An $O(h^{10})$ Methods For Numerical Solutions Of Some Differential Equations Occurring In Plate Deflection Theory

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Abstract

A tenth-order non-polynomial spline method for the solutions of two-point boundary value problem $u^{(4)}(x) + f(x, u(x)) = 0, u(a) = \lambda_1, u''(a) = \lambda_2, u(b) = \lambda_3, u''(b) = \lambda_4$, is constructed. Numerical method of tenth-order with end conditions of the order 10 is derived. The convergence analysis of the method has been discussed. Numerical examples are presented to illustrate the applications of method, and to compare the computed results with other known methods.

Keywords : Non-polynomial spline, Boundary formula, Convergence analysis.

1 Introduction

Consider the special nonlinear fourth-order boundary value problem given by:

$$u^{(4)}(x) + f(x, u(x)) = 0, a < x < b, a, b, x \in \Re, \tag{1}$$

with the following boundary conditions

$$u(a) = \lambda_1, u''(a) = \lambda_2, u(b) = \lambda_3, u''(b) = \lambda_4. \tag{2}$$

It is assumed that $f(x, u(x))$ is real and continuous on $[a, b]$, and $\lambda_i, i = 1, 2, 3$ and 4, are finite real constants. For details of the existence and uniqueness of the real valued function $u(x)$ which satisfies

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(1)-(2) see [1]. E. H. Twizell in [20] derived a fourth-order finite difference method for the numerical solution of (1)-(2). C. P. Katti [11] has given a sixth order finite difference method for the two-point boundary value problem (1) with first-order derivative boundary conditions. In the case of linear differential equations (1), The class of fourth-order method for numerical solutions of two-point boundary value problems have been obtained by some authors see Usmani [22]- [25], Usmani et al. [26], Rashidinia et al. [16]- [18] and references therein. Numerical methods based on the finite difference of the various orders by which the solution of (1) are approximated over a finite set of grid points have been developed by Chawla et al. [5]- [6], Jain et al. [9]- [10] and references therein. Daele et al. [7] introduced a new second order method for solving the boundary value problems (1) based on non-polynomial spline function. Al-Said et al. [2]- [3] have developed numerical methods for solutions of fourth-order obstacle problems with collocation, finite difference and spline techniques. S. S. Siddiqi and G. Akram [19] analyzed a system of fourth-order boundary value problems using non-polynomial spline functions. M.A. Ramadan and his coworkers [15] developed quintic non-polynomial spline solutions for fourth-order two-point boundary value problem. Siraj-ul-Islam et al. [21] developed numerical methods based on quartic non-polynomial splines for solution of of a system of third-order boundary-value problems. M.A.Khan et al. [13] have been developed and analyzed a class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems. Khan et al. [12] used parametric quintic spline function for the solution of a system of fourth-order boundary-value problems. Numerical methods for nonlinear fourth-order boundary value problems study by Mohamed Alihajji and Kamel Al-khaled [4]. Wazwaz [27] applied ADM meyhod for solving a special $2m$ order boundary value problem of the form $u^{(2m)}(x) = f(x, u), 0 < x < b$.

In this paper non-polynomial septic spline relations have been derived. We apply such non-polynomial septic spline functions that have polynomial and trigonometric parts to develop new numerical method for obtaining smooth approximations to the solutions (1)-(2). Non-polynomial septic spline formulation is derived in section 2. We develop the $O(h^{10})$ methods at end conditions in section 3. In section 4, convergence analysis is proved. Finally, in section 5, Numerical examples are given to illustrate the applications of the method. We introduce the set of grid points in the interval $[a, b]$

$$x_0 = a, \;\; x_l = a + (l)h, \;\; h = \frac{b-a}{N}, \;\; l = 1, 2, ..., N, \;\; x_N = b.$$

Non-polynomial septic spline function $S_l(x)$ which interpolates $u(x)$ at the mesh points x_l, $l = 1, 2, ..., N$, depends on a parameter τ and reduces to ordinary septic spline $S_l(x)$ in $[a, b]$ as $\tau \to 0$. For each segment $[x_l, x_{l+1}]$, $l = 1, 2, ..., N-1$, the septic spline $S_l(x)$, is defined as

$$S_l(x) = \sum_{i=0}^{5} a_{li}(x - x_l)^i + e_l \sin \tau(x - x_l) + f_l \cos \tau(x - x_l), \; l = 0, 1, 2, ..., N, \tag{3}$$

where $a_{li}, (i = 0, 1, 2, 3, 4, 5), e_l$ and f_l are constants and τ is free parameter.

