda)(z-\mu)} dz \\ = & \frac{1}{\lambda-\mu} \int_C \frac{e^z}{z-\lambda} dz - \frac{1}{\lambda-\mu} \int_C \frac{e^z}{z-\mu} dz && \text{(by Eq. (6))} \\ = & \frac{1}{\lambda-\mu} \int_C \frac{e^z}{z-\lambda} dz && \text{(by Cauchy integral theorem)} \\ = & \frac{2\pi i \cdot e^\lambda}{\lambda-\mu}. && \text{(by Cauchy integral formula)} \end{aligned}$$

Case 3. If $|r| > |\mu|$, then

$$\begin{aligned} & \int_C \frac{e^z}{(z-\lambda)(z-\mu)} dz \\ = & \frac{1}{\lambda-\mu} \int_C \frac{e^z}{z-\lambda} dz - \frac{1}{\lambda-\mu} \int_C \frac{e^z}{z-\mu} dz \\ = & \frac{2\pi i \cdot (e^\lambda - e^\mu)}{\lambda-\mu}. && \text{(by Cauchy integral formula)} \end{aligned}$$

□

The closed form of the definite integral (1) can be obtained below.

Theorem 1 *Suppose that r, θ, a, b, c, d are real numbers, $0 < a^2+b^2 < c^2+d^2$, $|r| \neq \sqrt{a^2+b^2}, \sqrt{c^2+d^2}$, and let $A = e^{r\cos\theta}[-r^2 \sin(r\sin\theta - \theta) - (b+d) r\cos(r\sin\theta) + (a+c) r \sin(r\sin\theta) + (ad+bc)\cos(r\sin\theta + \theta) - (ac-bd)\sin(r\sin\theta+\theta)]$, then the definite integral*

$$\int_0^{2\pi} \frac{A}{[(r\cos\theta - a)^2 + (r\sin\theta - b)^2][(r\cos\theta - c)^2 + (r\sin\theta - d)^2]} d\theta$$

$$= \begin{cases} 0 & if\, |r| < \sqrt{a^2+b^2}, \quad (7) \\ \frac{2\pi e^a[(b-d)\cos b-(a-c)\sin b]}{r[(a-c)^2+(b-d)^2]} & if \sqrt{a^2+b^2} < |r| < \sqrt{c^2+d^2}, \quad (8) \\ \frac{2\pi[(b-d)(e^a\cos b-e^c\cos d)-(a-c)(e^a\sin b-e^c\sin d)]}{r[(a-c)^2+(b-d)^2]} & if\, |r| > \sqrt{c^2+d^2}. \quad (9) \end{cases}$$

Proof. Let $z = re^{i\theta}$,$\lambda = a + ib$, and $\mu = c + id$, then

$$\frac{e^z}{(z-\lambda)(z-\mu)}$$

$$=\frac{e^{re^{i\theta}}}{[re^{i\theta}-(a+ib)][re^{i\theta}-(c+id)]}$$

$$=\frac{e^{r\cos\theta}\cdot e^{ir\sin\theta}}{[(r\cos\theta-a)+i(r\sin\theta-b)][(r\cos\theta-c)+i(r\sin\theta-d)]} \quad \text{(by Euler's formula)}$$

$$=\frac{e^{r\cos\theta}\cdot e^{ir\sin\theta}\cdot[re^{-i\theta}-(a-ib)][re^{-i\theta}-(c-id)]}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}$$

$$=\frac{e^{r\cos\theta}\{r^2e^{i(r\sin\theta-2\theta)}-[(a+c)-i(b+d)]re^{i(r\sin\theta-\theta)}+[(ac-bd)-i(ad+bc)]e^{ir\sin\theta}\}}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}\ . \quad (10)$$

Therefore, if $r \neq 0$, then

$$\int_0^{2\pi}\frac{A}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta = \frac{1}{r}\mathrm{Re}[\int_C \frac{e^z}{(z-\lambda)(z-\mu)}dz]. \quad (11)$$

