Korovkin-type theorems for abstract modular convergence

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Abstract

We give some Korovkin-type theorems on convergence and estimates of rates of approximations of nets of functions, satisfying suitable axioms, whose particular cases are filter/ideal convergence, almost convergence and triangular A-statistical convergence, where A is a non-negative summability method. Furthermore, we give some applications to Mellin-type convolution and bivariate Kantorovich-type discrete operators.

1 Introduction

The classical Bohman-Korovkin theorem is a result which yields uniform convergence in the space $\mathcal{C}([a,b])$ of all continuous real-valued functions defined on the compact subinterval $[a,b]$ of the real line, for a net $(T_w)_w$ of positive linear operators on $C([a,b])$, with the only hypothesis of convergence on the test functions 1, x, x^2 (see also [23, 26, 33, 34]). There have been several extensions the Korovkin theorem to the context of abstract functional spaces. For a related literature, see for instance [27, 32, 39, 42] for the case of L^p-spaces, [37, 40] for Orlicz spaces and [5, 10, 15] for general modular spaces. There have been also several studies about Korovkin-type theorems in the setting of convergence generated by summability matrices, statistical and filter convergence (see for example [2, 28, 29, 31, 38]) and with respect to "triangular A-statistical convergence", namely an extension of statistical convergence for double sequences of positive linear operators, where A is a suitable non-negative regular matrix (see also [4, 25]).

In this paper we deal with Korovkin-type results about convergence and quantitative theorems, and estimates of rates of approximation with respect to abstract

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convergences satisfying suitable axioms (see also [5, 16, 35]), including as particular cases convergence generated by summability (double infinite) matrices, filter convergence and almost convergence, which is not generated by any filter (see also [22]), and we consider the general case of a net of operators, acting on an abstract modular function space generated by a modular, extending earlier results proved in [4, 5, 20, 25, 28]. Note that, in the literature, it is often dealt with Korovkin-type theorems with respect to some of the above mentioned convergences, but without a general approach containing all of them. Our general results unify various previous theorems.

Furthermore, we show that the rates investigated in [4, 20, 25] are particular cases of those treated here with respect to the "axiomatic convergence", we present some examples of convergences and give comparison results between triangular convergence generated by a summability matrix method and filter convergence. In particular we see that, in general, the Ψ-A-statistical convergence studied in [4] and the filter convergence are such that neither contains the other. We give also some applications to moment kernels and bivariate Kantorovich-type discrete operators (for recent studies and developments, see also [1, 3, 6, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 41]).

2 Preliminaries

Let $(W, \succeq)$ be a directed set, and let us consider an axiomatic abstract convergence on W, defined as follows (see also [5, 16, 35]).

Definition 2.1 Let $\mathcal{T}$ be the set of all real-valued nets $(x_w)_{w \in W}$. A *convergence* is a pair $(\mathcal{S}, \ell)$, where $\mathcal{S}$ is a linear subspace of $\mathcal{T}$ and $\ell : \mathcal{S} \to \mathbb{R}$ is a function, satisfying the following axioms:

(a) $\ell((a_1\, x_w + a_2\, y_w)_w) = a_1\, \ell((x_w)_w) + a_2\, \ell((y_w)_w)$ for every pair of nets $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ and for each a_1, $a_2 \in \mathbb{R}$ (linearity).

(b) If $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ and there is $w^* \in W$ with $x_w \le y_w$ for every $w \succeq w^*$, then $\ell((x_w)_w) \le \ell((y_w)_w)$ (monotonicity).

(c) If $(x_w)_w$ is such that there is $w_* \in W$ with $x_w = l$ whenever $w \succeq w_*$, then $(x_w)_w \in \mathcal{S}$ and $\ell((x_w)_w) = l$.

(d) If $(x_w)_w \in \mathcal{S}$, then $(|x_w|)_w \in \mathcal{S}$ and $\ell((|x_w|)_w) = |\ell((x_w)_w)|$.

