Some applications of $_2\mathbf{F}_1(\alpha,\ \beta;\,;\gamma;\,;\mathbf{z})$

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Abstract

We apply the results of Koepf-(Masjed-Jamei), on polynomial solutions for differential equations of hypergeometric type, to Hermite, Laguerre, Daubechies and Chebyshev equations.

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1 Introduction

The Euler-Gauss hypergeometric function [1-7]:

$$_2\mathrm{F}_1\,(\alpha,\ \beta;\,;\gamma;\,;z) = 1+\ \frac{\alpha\beta}{\gamma}\ z+\ \frac{\alpha\,(\alpha+1)\,\beta(\ +1)}{\gamma(\gamma+1)}\ \frac{z^2}{2!}+\ \frac{\alpha\,(\alpha+1)\,(\alpha+2)\,\beta\,(\beta+1)\,(\beta+2)}{\gamma\,(\gamma+1)\,(\gamma+2)}\ \frac{z^3}{3!}+\ \dots \tag{1}$$

for $|z|\ <1,\quad \gamma\neq 0,-1,-2,\ \dots$, satisfies the differential equation:

$$z\,(1-z)\ \frac{d^2}{dz^2}{}_2\mathrm{F}_1+\ [\gamma-(\alpha+\beta+1)\,z]\ \frac{d}{dz}{}_2\mathrm{F}_1-\alpha\beta{}_2\mathrm{F}_1=0, \tag{2}$$

where α,β,γ,z are four degrees of freedom which we can select adequately to generate, via $_2\mathrm{F}_1$, important functions in mathematical physics. If we employ the Pochhammer [8]-Barnes [9, 10] symbol:

$$(\alpha)_0=1,\qquad (\alpha)_k=\ \alpha\,(\alpha+1)\,(\alpha+2)\ \cdots(\alpha+k-1)\,,\qquad k=1,\ 2,\ \dots \tag{3}$$

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then (1) adopts the form:

$$ {}_2\mathrm{F}_1\left(\alpha,\beta;;\gamma;;z\right)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\ \frac{z^k}{k!}\ . \tag{4}$$

the relation (5) into (2) permits to obtain the Kummer's differential equation:

$$ z\ \frac{d^2}{dz^2}\,{}_1\mathrm{F}_1+(\gamma-z)\,\frac{d}{dz}\,{}_1\mathrm{F}_1-\alpha\,{}_1\mathrm{F}_1=0\ . \tag{5}$$

Riemann [13] studied ${}_2\mathrm{F}_1$ and its connection with ${}_1\mathrm{F}_1$, emphasizing that the solutions of a differential equation are characterized by the position and nature of the singularities of the equation under analysis. Similarly, for all z:

$$ {}_0\mathrm{F}_1\left(\gamma;;z\right)=\ lim_{\ \alpha\longrightarrow\infty}\ {}_1\mathrm{F}_1\left(\alpha;;\gamma;;\frac{z}{\alpha}\right)=\sum_{k=0}^{\infty}\frac{1}{(\gamma)_k}\ \frac{z^k}{k!}=1+\ \frac{1}{\gamma}\ z+\ \frac{1}{\gamma(\gamma+1)}\ \frac{z^2}{2!}+\ \ldots\ \ , \tag{6}$$

Kummer [11] introduced the confluent hypergeometric function [12] (for all z):

$$ {}_1\mathrm{F}_1\left(\alpha;;\gamma;;z\right)=\ lim_{\ \beta\ \longrightarrow\infty}\ {}_2\mathrm{F}_1\left(\alpha,\beta;;\gamma;;\frac{z}{\beta}\right)\ =\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\gamma)_k}\ \frac{z^k}{k!}=1+\frac{\alpha}{\gamma}\ z+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\ \frac{z^2}{2!}+\ldots\ \ ;; \tag{7}$$

then (6) implies the differential equation:

$$ z\ \frac{d^2}{dz^2}\,{}_0\mathrm{F}_1+\ \gamma\ \frac{d}{dz}\,{}_0\mathrm{F}_1-{}_0\mathrm{F}_1=0. \tag{8}$$

Therefore, we can define the generalized hypergeometric functions [4, 7, 14]:

$$ {}_p\mathrm{F}_q\left(a_1,\ldots,a_p;;b_1,\ldots,b_q;;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\ \frac{z^k}{k!}\ , \tag{9}$$

verifying the equation [6, 15]:

$$ \left\{z\ \frac{d}{dz}\left[\left(z\ \frac{d}{dz}+b_1-1\right)\cdots\left(z\ \frac{d}{dz}+b_q-1\right)\right]-z\left(z\ \frac{d}{dz}+a_1\right)\cdots\left(z\ \frac{d}{dz}+a_p\right)\right\}{}_p\mathrm{F}_q=0\ , \tag{10}$$

thus (2), (6) and (8) are particular cases of (10). The expansion (9) is convergent when $p=q+1$ and $p\ \leq q$ for $|z|\ <1$ and all z, respectively; the series (9) has not convergence for any value of z when $p>q+1$.