Case 1. If $|r| < \sqrt{a^2+b^2}$, then Eq.(7) holds when $r = 0$, and hence we may assume that $r \neq 0$. Using Eqs. (3) and (11) yields

$$\int_0^{2\pi}\frac{A}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta = 0.$$

Case 2. If $\sqrt{a^2+b^2} < |r| < \sqrt{c^2+d^2}$, then

$$\int_0^{2\pi}\frac{A}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta$$

$$=\frac{1}{r}\ \mathrm{Re}[\frac{2\pi ie^{a+ib}}{(a+ib)-(c+id)}] \quad \text{(by Eqs. (4) and (11))}$$

$$=\frac{2\pi e^a}{r[(a-c)^2+(b-d)^2]}\quad \mathrm{Re}\ \{[(b-d)+i(a-c)]e^{ib}\}$$

$$=\frac{2\pi e^a[(b-d)\cos b-(a-c)\sin b]}{r[(a-c)^2+(b-d)^2]}.$$

International Journal of Mathematical Engineering and Science (IJMES)
Volume 3 Issue 3 (Year 2014) ISSN : 2277-6982
http://www.ijmes.com/

Case 3. If $|r| > \sqrt{c^2+d^2}$, then

$$\int_0^{2\pi} \frac{A}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta$$

$$=\frac{1}{r}\ \mathrm{Re}\Big[\frac{2\pi i(e^{a+ib}-e^{c+id})}{(a+ib)-(c+id)}\Big] \qquad \text{(by Eqs. (5) and (11))}$$

$$=\frac{2\pi}{r[(a-c)^2+(b-d)^2]}\quad \mathrm{Re}\ \{[(b-d)+i(a-c)](e^{a+ib}-e^{c+id})\}$$

$$=\frac{2\pi[(b-d)(e^a\cos b-e^c\cos d)-(a-c)(e^a\sin b-e^c\sin d)]}{r[(a-c)^2+(b-d)^2]}. \qquad \square$$

Next, we determine the closed form of the definite integral (2).

Theorem 2 *If the assumptions are the same as Theorem 1, and let* $B = e^{r\cos\theta}[r^2\cos(r\sin\theta-\theta) - (b+d)r\sin(r\sin\theta) - (a+c)r\cos(r\sin\theta) + (ad+bc)\sin(r\sin\theta+\theta) + (ac-bd)\cos(r\sin\theta+\theta)]$*, then the definite integral*

$$\int_0^{2\pi} \frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta$$

$$=\begin{cases} 0 & if\ |r| < \sqrt{a^2+b^2}, \qquad (12)\\ \frac{2\pi e^a[(b-d)\sin b+(a-c)\cos b]}{r[(a-c)^2+(b-d)^2]} & if \sqrt{a^2+b^2} < |r| < \sqrt{c^2+d^2}, \qquad (13)\\ \frac{2\pi[(b-d)(e^a\sin b-e^c\sin d)+(a-c)(e^a\cos b-e^c\cos d)]}{r[(a-c)^2+(b-d)^2]} & if\ |r| > \sqrt{c^2+d^2}. \qquad (14)\end{cases}$$

Proof. If $r \neq 0$, then using Eq. (10) yields

$$\int_0^{2\pi} \frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta = \frac{1}{r}\mathrm{Im}\Big[\int_C \frac{e^z}{(z-\lambda)(z-\mu)}dz\Big]. \qquad (15)$$

Case 1. If $|r| < \sqrt{a^2+b^2}$. Also, we may assume that $r \neq 0$. Using Eqs. (3) and (15) yields

$$\int_0^{2\pi} \frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta = 0.$$