Let u_l be an approximation to $u(x_l)$, obtained by the segment $S_l(x)$ of the mixed spline function passing through the points (x_l, u_l) and (x_{l+1}, u_{l+1}), to obtain the necessary conditions for the coefficients introduced in (3), we do not only require that $S_l(x)$ satisfies interpolatory conditions at

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x_l, x_{l+1} but also the continuity of second, four and six derivatives at the common nodes (x_l, u_l).

To derive expression for the coefficients, we first denote:

$$\begin{cases} S_l(x_l) = u_l, S_l''(x_l) = M_l, S_l^{(4)}(x_l) = N_l, S_l^{(6)}(x_l) = L_l, \\ S_l(x_{l+1}) = u_{l+1}, S_l''(x_{l+1}) = M_{l+1}, S_l^{(4)}(x_{l+1}) = N_{l+1}, S_l^{(6)}(x_{l+1}) = L_{l+1}. \end{cases} \tag{4}$$

Using the continuity of first, third and fifth derivatives at (x_l, u_l), that are $S_{l-1}^{(\xi)}(x_l) = S_l^{(\xi)}(x_l), \xi = 1, 3$ and 5, and also by elimination of M_l, L_l' we obtain the following relations between N_l and u_l:

$$h^4 \left(\alpha_1 N_{l-3} + \alpha_2 N_{l-2} + \alpha_3 N_{l-1} + \alpha_4 N_l + \alpha_3 N_{1+l} + \alpha_2 N_{2+l} + \alpha_1 N_{3+l}\right) = -$$
$$\left(u_{l-3} + \beta_1 u_{l-2} + \beta_2 u_{l-1} + \beta_3 u_l + \beta_2 u_{1+l} + \beta_1 u_{2+l} + u_{3+l}\right), l = 3, ..., N-3, \tag{5}$$

where

$$\alpha_1 = \frac{\left(120\theta - 20\theta^3 + \theta^5 - 120Sin[\theta]\right)}{120\theta^4(-\theta + Sin[\theta])},$$

$$\alpha_2 = -2\frac{\left(240\theta + 20\theta^3 - 13\theta^5 + \theta\left(120 - 20\theta^2 + \theta^4\right)Cos[\theta] - 360Sin[\theta]\right)}{120\theta^4(-\theta + Sin[\theta])},$$

$$\alpha_3 = \frac{\left(840\theta + 100\theta^3 + 67\theta^5 + \left(960\theta + 80\theta^3 - 52\theta^5\right)Cos[\theta] - 1800Sin[\theta]\right)}{120\theta^4(-\theta + Sin[\theta])},$$

$$\alpha_4 = -4\frac{\left(240\theta + 20\theta^3 - 13\theta^5 + 3\theta\left(120 + 20\theta^2 + 11\theta^4\right)Cos[\theta] - 600Sin[\theta]\right)}{120\theta^4(-\theta + Sin[\theta])},$$

$$\beta_1 = \frac{240\theta^4(2\theta + \theta Cos[\theta] - 3Sin[\theta])}{120\theta^4(-\theta + Sin[\theta])}, \beta_2 = -\frac{120\theta^4(7\theta + 8\theta Cos[\theta] - 15Sin[\theta])}{120\theta^4(-\theta + Sin[\theta])},$$

$$\beta_3 = \frac{480\theta^4(2\theta + 3\theta Cos[\theta] - 5Sin[\theta])}{120\theta^4(-\theta + Sin[\theta])}, \theta = \tau h.$$