(e) Let $(x_w)_w$, $(y_w)_w$, $(z_w)_w$, satisfying $(x_w)_w$, $(z_w)_w \in \mathcal{S}$, $\ell((x_w)_w) = \ell((z_w)_w)$ and suppose that there is $\overline{w} \in W$ with $x_w \le y_w \le z_w$ for every $w \ge \overline{w}$. Then $(y_w)_w \in \mathcal{S}$.

Note that $\mathcal{S}$ is the space of all convergent nets, ℓ will be the "limit" according to this approach, and we will denote by the symbol $(\ell) \lim\limits_{w} x_w$ the quantity $\ell((x_w)_w)$.

We now give the axiomatic definition of the operators "limit superior" and "limit inferior" related with a convergence $(\mathcal{S}, \ell)$ (see also [5]).

Definition 2.2 Let $\mathcal{T}$, $\mathcal{S}$ be as above. We define two functions $\overline{\ell}$, $\underline{\ell} : \mathcal{T} \to \widetilde{\mathbb{R}}$, satisfying the following axioms:

(f) If $(x_w)_w$, $(y_w)_w \in \mathcal{T}$, then $\underline{\ell}((x_w)_w) \leq \overline{\ell}((x_w)_w)$ and $\overline{\ell}((x_w)_w) = -\underline{\ell}((-x_w)_w)$.

(g) If $(x_w)_w \in \mathcal{T}$, then

(i) $\overline{\ell}((x_w + y_w)_w) \leq \overline{\ell}((x_w)_w) + \overline{\ell}((y_w)_w)$ (subadditivity);

(ii) $\underline{\ell}((x_w + y_w)_w) \geq \underline{\ell}((x_w)_w) + \underline{\ell}((y_w)_w)$ (superadditivity).

(h) If $(x_w)_w$, $(y_w)_w \in \mathcal{T}$ and $x_w \leq y_w$ definitely, then $\overline{\ell}((x_w)_w) \leq \overline{\ell}((y_w)_w)$ and $\underline{\ell}((x_w)_w) \leq \underline{\ell}((y_w)_w)$ (monotonicity).

(j) A net $(x_w)_w \in \mathcal{T}$ belongs to $\mathcal{S}$ if and only if $\overline{\ell}((x_w)_w) = \underline{\ell}((x_w)_w)$.

We will denote by the symbols $(\ell) \limsup_w x_w$ and $(\ell) \liminf_w x_w$ the quantities $\overline{\ell}((x_w)_w)$ and $\overline{\ell}((x_w)_w)$, respectively.

We define two tools, in order to "compare" two nets belonging to $\mathcal{S}$.

Definition 2.3 Let $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ with $(\ell) \lim_w x_w = (\ell) \lim_w y_w = 0$ and $y_w \neq 0$ for every $w \in W$. We say that $x_w = o(y_w)$ iff $(\ell) \lim_w \dfrac{|x_w|}{|y_w|} = 0$, and that $x_w = O(y_w)$ iff $(\ell) \limsup_w \dfrac{|x_w|}{|y_w|} \in \mathbb{R}$.

Let $G = (G, d)$ be a metric space, $\mathcal{B}$ be the σ-algebra of all Borel subsets of G, and μ be a positive finite regular measure defined on $\mathcal{B}$. Let $L^0(G)$ be the space of all real-valued μ-measurable functions on G with identification up to sets of measure μ zero, $\mathcal{C}_b(G)$ be the space of all real-valued continuous and bounded functions on G, $\mathcal{C}_c(G)$ be the subspace of $\mathcal{C}_b(G)$ of all functions with compact support on G and $\mathrm{Lip}(G)$ be the space of all real-valued Lipschitz functions on G.

We now recall the notion of modular space (see also [14]).