Case 2. If $\sqrt{a^2+b^2} < |r| < \sqrt{c^2+d^2}$, then

$$\int_0^{2\pi} \frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta$$

$$=\frac{1}{r}\ \mathrm{Im}\Big[\frac{2\pi i e^{a+ib}}{(a+ib)-(c+id)}\Big] \qquad \text{(by Eqs. (4) and (15))}$$

$$=\frac{2\pi e^{a}}{r[(a-c)^2+(b-d)^2]}\mathrm{Im}\{[(b-d)+i(a-c)](\cos b+i\sin b)\}$$

$$=\frac{2\pi e^{a}[(b-d)\sin b+(a-c)\cos b]}{r[(a-c)^2+(b-d)^2]}.$$

Case 3. If $|r|>\sqrt{c^2+d^2}$, we have

$$\int_0^{2\pi}\frac{B}{[(r\cos\theta-a)^2+(r\sin\theta-b)^2][(r\cos\theta-c)^2+(r\sin\theta-d)^2]}d\theta$$

$$=\frac{1}{r}\ \mathrm{Im}\Big[\frac{2\pi i(e^{a+ib}-e^{c+id})}{(a+ib)-(c+id)}\Big] \qquad \text{(by Eqs. (5) and (15))}$$

$$=\frac{2\pi}{r[(a-c)^2+(b-d)^2]}\ \mathrm{Im}\{[(b-d)+i(a-c)](e^{a+ib}-e^{c+id})\}$$

$$=\frac{2\pi[(b-d)(e^{a}\sin b-e^{c}\sin d)+(a-c)(e^{a}\cos b-e^{c}\cos d)]}{r[(a-c)^2+(b-d)^2]}. \qquad \square$$

3 Examples

In the following, for the two types of definite integrals in this study, we provide some examples and use Theorems 1 and 2 to obtain their closed forms. On the other hand, Maple is used to calculate the approximations of these definite integrals and their solutions for verifying our answers.

3.1 Example

In Theorem 1, let $r=3, a=2, b=-4, c=6$, and $d=-3$, then using Eq. (7) yields

$$\int_0^{2\pi}\frac{e^{3\cos\theta}[-9\sin(3\sin\theta-\theta)+21\cos(3\sin\theta)+24\sin(3\sin\theta)-30\cos(3\sin\theta+\theta)]}{[(3\cos\theta-2)^2+(3\sin\theta+4)^2][(3\cos\theta-6)^2+(3\sin\theta+3)^2]}d\theta=0\ . \tag{16}$$

Next, we use Maple to verify the correctness of Eq. (16).
>evalf(int(exp(3*cos(theta))*(-9*sin(3*sin(theta)-theta)+21*cos(3*sin(theta))+24*sin(3*sin(theta))-30*cos(3*sin(theta)+theta))/(((3*cos(theta)-2)^2+(3*sin(theta)+4)^2)*((3*cos(theta)-6)^2+(3*sin(theta)+3)^2)),theta=0..2*Pi),18);

0.

If $r=-5, a=-3, b=2, c=-7$, and $d=4$ in Theorem 1, then by Eq. (8) we obtain

$$\int_0^{2\pi}\frac{e^{-5\cos\theta}[-25\sin(-5\sin\theta-\theta)+30\cos(-5\sin\theta)+50\sin(-5\sin\theta)-26\cos(-5\sin\theta+\theta)-13\sin(-5\sin\theta+\theta)]}{[(-5\cos\theta+3)^2+(-5\sin\theta-2)^2][(-5\cos\theta+7)^2+(-5\sin\theta-4)^2]}d\theta$$

$$=\frac{\pi(\cos 2+2\sin 2)}{25e^{3}}. \tag{17}$$

We also use Maple to verify the correctness of Eq. (17).
>evalf(int(exp(-5*cos(theta))*(-25*sin(-5*sin(theta)-theta)+30*cos(-5*sin(theta))+50*sin(-5*sin(theta))-26*cos(-5*sin(theta)+theta)-13*sin(-5*sin(theta)+theta))/(((-5*cos(theta)+3)^2+(-5*sin(theta)-2)^2)*((-5*cos(theta)+7)^2+(-5*sin(theta)-4)^2)),theta=0..2*Pi),18);

