A reconstruction method for the gradient of a function in two-dimensional space

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Abstract

Numerical differentiation is a classical ill-posed problem. In image processing, sometimes we have to compute the gradient of an image. This involves a problem of numerical differentiation. In this paper we present a truncation method to compute the gradient of a two-variables function which can be considered as an image. A Hölder-type stability estimate is obtained. Numerical examples show that the proposed method is effective and stable.

1 Introduction

Numerical differentiation problems arise in several contexts and have important applications in science and engineering [1-12]. The problem of numerical differentiation is well known to be ill-posed in the sense that a small perturbation in given data can induce a large error in the gradient function. For ill-posed problems, due to the ill-posdness, some regularization technique should be employed. In the existing literature, various aspects of the problem have been treated. We cannot give here an exhaustive survey. Let us review some computational methods, e.g., difference methods, spline methods, Tikhonov regularization methods. In this study, we consider a problem of numerical differentiation which arises in the image processing. For purposes of theoretical analysis, it is helpful to think of images $f(x, y)$ as functions in $L^2(\mathbb{R}^2)$, i.e., functions such that

$$\|f\| = (\int_{\mathbb{R}^2} |f(x,y)|^2 dxdy)^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|$ represents the L^2-norm. Denote 2D Fourier transform of $f(x, y)$ by

$$\hat{f}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(\xi x+\eta y)}dxdy. \tag{2}$$

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In imaging processing, we need to compute $\|\nabla f\|^2$. Then from Parseval's theorem, it yields

$$\int_{\mathbb{R}^2} |\nabla f|^2 dxdy = \int_{\mathbb{R}^2} \left|\frac{\partial f}{\partial\ x}\right|^2 + \left|\frac{\partial\ f}{\partial y}\right|^2 dxdy$$
$$= \int_{\mathbb{R}^2} (\xi^2 + \eta^2)|\hat{f}(\xi,\eta)|^2 d\xi d\eta. \tag{3}$$

Usually, the available data is the noisy images $f_\delta(x, y)$ with Fourier transform $\hat{f}_\delta(\xi,\eta)$. Hence, we have to compute $\|\nabla f_\delta\|^2$. According to (3), we have

$$\int_{\mathbb{R}^2} |\nabla f_\delta|^2 dxdy = \int_{\mathbb{R}^2} (\xi^2 + \eta^2)|\hat{f}_\delta(\xi,\eta)|^2 d\xi d\eta. \tag{4}$$

To make the last integral converge, the function $\hat{f}_\delta(\xi,\eta)$ must decay sufficiently fast at infinity. But in practice, this is not true for a noisy image. Therefore, this is an ill-posed problem. We need to introduce a regularization method. The idea is very simple and natural: since the ill-posedness of numerical differentiation is caused by the high frequency component, we cut off them. Actually, such a similar idea of solving numerical differentiation appeared in [13], but the literature is devoted to the one-dimensional space. In the present paper, as a remedy, we discuss the stability of gradient of two-variables function by employing the Fourier truncation method. Some numerical tests show that the proposed method is effective and stable.

This paper is organized as follows. In Section A, we analyze the ill-posedness of numerical differentiation and propose Fourier truncation method. In Section B, we discuss the stability of the truncation method and obtain a Hölder-type convergence estimate of the approximate gradient of the function . In Section C, we present some illustrative numerical examples.

2 Regularization

In this section we simply analyze the ill-posedness of numerical differentiation and discuss how to stabilize the gradient of the function. We set a function $f(x, y) \in H^p(\mathbb{R}^2)$ which is a Sobolev space, $p \geq 1$. In order to compute $\|\nabla f\|$, let

$$\hat{g}(\xi,\eta) = (\xi^{\ 2} + \eta^{\ 2})^{\frac{1}{2}}|\hat{f}(\xi,\eta)|, \tag{5}$$

i.e.,

$$g(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi^{\ 2} + \eta^{\ 2})^{\frac{1}{2}}|\hat{f}(\xi,\eta)|e^{i(\xi x+\eta y)}\mathrm{d}\xi\mathrm{d}\eta. \tag{6}$$

According to (3), we have $\|\nabla f\| = \|g\|$.