When $\tau \to 0, (\tau h = \theta)$, that $\theta \to 0$, then:
$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3) \to (-\frac{1}{840}, -\frac{1}{7}, -\frac{397}{280}, -\frac{302}{105}, 0, -9, 16)$, and the relations defined by (5) reduce into septic polynomial spline function [28]. Now by using the spline relation (5) and discretize the given system (1) at the grid points x_l. We obtain $(N-5)$ nonlinear equation in the $(N-1)$ unknowns u_l, $l = 1, 2, ..., N-1$ as

$$(u_{l-3} - \alpha_1 h^4 f(x_{l-3}, u_{l-3})) + (\beta_1 u_{l-2} - \alpha_2 h^4 f(x_{l-2}, u_{l-2})) +$$
$$(\beta_2 u_{l-1} - \alpha_3 h^4 f(x_{l-1}, u_{l-1})) + (\beta_3 u_l - \alpha_4 h^4 f(x_l, u_l)) +$$
$$(\beta_2 u_{l+1} - \alpha_3 h^4 f(x_{l+1}, u_{l+1})) + (\beta_1 u_{l+2} - \alpha_2 h^4 f(x_{l+2}, u_{l+2})) +$$
$$(u_{l+3} - \alpha_1 h^4 f(x_{l+3}, u_{l+3})) + t_l = 0, \qquad l = 3(1)N-3. \tag{6}$$

by the Taylor expansion the local truncation errors $t_l, l = 3, ..., N-3$, associated with our method are given by

$$\begin{aligned}
t_l = (2 + 2\beta_1 + 2\beta_2 + \beta_3)\, u_l + (9 + 4\beta_1 + \beta_2)\, h^2 u_l^{(2)} + \\
\frac{1}{12}(81 + 24\alpha_1 + 24\alpha_2 + 24\alpha_3 + 12\alpha_4 + 16\beta_1 + \beta_2)\, h^4 u_l^{(4)} + \\
\frac{1}{360}(729 + 3240\alpha_1 + 1440\alpha_2 + 360\alpha_3 + 64\beta_1 + \beta_2)\, h^6 u_l^{(6)} + \\
\frac{1}{20160}(6561 + 136080\alpha_1 + 26880\alpha_2 + 1680\alpha_3 + 256\beta_1 + \beta_2)\, h^8 u_l^{(8)} + \\
\frac{1}{1814400}(59049 + 3674160\alpha_1 + 322560\alpha_2 + 5040\alpha_3 + 1024\beta_1 + \beta_2)\, h^{10} u_l^{(10)} + \\
\frac{1}{239500800}(531441 + 77944680\alpha_1 + 3041280\alpha_2 + 11880\alpha_3 + 4096\beta_1 + \beta_2)\, h^{12} u_l^{(12)} + \\
\frac{1}{43589145600}(4782969 + 1418593176\alpha_1 + 24600576\alpha_2 + 24024\alpha_3 + 16384\beta_1 + \beta_2) \\
h^{14} u_l^{(14)} + O(h^{15}).
\end{aligned} \tag{7}$$

For different choices of parameters $\alpha_\zeta, \zeta = 1, 2, 3, 4$ and $\beta_\zeta, \zeta = 1, 2, 3$ we can obtain classes of methods such as:

Fourth-order method

For $\alpha_1 = -\frac{1}{840}, \alpha_2 = -\frac{1}{7}, \alpha_3 = -\frac{397}{280}, \alpha_4 = -\frac{302}{105}, \beta_1 = 0, \beta_2 = -9$, and $\beta_3 = 16$, gives $t_l = \frac{1}{120} h^8 u_l^{(8)} + O(h^9), \quad l = 3, ..., N-3$.

Tenth-order method

For $\alpha_1 = -\frac{113}{206640}, \alpha_2 = -\frac{5347}{34440}, \alpha_3 = -\frac{18593}{13776}, \alpha_4 = -\frac{29371}{10332}, \beta_1 = -\frac{6}{41}, \beta_2 = -\frac{345}{41}$, and $\beta_3 = \frac{620}{41}$, gives $t_l = \frac{383}{109105920} h^{14} u_l^{(14)} + O(h^{15}), \quad l = 3, ..., N-3$.