$$0.00877431438238486168$$

>evalf(Pi*(cos(2)+2*sin(2))/(25*exp(3)),18);

$$0.00877431438238486168$$

On the other hand, let $r = 9, a = 2, b = -3, c = 4$, and $d = -5$ in Theorem 1, it follows from Eq. (9) that

$$\int_0^{2\pi} \frac{e^{9\cos\theta}[-81\sin(9\sin\theta-\theta)+72\cos(9\sin\theta)+54\sin(9\sin\theta)-22\cos(9\sin\theta+\theta)+7\sin(9\sin\theta+\theta)]}{[(9\cos\theta-2)^2+(9\sin\theta+3)^2][(9\cos\theta-4)^2+(9\sin\theta+5)^2]}d\theta$$

$$=\frac{\pi[e^2(\cos 3-\sin 3)-e^4(\cos 5-\sin 5)]}{18}. \tag{18}$$

>evalf(int(exp(9*cos(theta))*(-81*sin(9*sin(theta)-theta)+72*cos(9*sin(theta))+54*sin(9*sin(theta))-22*cos(9*sin(theta)+theta)+7*sin(9*sin(theta)+theta))/(((9*cos(theta)-2)^2+(9*sin(theta)+3)^2)*((9*cos(theta)-4)^2+(9*sin(theta)+5)^2)),theta=0..2*Pi),18);

$$-13.2995442906699572$$

>evalf(Pi*(exp(2)*(cos(3)-sin(3))-exp(4)*(cos(5)-sin(5)))/18,18);

$$-13.2995442906699576$$

3.2 Example

If $r = 2, a = -3, b = 2, c = -5$,and $d = 1$ in Theorem 2, it follows from Eq. (12) that

$$\int_0^{2\pi} \frac{e^{2\cos\theta}[4\cos(2\sin\theta-\theta)-6\sin(2\sin\theta)+16\cos(2\sin\theta)-13\sin(2\sin\theta+\theta)+13\cos(2\sin\theta+\theta)]}{[(2\cos\theta+3)^2+(2\sin\theta-2)^2][(2\cos\theta+5)^2+(2\sin\theta-1)^2]}d\theta = 0. \tag{19}$$

Using Maple to verify the correctness of Eq. (19) as follows:
>evalf(int(exp(2*cos(theta))*(4*cos(2*sin(theta)-theta)-6*sin(2*sin(theta))+16*cos(2*sin(theta))-13*sin(2*sin(theta)+theta)+13*cos(2*sin(theta)+theta))/(((2*cos(theta)+3)^2+(2*sin(theta)-2)^2)*((2*cos(theta)+5)^2+(2*sin(theta)-1)^2)),theta=0..2*Pi),18);

$$0.$$

Let $r = 6, a = 4, b = -3, c = 8$, and $d = -2$ in Theorem 2, then by Eq. (13) we have

$$\int_0^{2\pi} \frac{e^{6\cos\theta}[36\cos(6\sin\theta-\theta)+30\sin(6\sin\theta)-72\cos(6\sin\theta)-32\sin(6\sin\theta+\theta)+26\cos(6\sin\theta+\theta)]}{[(6\cos\theta-4)^2+(6\sin\theta+3)^2][(6\cos\theta-8)^2+(6\sin\theta+2)^2]}d\theta$$

$$=\frac{\pi e^4(\sin 3-4\cos 3)}{51}. \tag{20}$$

>evalf(int(exp(6*cos(theta))*(36*cos(6*sin(theta)-theta)+30*sin(6*sin(theta))-72*cos(6*sin(theta))-32*sin(6*sin(theta)+theta)+26*cos(6*sin(theta)+theta))/(((6*cos(theta)-4)^2+(6*sin(theta)+3)^2)*((6*cos(theta)-8)^2+(6*sin(theta)+2)^2)),theta=0..2*Pi),18);