From the right hand side of (5), we know that $\xi^2 + \eta^2$ can be seen as an amplification factor of $\hat{f}(\xi,\eta)$. Therefore,when we consider our problem in $L^2(\mathbb{R}^2)$, the exact data function, $\hat{f}(\xi,\eta)$ must decay rapidly as $(\xi^2+\eta^2)\to\infty$. But in practice the input data is affected by noise. We assume the noisy data function $f_\delta \in L^2(\mathbb{R}^2)$ satisfies

$$\|f-f_\delta\| \le \delta, \tag{7}$$

where $\delta>0$ denotes the noise level. Thus, if we try to obtain the gradient of the function, high frequency components in the error are magnified and can destroy the solution. In this sense it is impossible to solve the problem using classical numerical methods and requires special techniques to be employed. In the forthcoming section, we will present our regularization method to treat the ill-posed problem. In addition we impose an a-priori bound on the input data(this is necessary for ill-posed problems), i.e.,

$$\|f\|_p \le E, p>1, \tag{8}$$

where $E>0$ is a constant, $\|f(x,y)\|_p$ denotes the norm in Sobolev space $H^p(\mathbb{R}^2)$ defined by

$$\|f(x,y)\|_p := (\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+\xi^2+\eta^2)^p|\hat{f}(\xi,\eta)|^2\mathrm{d}\xi\mathrm{d}\eta)^{\frac{1}{2}}.$$

Since the ill-posedness of the problem is caused by the high frequency components, a natural way to stabilize the problem is to eliminate all high frequencies and instead consider (6) only for $\xi^2+\eta^2\le\xi_{max}$ with the noisy data f_δ, where ξ_{max} is a regularization parameter. Then we get a regularized solution with noisy data

$$g^\delta_{\xi_{max}}(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}^2}(\xi^2+\eta^2)^{\frac{1}{2}}X_{max}|\hat{f}_\delta(\xi,\eta)|e^{i(\xi x+\eta y)}\mathrm{d}\xi\mathrm{d}\eta, \tag{9}$$

where X_{max} is the characteristic function satisfies

$$X_{max}(\xi,\eta)=\begin{cases}1, & (\xi^2+\eta^2)^{\frac{1}{2}}\le\xi_{max},\\ 0, & (\xi^2+\eta^2)^{\frac{1}{2}}>\xi_{max}.\end{cases}$$

In the following sections we will derive an error estimate and discuss how to compute it numerically.

3 Error estimate

In this section we derive a bound on the difference between (6) and (9). We assume that we have an a priori bound on the exact input data, $\|f\|_p \le E$ (see(8)). The relation between any two regularized solution (9) is given by the following lemma.

Lemma 1. Suppose that we have two regularized functions $g^\delta_{\xi_{max}}(x,y)$ and $g_{\xi_{max}}(x,y)$ defined by (9) with data f_δ and f,satisfying $\|f-f_\delta\|\le\delta$. If we select $\xi_{max}=(\frac{E}{\delta})^{\frac{1}{p}}, p>1$, then we can get the error bound

$$\|g_{\xi_{max}}(x,y)-g^\delta_{\xi_{max}}(x,y)\| \le E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}. \tag{10}$$

Proof. From the Parseval theorem we have

$$\begin{aligned}
&\|g_{\xi_{max}}(x,y) - g^{\delta}_{\xi_{max}}(x,y)\|^2_{L^2(R^2)} \\
&= \|\hat{g}_{\xi_{max}}(\xi,\eta) - \hat{g}^{\delta}_{\xi_{max}}(\xi,\eta)\|^2_{L^2(\mathbb{R}^2)} \\
&= \iint\limits_{(\xi^2+\eta^2)^{\frac{1}{2}} \le \xi_{\max}} (\xi^2+\eta^2)(|\hat{f}(\xi,\eta)| - |\hat{f}_\delta(\xi,\eta)|)^2 \mathrm{d}\xi\mathrm{d}\eta \\
&\le \xi^2_{\max}\|\hat{f} - \hat{f}_\delta\|^2 = \xi^2_{\max}\|f - f_\delta\|^2 \le \xi^2_{\max}\delta^2.
\end{aligned}$$

Using $\xi_{max} = (\frac{E}{\delta})^{\frac{1}{p}}$, we obtain

$$\|g_{\xi_{max}}(x,y) - g^{\delta}_{\xi_{max}}(x,y)\| \le \ E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}.$$

Form Lemma 1 we see that the function defined by (9) depends continuously on the input data f. Next we will investigate the difference between (6) and (9) with the same exact data f.