2 End condition

To obtain the unique solution of the nonlinear system (6) we need four more equations. By using Taylor series and method of undetermined coefficients the boundary formulas associate with boundary conditions for the tenth-order method can be determine as follows. In order to obtain the tenth-order boundary formula we define the following identities

$$\begin{aligned}
&\textstyle\sum_{k=0}^{4} \gamma_k u_k + \mu_1 h^2 u_0'' = h^4 \sum_{k=0}^{9} \eta_k u_k^{(4)} + t_1 h^{14} u_0^{(14)} + O(h^{(15)}), i = 1 \\
&\textstyle\sum_{k=0}^{5} \nu_k u_k + \mu_2 h^2 u_0'' = h^4 \sum_{k=0}^{9} \sigma_k u_k^{(4)} + t_2 h^{14} u_0^{(14)} + O(h^{(15)}), i = 2, \\
&\textstyle\sum_{k=0}^{5} \nu_k u_{N-k} + \mu_2 h^2 u_N'' = h^4 \sum_{k=0}^{9} \sigma_k u_{N-k}^{(4)} + t_{N-2} h^{14} u_N^{(14)} + O(h^{(15)}), i = N-2, \\
&\textstyle\sum_{k=0}^{4} \gamma_k u_{N-k} + \mu_1 h^2 u_N'' = h^4 \sum_{k=0}^{9} \eta_k u_{N-k}^{(4)} + t_{N-1} h^{14} u_N^{(14)} + O(h^{(15)}), i = N-1,
\end{aligned} \tag{8}$$

by using Taylor 's expansion we obtain the unknown coefficients in (8) as follows:
$(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \mu_1) = (-7, 16, -10, 0, 1, 4)$,

$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \mu_2) = (2, -10, 16, -9, 0, 1, 1)$,

$(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9,) =$
$(\frac{101231}{427680}, \frac{2597713}{777600}, \frac{73069}{1871100}, \frac{34130843}{14968800}, -\frac{17915087}{7484400}, \frac{11439803}{5987520}, -\frac{990049}{935550}, \frac{828989}{2138400}, -\frac{1266343}{14968800}, \frac{71323}{8553600})$,

$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9) =$
$(\frac{5605147}{95800320}, \frac{15751493}{10886400}, \frac{329900749}{119750400}, \frac{100658183}{59875200}, -\frac{48060091}{239500800}, \frac{174017}{598752}, -\frac{19551923}{119750400}, \frac{3625253}{59875200}, -\frac{912767}{68428800}, \frac{158633}{119750400})$,

$(t_1 = t_{N-1} = -\frac{3900331}{530712000}), \quad (t_2 = t_{N-2} = -\frac{231752503}{186810624000})$.

3 Convergence analysis

In this section, we investigate the convergence analysis of the fourth-order method and also in the same way we can prove the convergence analysis for any of the other methods. The equations (6) along with boundary condition (8) yields the nonlinear system of equations, and may be written in matrix form as

$$A_0 U^{(1)} + h^4 B \mathbf{f}^{(1)}(U^{(1)}) = R^{(1)}, \tag{9}$$

in (9) the matrices A_0 and B are order $N-1$ and are given by

$$A_0 = P^3 + 6P^2, \tag{10}$$

$P = (p_{ij})$ is monotone three diagonal matrix defined by

$$p_{ij} = \begin{cases} 2 & i = j = 1, 2, 3, ..., N-1, \\ -1 & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