13.7929424380305919

>evalf(Pi*exp(4)*(sin(3)-4*cos(3))/51,18);

13.7929424380305920

In addition, if $r=11, a=-1, b=5, c=-6$, and $d=7$ in Theorem 2, it follows from Eq. (14) that

$$\int_0^{2\pi}\frac{e^{11\cos\theta}[121\cos(11\sin\theta-\theta)-132\sin(11\sin\theta)+77\cos(11\sin\theta)-37\sin(11\sin\theta+\theta)-29\cos(11\sin\theta+\theta)]}{[(11\cos\theta+1)^2+(11\sin\theta-5)^2][(11\cos\theta+6)^2+(11\sin\theta-7)^2]}d\theta$$

$$=\frac{2\pi[\exp(-1)\cdot(5\cos 5-2\sin 5)-\exp(-6)\cdot(5\cos 7-2\sin 7)]}{319}. \tag{21}$$

>evalf(int(exp(11*cos(theta))*(121*cos(11*sin(theta)-theta)-132*sin(11*sin(theta))+77*cos(11*sin(theta))-37*sin(11*sin(theta)+theta)-29*cos(11*sin(theta)+theta))/(((11 *cos(theta)+1)^2+(11*sin(theta)-5)^2)*((11*cos(theta)+6)^2+(11*sin(theta)-7)^2)),theta=0..2*Pi),18);

0.0240537235835385117

>evalf(2*Pi*(exp(-1)*(5*cos(5)-2*sin(5)-exp(-6)*(5*cos(7)-2*sin(7))/319,18);

0.0240537235835385119

4 Conclusion

In this study, we mainly use Cauchy integral theorem and Cauchy integral formula to solve some definite integrals. In fact, the applications of the two methods are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

References

[1] A. A. Adams, H. Gottliebsen, S. A. Linton, and U. Martin,“ Automated theorem proving in support of computer algebra: symbolic definite integration as a case study,” *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation*, Canada, 1999, pp. 253-260.