2 Differential equations with hypergeometric structure and their polynomials solutions

The Refs. [16, 17] consider the differential equation: In Sec. 2 we exhibit the results of Koepf-(Masjed-Jamei) [16, 17] on polynomial solutions for differential equations of hypergeometric type, and the Sec. 3

has applications of those results to Daubechies polynomials [18, 19] and Chebyshev's associated polynomials [20-22].

$$(ax^2+bx+c){y_n}'' + (xd+e){y_n}' - n(na+d-a)y_n = 0, \quad n = 1, 2, ... \tag{11}$$

whose polynomials solutions (non-normalized) can be written in the forms $\left(\triangle = \sqrt{b^2-4ac}\right)$:

$$y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2a}{b+\triangle}\right)^{k-n} {}_2\mathrm{F}_1\left(k-n,\ \frac{2ae-bd}{2a\triangle}+1-\frac{d}{2a}-n;;2-\frac{d}{a}-2n;;\ \frac{2\triangle}{b+\triangle}\right) x^k \ , \tag{12}$$

$$= \ \triangle^n \ \frac{\left(\frac{(b+\triangle)d-2ae}{2a\triangle}\right)_n}{(-a)^n \left(n-1+\frac{d}{a}\right)_n} \ {}_2\mathrm{F}_1\left(-n,\ n+\frac{d}{a}-1;;\ \frac{(b+\triangle)\,d-2ae}{2a\triangle};;\ \frac{2ax+b+\triangle}{2\triangle}\right), \tag{13}$$

with the particular cases:
i). $a = 0$,

$$(bx+c)y_n^{''} + (xd+e)y_n^{'} - dny_n = 0 \tag{14}$$

$$y_n\,(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{c}\right)^{k-n} {}_2\mathrm{F}_0\left(k-n,\ \frac{cd-be}{b^2}+1-n;;\ \frac{b^2}{cd}\right) x^k, \qquad c, d \neq 0, \tag{15}$$

$$= \ \left(\frac{b}{d}\right)^n \left(\frac{eb-cd}{b^2}\right)_n {}_1\mathrm{F}_1\left(-n;;\ \frac{be-cd}{b^2};;-\frac{dbx+cd}{b^2}\right) \ , \tag{16}$$

ii). $a = b = 0$,

$$cy_n^{''} + (xd+e)y_n^{'} - dny_n = 0 \tag{17}$$

$$y_n(x) = \left(x+\frac{e}{d}\right)^n {}_2\mathrm{F}_1\left(\frac{-n}{2}, \frac{1-n}{2}; \frac{2cd}{(xd+e)^2}\right), \ d \neq 0. \tag{18}$$

For example, the Chebyshev's equation [4, 7, 23]:

$$(1-x^2)y_n'' - xy_n' + n^2 y_n = 0, \tag{19}$$

has the structure (11) with a = - c = d = -1, b = e = 0, then (13) implies the expression:

$$y_n(x) = \ 2^{1-n} {}_2\mathrm{F}_1\left(-n,\ n;;\ \frac{1}{2};;\ \frac{1-x}{2}\right) = \ 2^{1-n}\ \mathrm{T}_n\,(x)\,, \tag{20}$$

where $\mathrm{T}_n(x)$ are the Chebyshev polynomials of the first kind [23]. Therefore, we multiply (11) by 2^{n-1} to obtain the alternative relation:

$$T_n(x) = 2^{n-1} \sum_{k=0}^{n} (-1)^{k-n} \binom{n}{k} {}_2\mathrm{F}_1\left(k-n,\ \frac{1}{2}-n; 1-2n; 2\right) x^k \ . \tag{21}$$

The Laguerre's differential equation [4, 7, 24]:

$$xy_n'' + (1-x)y_n' + ny_n = 0, \tag{22}$$

corresponds to $a = c = 0$, $b =$ - $d = e = 1$, then (16) gives the Laguerre polynomials in terms of the Kummer hypergeometric function:

$$y_n(x) = (-1)^n \; n! \; \mathrm{L}_n(x) = (-1)^n \; n! \; {}_1\mathrm{F}_1(-n; 1; x) \,. \tag{23}$$

For the Hermite's equation [4, 7, 24, 25]:

$$y_n'' - 2xy_n' + 2ny_n = 0, \tag{24}$$

from (18) with $a = b = e = 0$, $c = 1$, $d =$ -2:

$$y_n(x) = 2^{-n} \; \mathrm{H}_n(x) = \left(\frac{x}{2}\right)^n {}_2\mathrm{F}_0\left(-\frac{n}{2}, \; \frac{1-n}{2} \; ;; \; -\frac{1}{x^2}\right) \,. \tag{25}$$

In the next Section we shall apply the results of Koepf-(Masjed-Jamei)[16, 17] to Daubechies polynomials [18, 19] and Chebyshev's associated polynomials [20-22], and thus to deduce expressions for them via Euler-Gauss hypergeometric function.