By using Henrici [29] the matrix P is a monotone matrix and we have

$$\|(P)^{-1}\| \leq \frac{(b-a)^2}{8h^2}. \tag{12}$$

and the matrix B in case of fourth-order method defined by

$$B = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 & \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \\ \frac{120}{840} & \frac{1191}{840} & \frac{2416}{840} & \frac{1191}{840} & \frac{120}{840} & \frac{1}{840} & & & & \\ \frac{1}{840} & \frac{120}{840} & \frac{1191}{840} & \frac{2416}{840} & \frac{1191}{840} & \frac{120}{840} & \frac{1}{840} & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{840} & \frac{120}{840} & \frac{1191}{840} & \frac{2416}{840} & \frac{1191}{840} & \frac{120}{840} & \frac{1}{840} \\ & & & & \frac{1}{840} & \frac{120}{840} & \frac{1191}{840} & \frac{2416}{840} & \frac{1191}{840} & \frac{120}{840} \\ & \sigma_9 & \sigma_8 & \sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \\ & \eta_9 & \eta_8 & \eta_7 & \eta_6 & \eta_5 & \eta_4 & \eta_3 & \eta_2 & \eta_1 \end{pmatrix} \tag{13}$$

We get that

$$A_0 = P^3 + 6P^2, \tag{14}$$

where A_0 is seven-diagonal matrix thus we have the following theorem.
Theorem 4.1 If $M = P^n + \lambda P^{n-1}$, where P is given by (11) and $\lambda \in \mathbb{R}^+, n \in \mathbb{N}$, then M is a monotone matrix and

$$M^{-1} = \frac{1}{\lambda}[P^{^{-(n-1)}} - \frac{P^{^{-(n-2)}}}{\lambda}(I + \frac{P}{\lambda})^{-1}], \quad \|M^{-1}\| \leq \frac{1}{\lambda}(\frac{(b-a)^2}{8h^2})^{(n-1)}.$$

Proof.
The matrix P is a monotone matrix see Henrici [29] and we have

$$M = P^n + \lambda P^{n-1} \Longrightarrow M^{-1} = [P^n + \lambda P^{n-1})]^{-1} = [P + \lambda I)]^{-1} P^{^{-(n-1)}}$$

$$M^{-1} = \frac{1}{\lambda}[I + \frac{P}{\lambda}]^{-1} P^{^{-(n-1)}} = \frac{1}{\lambda}[I - \frac{P}{\lambda} + (\frac{P}{\lambda})^2 - (\frac{P}{\lambda})^3 + ...]P^{^{-(n-1)}} =$$

$$\frac{1}{\lambda}[I - \frac{P}{\lambda}(I - \frac{P}{\lambda} + (\frac{P}{\lambda})^2 - ...)]P^{^{-(n-1)}} = \frac{1}{\lambda}[P^{^{-(n-1)}} - \frac{P^{^{-(n-2)}}}{\lambda}(I + \frac{P}{\lambda})^{-1}] =$$

$$M^{-1} = \frac{1}{\lambda}[P^{^{-(n-1)}} - \frac{P^{^{-(n-2)}}}{\lambda}(I + \frac{P}{\lambda})^{-1}] < \frac{1}{\lambda}[P^{^{-(n-1)}}],$$

$$\text{by using (12) we get} \quad \|M^{-1}\| \leq \frac{1}{\lambda}(\frac{(b-a)^2}{8h^2})^{(n-1)}.$$

by using theorem 4.1 we obtain

$$\|A_0^{-1}\| \leq \frac{(b-a)^4}{384h^4}, \tag{15}$$

The matrixs $\mathbf{f}^{(1)}$ and $R^{(1)}$ each have $N-1$ components and are given by

$$\mathbf{f}^{(1)} = (f_1^{(1)}, ..., f_{N-1}^{(1)})^t \tag{16}$$

where $\mathbf{f}_l^{(1)}(U^{(1)}) = f(x_l, u_l^{(1)}), l = 1, 2, ..., N-1$, and

$$R^{(1)} = \begin{pmatrix} -\gamma_0\lambda_1 - \mu_1 h^2\lambda_2 + h^4\eta_0 f(x_0, \lambda_1), \\ -\nu_0\lambda_1 - \mu_2 h^2\lambda_2 + h^4\sigma_0 f(x_0, \lambda_1), \\ -\lambda_1 + \frac{h^4}{840} f(x_0, \lambda_1), \\ 0 \\ \vdots \\ 0 \\ -\lambda_3 + \frac{h^4}{840} f(x_N, \lambda_3), \\ -\nu_0\lambda_3 - \mu_2 h^2\lambda_4 + h^4\sigma_0 f(x_N, \lambda_3), \\ -\gamma_0\lambda_3 - \mu_1 h^2\lambda_4 + h^4\eta_0 f(x_N, \lambda_3), \end{pmatrix}. \tag{17}$$