[2] M. A. Nyblom, “On the evaluation of a definite integral involving nested square root functions,” *Rocky Mountain Journal of Mathematics*, vol. 37, no. 4, pp. 1301-1304, 2007.
[3] C. Oster, “Limit of a definite integral, *SIAM Review*, vol. 33, no. 1, pp. 115-116, 1991.
[4] C. -H. Yu, “Solving some definite integrals using Parseval’s theorem, ” *American Journal of Numerical Analysis*, vol. 2, no. 2, pp. 60-64, 2014.
[5] C. -H. Yu, “Some types of integral problems, ” *American Journal of Systems and Software*, vol. 2, no. 1, pp. 22-26, 2014.
[6] C. -H. Yu, “Using Maple to study the double integral problems, ” *Applied and Computational Mathematics*, vol. 2, no.2, pp. 28-31, 2013.
[7] C. -H. Yu, “A study on double Integrals, ” *International Journal of Research in Information Technology*, vol. 1, no. 8, pp. 24-31, 2013.
[8] C. -H. Yu, “Application of Parseval’s theorem on evaluating some definite integrals, ” *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 1, pp. 1-5, 2014.
[9] C. -H. Yu,“ Evaluation of two types of integrals using Maple, ” *Universal Journal of Applied Science*, vol. 2, no. 2, pp. 39-46, 2014.
[10] C. -H. Yu, “Studying three types of integrals with Maple, ” *American Journal of Computing Research Repository*, vol. 2, no. 1, pp. 19-21, 2014.
[11] C. -H. Yu, “The application of Parseval’s theorem to integral problems,” *Applied Mathematics and Physics*, vol. 2, no. 1, pp. 4-9, 2014.
[12] C. -H. Yu, “A study of some integral problems using Maple, ” *Mathematics and Statistics*, vol. 2, no. 1, pp. 1-5, 2014.
[13] C. -H. Yu, “Solving some definite integrals by using Maple, ” *World Journal of Computer Application and Technology*, vol. 2, no. 3, pp. 61-65, 2014.
[14] C. -H. Yu, “Using Maple to study two types of integrals, ” *International Journal of Research in Computer Applications and Robotics*, vol. 1, issue. 4, pp. 14-22, 2013.
[15] C. -H. Yu, “Solving some integrals with Maple,” *International Journal of Research in Aeronautical and Mechanical Engineering*, vol. 1, issue. 3, pp. 29-35, 2013.
[16] C. -H. Yu, “A study on integral problems by using Maple, ” *International Journal of Advanced Research in Computer Science and Software Engineering*, vol. 3, issue. 7, pp. 41-46, 2013.
[17] C. -H. Yu, “Evaluating some integrals with Maple, ” *International Journal of Computer Science and Mobile Computing*, vol. 2, issue. 7, pp. 66-71, 2013.
[18] C. -H. Yu, “Application of Maple on evaluation of definite integrals,” *Applied Mechanics and Materials*, vols. 479-480 (2014), pp.823-827, 2014.
[19] C. -H. Yu,“ Application of Maple on the integral problems,” *Applied Mechanics and Materials*, vols. 479-480 (2014), pp. 849-854, 2014.
[20] C. -H. Yu, “Using Maple to study the integrals of trigonometric functions,” *Proceedings of the 6th IEEE/International Conference on Advanced Infocomm Technology*, Taiwan, 2013, no. 00294.
[21] C. -H. Yu, “A study of the integrals of trigonometric functions with Maple, ” Proceedings of the Institute of Industrial Engineers Asian Conference, Taiwan, 2013, Springer, vol. 1, pp. 603-610.
[22] C. -H. Yu, “Application of Maple on the integral problem of some type of rational functions, ”(in Chinese) *Proceedings of the Annual Meeting and Academic Conference for Association of IE*, Taiwan, 2012, D357-D362.

[23] C. -H. Yu, "Application of Maple on some integral problems, " (in Chinese) *Proceedings of the International Conference on Safety and Security Management and Engineering Technology*, Taiwan, 2012, pp. 290-294.
[24] C. -H. Yu, "Application of Maple on some type of integral problem, " (in Chinese) *Proceedings of the Ubiquitous-Home Conference*, Taiwan, 2012, pp.206-210.
[25] C. -H. Yu, "Application of Maple on evaluating the closed forms of two types of integrals," (in Chinese) *Proceedings of the 17th Mobile Computing Workshop*, Taiwan, 2012, ID16.
[26] C. -H. Yu, "Application of Maple: taking two special integral problems as examples," (in Chinese) *Proceedings of the 8th International Conference on Knowledge Community*, Taiwan, 2012, pp. 803-811.
[27] C. -H. Yu," Evaluating some types of definite integrals, " *American Journal of Software Engineering*, vol. 2, no. 1, pp. 13-15, 2014.
[28] C. -H. Yu and B. -H. Chen, "Solving some types of integrals using Maple, " *Universal Journal of Computational Mathematics*, vol. 2, no. 3, pp. 39-47, 2014.
[29] C. -H. Yu and S. -D. Sheu, "Using area mean value theorem to solve some double integrals," *Turkish Journal of Analysis and Number Theory*, vol. 2, issue. 3, pp. 75-79, 2014.
[30] C. -H. Yu and S. -D. Sheu, "Infinite series forms of double integrals, " *International Journal of Data Envelopment Analysis and *Operations Research**, vol. 1, issue. 2, pp. 16-20, 2014.
[31] J. E. Marsden, Basic complex analysis, W. H. Freeman and Company, San Francisco, 1973.