Lemma 2. Let $g(x,y)$ and $g_{\xi_{\max}}(x,y)$ be the (6) and (9) with the same exact data f, and let $\xi_{\max} = (\frac{E}{\delta})^{\frac{1}{p}}, p > 1$. Suppose that $\|f\|_p \le \ E$, then

$$\|g(x,y) - g_{\xi_{\max}}(x,y)\| \le \ E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}. \tag{11}$$

Proof. As in Lemma 1 we start with the Parseval relation, and using the fact $(|\xi|^2 + |\eta|^2)^{\frac{1}{2}} > \xi_{\max}$, we get

$$\begin{aligned}
&\|g(x,y) - g_{\xi_{\max}}(x,y)\|^2 = \|\hat{g}(\xi,\eta) - \hat{g}_{\xi_{\max}}(\xi,\eta)\|^2 \\
&= \iint_{(|\xi|^2+|\eta|^2)^{\frac{1}{2}} > \xi_{\max}} (\xi^2+\eta^2)|\hat{f}(\xi,\eta)|^2 \mathrm{d}\xi\mathrm{d}\eta \\
&= \iint_{(\xi^2+\eta^2)^{\frac{1}{2}} > \xi_{\max}} \frac{(\xi^2+\eta^2)}{(1+\xi^2+\eta^2)^p}(1+\xi^2+\eta^2)^p|\hat{f}(\xi,\eta)|^2 d\xi d\eta \\
&\le E^2 \sup_{(\xi^2+\eta^2)^{\frac{1}{2}} > \xi_{\max}} \frac{(\xi^2+\eta^2)}{(1+\xi^2+\eta^2)^p} \\
&\le E^2\xi^{2(1-p)}_{\max}.
\end{aligned}$$

Now we use the bound $\|f\|_p \le \ E$ (see(8)), and as before we have $\xi_{\max} = (\frac{E}{\delta})^{\frac{1}{p}}$ which leads to the error bound

$$\|g(x,y) - g_{\xi_{\max}}(x,y)\| \le \ E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}.$$

Now we are ready to formulate the main result of this section:

Theorem 1. Suppose that $g(x,y)$ is given by (6) with exact data f and that $g^{\delta}_{\xi_{\max}}(x,y)$ is given by (9) with noisy data f_δ. If we have a bound $\|f\|_p \le \ E$, and the measured function f_δ satisfies

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$\|f - f_\delta\| \le \delta$, and if we choose $\xi_{\max} = (\frac{E}{\delta})^{\frac{1}{p}}$ where $p > 1$, then we get the error bound

$$\|g(x,y) - g^{\delta}_{\xi_{\max}}(x,y)\| \le \ 2E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}. \tag{12}$$

Proof. Let $g_{\xi_{\max}}(x,y)$ be defined by (9) with exact data f. Then by using the triangle inequality and the two previous lemmas we get

$$\begin{aligned}
&\|g(x,y) - g^{\delta}_{\xi_{\max}}(x,y)\| \\
&\le \|g(x,y) - g_{\xi_{\max}}(x,y)\| + \|g_{\xi_{\max}}(x,y) - g^{\delta}_{\xi_{\max}}(x,y)\| \\
&\le E^{\frac{1}{p}}\delta^{1-\frac{1}{p}} + E^{\frac{1}{p}}\delta^{1-\frac{1}{p}} \\
&= 2E^{\frac{1}{p}}\delta^{1-\frac{1}{p}}.
\end{aligned}$$

From Theorem 1 we find that (9) is an approximation of the exact gradient function $g(x,y)$. The approximation error depends continuously on the measurement error.
Remark 1. From the the triangle inequality $\big|\|g(x,y)\| - \|g^{\delta}_{\xi_{\max}}(x,y)\|\big| \le \|g(x,y) - g^{\delta}_{\xi_{\max}}(x,y)\|$, we can see that if $\delta \to 0$, then $\|g^{\delta}_{\xi_{\max}}(x,y)\| \to \|g(x,y)\| = \|\nabla f\|$.

4 Numerical examples

Numerical implementation is completed by Matlab in IEEE double precision with unit round-off $1.1\cdot 10^{-16}$. The test interval is $(x,y) \in [0,1] \times [0,1]$ and the total number of test points is 50×50. The regularized solutions were computed by the 2D discrete Fast Fourier Transform (2D FFT) [14] and 2D inverse discrete Fast Fourier Transform (2D IFFT) according to formula (9). The regularization parameter $\xi_{\max}$ is chosen by Theorem 1. In the following numerical test, we give comparison between the numerical solution and exact solution $g(x,y)$.

Example 1. Consider a function

$$f(x,y) = e^{-(x-0.5)^2-(y-0.5)^2}.$$

Fig.1(1) shows the regularization solution with 1% random noise and $\xi_{\max} = 50$.

Fig.1(2) shows the exact solution $g(x,y)$.