We suppose that

$$A_0\overline{U}^{(1)} + h^4B\mathbf{f}^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)}, \tag{18}$$

where the vector $\overline{U}^{(1)} = u(x_l), l = 1, 2, ..., N-1.$ is the exact solution and $t^{(1)} = [t_1^{(1)}, t_2^{(1)}, ..., t_{N-1}^{(1)}]^T$, is the vector of order $N-1$ of local truncation errors. From (9) and (18) we have:

$$[A]E^{(1)} = [A_0 + h^4BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \tag{19}$$

where

$$E^{(1)} = \overline{U}^{(1)} - U^{(1)} = [e_1^{(1)}, e_2^{(1)}, ..., e_{N-1}^{(1)}]^T,$$

$$\mathbf{f}^{(1)}(\overline{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)}, \tag{20}$$

and $F_k(U^{(1)}) = \text{diag}\{\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\}, l = 1, 2, ..., N-1$, is a diagonal matrix of order $N-1$.

Lemma 4.1 If M is a square matrix of order N and $\|\text{M}\| < 1$, then $(I + \text{M})^{-1}$ exist and $\|(I + \text{M})^{-1}\| \leq \frac{1}{(1-\|\text{M}\|)}$.

Lemma 4.2 The matrix $[A_0 + h^4BF_k(U^{(1)})]$ in (19) is nonsingular, provided $Y < \frac{5748019200}{172185917(b-a)^4}$, where $Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = 1, 2, ..., N-1$. (The norm referred to is the L_∞ norm).

Proof:

We know that $[A_0 + h^4BF_k(U^{(1)})] = A_0[I + h^4A_0^{-1}BF_k(U^{(1)})]$, we need to show that inverse of $[I + h^4A_0^{-1}BF_k(U^{(1)})]$ exist. By using lemma 4.1, we have

$$h^4\|A_0^{-1}BF_k(U^{(1)})\| \leq h^4\|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\| < 1, \tag{21}$$

by using (13) we obtain $\|B\| \leq \frac{172185917}{14968800}$ and also we have $\|F_k(U^{(1)})\| \leq Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = 1, 2, ..., N-1$, and then by using (15) and (21) we obtain

$$Y < \frac{5748019200}{172185917(b-a)^4}.$$

As a consequence of Lemmas 4.2 and 4.1 the nonlinear system (9) has a unique solution if $Y < \frac{5748019200}{172185917(b-a)^4}$.

Theorem 4.3 Let $u(x_l)$ be the exact solution of the boundary value problem (1) with boundary conditions (2) and we assume u_l, $l = 1, 2, ..., N-1$ be the numerical solution obtained by solving the nonlinear system (9). Then we have:

$$\|E^{(1)}\| \equiv O(h^4), (\text{provided}, Y < \frac{5748019200}{172185917(b-a)^4}, \text{for fourth-order method})$$

Proof: We can write the error equation (19) in the following form

$$E^{(1)} = (A_0 + h^4 BF_k(U^{(1)}))^{-1} t^{(1)} = (I + h^4 A_0^{-1} BF_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)},$$
$$\|E^{(1)}\| \leq \|(I + h^4 A_0^{-1} BF_k(U^{(1)}))^{-1}\| \|A_0^{-1}\| \|t^{(1)}\| \|,$$