3 Polynomials of Daubechies and associated polynomials of Chebyshev

The wavelets [26] are important in science, engineering and technology; in particular, the construction of Daubechies wavelets [27] depends of the zeros of the polynomials $\mathrm{D}_r(x)$ defined by $(|x| \leq 1)$:

$$D_n(x) = \sum_{k=0}^{n} \binom{n+k}{k} x^k, \tag{26}$$

that is, $\mathrm{D}_0 = 1, \;\; \mathrm{D}_1 = 1 + x, \;\; \mathrm{D}_2 = 1 + 3x + 6x^2,$ etc.
In [19, 28] was proved the connection:

$$\mathrm{D}_n(x) = \lim_{\lambda \longrightarrow 0} \; {}_2\mathrm{F}_1(-n, \; n+1;; \; -n+\lambda;; x) \; , \tag{27}$$

then (2) gives the corresponding differential equation for Daubechies polynomials:

$$x(1-x) \; \frac{d^2}{dx^2}\mathrm{D}_n - (2x+n)\frac{d}{dx}\mathrm{D}_n + \; n(n+1)\,\mathrm{D}_n = \; 0 \,. \tag{28}$$

However, now we can study (28) employing the results of [16, 17] because it has the hypergeometric type (11) with $a =$ - $b =$ -1, $c = 0$, $d =$ -2, $e =$ - n, therefore:

$$D_n(x) = \frac{(2n)!}{(n!)^2} \sum_{k=0}^{n} (-1)^{k-n} \binom{n}{k} {}_2\mathrm{F}_1(k-n, \; -1-2n; \; -2n; 1)\, x^k, \tag{29}$$

as an alternative to (27). The expression (29) is in harmony with (26) because it is simple to show the identity:

$${}_2\mathrm{F}_1(k-n, \; -1-2n; \; -2n; 1) = \; (-1)^{n-k} \frac{(n+k)!\,(n-k)!}{(2n)!}, \tag{30}$$

in fact, if in the Gauss formula [4, 7]:

$${}_2\mathrm{F}_1(\alpha, \beta;; \gamma;; 1) = \; \frac{\Gamma(\gamma) \;\; \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \; , \tag{31}$$

where $\Gamma(z)$ is the Gamma function of Euler [29], we use $\alpha = k - n, \ \ \beta = -1 - 2n, \ \ c = -2n$ then we deduce (30).
In [20-22] were introduced and studied the Chebyshev's associated polynomials $\mathrm{T}^n_m(x), \quad |x| \leq 1$, n = 0,1, . . . , m, via the relationship:

$$T^n_m(x) = (-1)^n \binom{2m-n}{n} {}_2\mathrm{F}_1\left(-n,\ 2m-n;\ m-n+\frac{1}{2}; \frac{1-x}{2}\right), \tag{32}$$

that is
$\mathrm{T}^0_3 = 1, \ \ \mathrm{T}^1_3 = -5x, \quad \mathrm{T}^2_3 = 8x^2 - 2, \ \ \mathrm{T}^3_3 = -\mathrm{T}_3, \quad \mathrm{T}^3_5 = -56x^3 + 21x, \ \ \mathrm{T}^4_5 = 48x^4 - 36x^2 + 3,$ *etc.*,

$$\left(1-x^2\right) \frac{d^2}{dx^2}\mathrm{T}^n_m - (2m-2n+1)\ x\ \frac{d}{dx}\mathrm{T}^n_m + n\,(2m-n)\,\mathrm{T}^n_m = 0\ , \tag{33}$$

with the structure (11) for a = - c = -1, b = e = 0, d = 2n - 2m -1, thus (12) gives an alternative expression to (32):

$$T^n_m(x) = 2^{n-1}\frac{(2m-n)(m-1)!}{n!(m-n)!}\sum_{k=0}^{n}(-1)^{k-n}\binom{n}{k}{}_2\mathrm{F}_1\left(k-n,\ -1-2n;\ -2n;1\right)x^k. \tag{34}$$

In this work we show some applications of the attractive results of Koepf-(Masjed-Jamei) [16, 17] on polynomial solutions for differential equations of hypergeometric type, and it is evident that those results have immediate importance in the analysis of the Schrdinger equation for several potentials of interest in quantum mechanics.

4 References

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