Example 2. Consider a function

$$f(x,y) = \sin(2\pi x)\sin(2\pi y).$$

Fig.2(1) shows the regularization solution with 1% random noise and $\xi_{\max} = 120$.

Fig.2(2) shows the exact solution $g(x,y)$.

5 Conclusion

We have proposed an efficient numerical method for solving a classical ill-posed problem-numerical differentiation. We have proved that the numerical method is stable and given a Hölder-type error

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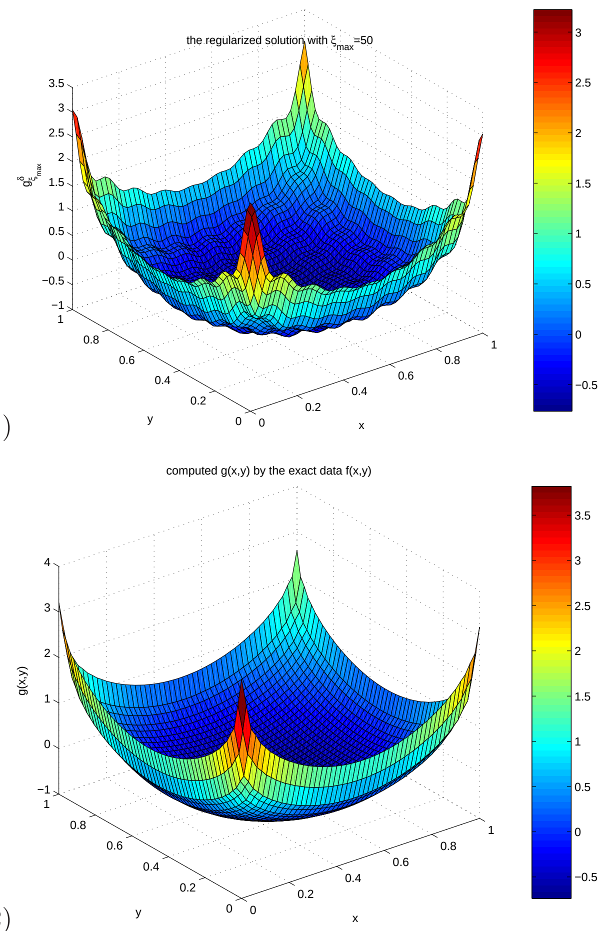
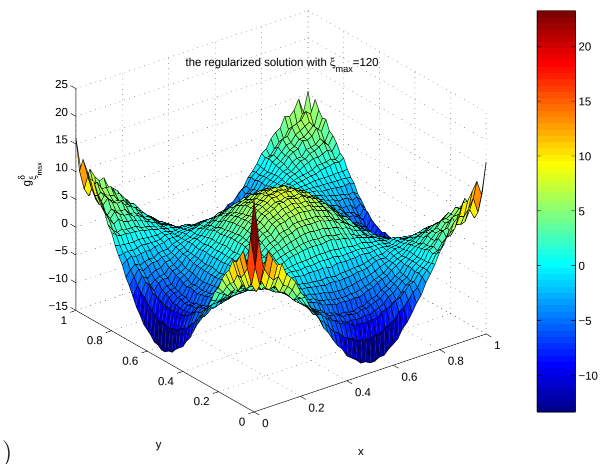


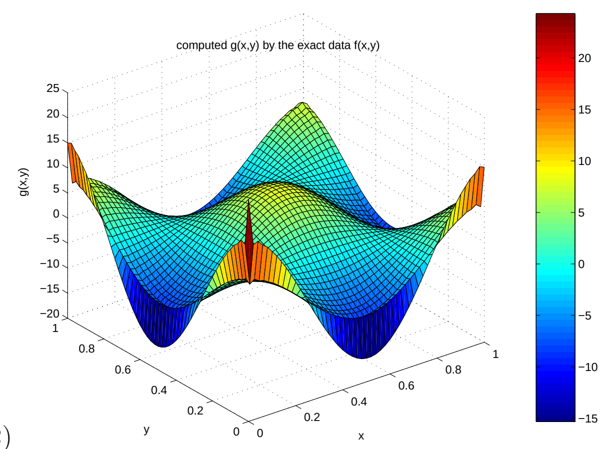
(1)

(2)

Figure 1: (1). The regularized solution; (2). The exact solution.



(1)



(2)

Figure 2: (1). The regularized solution; (2). The exact solution.

estimate. The method is based on Fourier truncation in the frequency space. The theoretical estimate is simple and the algorithm is effective. The method will be expected to deal with other ill-posed problems when we find the ill-posedness of the problem is caused by the high frequency components.

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