It follows that

$$\|E^{(1)}\| \leq \frac{\|A_0^{-1}\| \|t^{(1)}\|}{1 - h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\|}, \tag{22}$$

provided that $h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1$. Also we have

$$\|t^{(1)}\| \leq \frac{1}{120} h^8 M_8, \tag{23}$$

$$\alpha_1 = -\frac{1}{840}, \alpha_2 = -\frac{1}{7}, \alpha_3 = -\frac{397}{280}, \alpha_4 = -\frac{302}{105}, \beta_1 = 0, \beta_2 = -9, \beta_3 = 16,$$

where $M_8 = \max|u^{(8)}(\xi)|, a \leq \xi \leq b$.
Substituting $\|A_0^{-1}\|, \|F_k(U^{(1)})\|$, $\|B\|$ and $\|t^{(1)}\|$ from above relations in (22) and simplifying we obtain

$$\|E^{(1)}\| \leq \frac{124740(a-b)^4 h^4 M_8}{5748019200 - 172185917(a-b)^4 Y} \equiv O(h^4), \tag{24}$$

It is a fourth-order convergent method provided

$$Y < \frac{5748019200}{172185917(b-a)^4}. \tag{25}$$

□

Corollary
In the same way we can prove the convergence analysis for tenth-other method and

$$\|E^{(1)}\| \equiv O(h^{10}), \tag{26}$$

4 Numerical results

In this section we present the results obtained by applying the numerical methods discussed in pervious sections to the following two-point boundary-value problems.

Examples 1-4 has been solved using our methods with different values of $\alpha_1 = -\frac{113}{206640}, \alpha_2 = -\frac{5347}{34440}, \alpha_3 = -\frac{18593}{13776}, \alpha_4 = -\frac{29371}{10332}, \beta_1 = -\frac{6}{41}, \beta_2 = -\frac{345}{41}, \beta_3 = \frac{620}{41}$, and also compared the obtained solution with the exact solution. The maximum absolute errors in solutions of tenth-order method are tabulated in Tables 1. The maximum absolute errors in solutions of examples 1-4 are compared

Table 1: Maximum absolute errors in solution with tenth-order method

m	Example 3	Example 2	Example 4	Example 1
4	2.245×10^{-16}	1.102×10^{-10}	7.026×10^{-17}	1.065×10^{-14}
5	5.622×10^{-20}	1.046×10^{-13}	1.593×10^{-20}	2.366×10^{-18}
6	1.263×10^{-22}	5.480×10^{-17}	3.278×10^{-24}	4.770×10^{-22}
7	1.645×10^{-27}	1.955×10^{-20}	3.267×10^{-28}	4.385×10^{-26}
8	1.091×10^{-30}	5.158×10^{-24}	4.097×10^{-31}	6.406×10^{-29}
9	1.719×10^{-33}	6.286×10^{-28}	5.639×10^{-34}	8.631×10^{-32}
10	1.840×10^{-36}	6.157×10^{-31}	5.907×10^{-37}	9.007×10^{-35}

Table 2: Maximum absolute errors in solution with fourth-order method

m	Example 3	Example 2	Example 4	Example 1
4	8.352×10^{-9}	7.834×10^{-8}	3.393×10^{-7}	2.324×10^{-8}
5	5.625×10^{-10}	6.3106×10^{-9}	2.273×10^{-8}	1.589×10^{-9}
6	3.581×10^{-11}	4.280×10^{-10}	1.444×10^{-9}	1.016×10^{-10}
7	2.247×10^{-12}	2.746×10^{-11}	9.062×10^{-11}	6.387×10^{-12}
8	1.406×10^{-13}	1.728×10^{-12}	5.667×10^{-12}	3.997×10^{-13}
9	8.792×10^{-15}	1.082×10^{-13}	3.543×10^{-13}	2.499×10^{-14}
10	5.495×10^{-16}	6.766×10^{-15}	2.214×10^{-14}	1.562×10^{-15}

with methods in [1,9,15,17,20,23,30]. Examples 1-4 has been solved using fourth-order method and the maximum absolute errors in solutions are tabulated in Tables 2.

Example 1: We consider the differential equation

$$u^{(4)} - 5u^3 = 96x\mathrm{Cos}(x) - 16x\left(-1+x^2\right)\mathrm{Cos}(x) + 24\mathrm{Sin}(x) - 48x^2\mathrm{Sin}(x)$$
$$-24\left(-1+x^2\right)\mathrm{Sin}(x) + \left(-1+x^2\right)^2\mathrm{Sin}(x) - 5\left(-1+x^2\right)^6\mathrm{Sin}(x)^3,$$
$$0 < x < 1, \tag{27}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 8\ \mathrm{Sin}(1). \tag{28}$$

The analytical solution is $u(x) = (x^2-1)^2\mathrm{Sin}(x)$.
Example 2: Consider the differential equation

$$u^{(4)} - 6e^{4u} = -\frac{12}{(1+x)^4}, 0 < x < 1, \tag{29}$$

with the boundary conditions:

$$u(0) = 0, u(1) = \ln(2), u''(0) = -1, \quad u''(0) = \frac{-1}{4}. \tag{30}$$

The analytical solution is $u(x) = \ln(1+x)$.

Example 3: Consider the differential equation

$$u^{(4)} + xu = -(8 + 7x + x^3)e^x, \quad 0 < x < 1, \tag{31}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = -4e. \tag{32}$$

The analytical solution for this boundary value problem is $u(x) = x(1-x)e^x$.
Example 4: Consider the following problem ,

$$u^{(4)} - u = -8x\text{Cos}(x) - 12\text{Sin}(x), \quad 0 < x < 1, \tag{33}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 4\text{Cos}(1) + 2\text{Sin}(1). \tag{34}$$

The exact solution is given by $u(x) = (x^2 - 1)\text{Sin}(x)$. Examples 1-4 solved by using non-polynomial septic spline method with step lengths $h = 2^{-m}, m = 4, ..., 10$, with fourth and tenth order and also the maximum absolute errors in solutions for our method are listed in tables 1-2.

Table 3: Maximum absolute errors in solution Example 2.

m	Second-order in [20]	Fourth-order in [20]	Method A in [1]	Method B in [1]
3	1.9×10^{-4}	3.7×10^{-6}	1.4×10^{-5}	1.4×10^{-5}
4	4.6×10^{-5}	2.9×10^{-7}	8.3×10^{-7}	8.3×10^{-7}
5	1.1×10^{-5}	1.9×10^{-8}	5.4×10^{-8}	5.4×10^{-8}

Table 4: Maximum absolute errors in solution Example 3.

m	Sixth-order in [17]	Sixth-order in [9]	Sixth-order in [23]	Sixth-order in [23]
3	2.47×10^{-9}	1.91×10^{-7}	2.66×10^{-6}	3.86×10^{-7}
4	3.93×10^{-11}	3.12×10^{-9}	4.68×10^{-8}	6.59×10^{-9}
5	3.25×10^{-13}	4.98×10^{-11}	7.72×10^{-10}	1.05×10^{-10}
6	-	-	8.01×10^{-12}	9.81×10^{-12}

Table 5: Maximum absolute errors in solution Example 4 and 5 in [15].

m	Example 4 Fourth-order (1)	Example 4 Fourth-order (2)	Example 5 Fourth-order (1)	Example 5 Fourth-order (2)
3	1.91×10^{-7}	2.09×10^{-7}	5.96×10^{-8}	6.48×10^{-8}
4	3.12×10^{-9}	7.92×10^{-9}	9.10×10^{-10}	2.29×10^{-9}
5	5.02×10^{-11}	1.27×10^{-9}	1.42×10^{-12}	3.66×10^{-10}

Table 6: Maximum absolute errors in solution in [30]

m	Example 2	Example 1
3	3.38556×10^{-7}	3.56386×10^{-8}
4	1.18519×10^{-9}	5.78254×10^{-10}
5	3.83505×10^{-11}	1.30425×10^{-11}
6	1.01539×10^{-12}	2.18424×10^{-13}
7	1.81576×10^{-14}	2.61800×10^{-16}

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