Definitions 2.4 (a) A functional $\rho : L^0(G) \to \widetilde{\mathbb{R}_0^+}$ is called a *modular* on $L^0(G)$ iff it satisfies the following conditions:

i) $\rho[f] = 0 \Longleftrightarrow f = 0$ μ-almost everywhere on G;

ii) $\rho[-f] = \rho[f]$ for every $f \in L^0(G)$;

iii) $\rho[af + bg] \leq \rho[f] + \rho[g]$ whenever f, $g \in L^0(G)$ and $a \geq 0$, $b \geq 0$ with $a + b = 1$.

(b) A modular ρ is said to be *convex* iff it satisfies conditions i), ii) and

iii') $\rho[af + bg] \leq a\rho[f] + b\rho[g]$ for all f, $g \in L^0(G)$ and for every a, $b \geq 0$ with $a + b = 1$.

(c) Let $Q \geq 1$ be a real constant. We say that a modular ρ is *Q-quasi semiconvex* if $\rho[a\, f] \leq Q\, a\, \rho[Q\, f]$ for all $f \in L^0(G)$, $f \geq 0$ and $0 < a \leq 1$ (see also [10]).

(d) A modular ρ is *monotone* iff $\rho[f] \leq \rho[g]$ for all $f,\, g \in L^0(G)$ with $|f| \leq |g|$.

(e) A modular ρ is *finite* iff χ_A (the characteristic function associated with A) belongs to $L^\rho(G)$ whenever $A \in \mathcal{B}$ with $\mu(A) < +\infty$.

(f) A modular ρ is *strongly finite* iff χ_A belongs to $E^\rho(G)$ for each $A \in \mathcal{B}$ with $\mu(A) < +\infty$.

(g) A modular ρ is said to be *absolutely continuous* iff there is a positive constant a with the property: for all $f \in L^0(G)$ with $\rho[f] < +\infty$,

i) for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\rho[af\chi_{G\setminus A}] \leq \varepsilon$,

ii) for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho[af\,\chi_B] \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

(h) The *modular space* $L^\rho(G)$ generated by ρ is

$$L^\rho(G) = \{f \in L^0(G) : \lim_{\lambda \to 0^+} \rho[\lambda f] = 0\},$$

where the limit is intended in the usual sense, and the *space of the finite elements of* $L^\rho(G)$ is

$$E^\rho(G) = \{f \in L^\rho(G) : \rho[\lambda f] < +\infty \text{ for all } \lambda > 0\}.$$

Example 2.5 Let Φ be the set of all continuous non-decreasing functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varphi(0) = 0$, $\varphi(u) > 0$ for any $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$ in the usual sense, and let $\widetilde{\Phi}$ be the set of all convex functions belonging to Φ.

For every $\varphi \in \Phi$ (resp. $\widetilde{\Phi}$), the functional ρ^φ defined by

$$\rho^\varphi[f] = \int_G \varphi(|f(s)|)\, d\mu(s), \quad f \in L^0(G), \tag{1}$$

is a modular (resp. convex modular) on $L^0(G)$ and

$$L^\varphi(G) := \{f \in L^0(G) : \rho^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$$

is the *Orlicz space* generated by φ, and satisfies all above properties (see also [14]).

We now define the modular and strong convergences in the context of the axiomatic convergence in Definition 2.1 (for the classical case and filter convergence see [14] and [5, 6, 17] respectively).

A net $(f_w)_w$ of functions in $L^\rho(G)$ is (ℓ)-*modularly convergent* to $f \in L^\rho(G)$ if there is a $\lambda > 0$ with

$$(\ell) \lim_w \rho[\lambda(f_w - f)] = 0.$$

A net $(f_w)_w$ in $L^\rho(G)$ is (ℓ)-*strongly convergent* to $f \in L^\rho(G)$ if

$$(\ell) \lim_w \rho[\lambda(f_w - f)] = 0 \text{ for every } \lambda > 0.$$

Given a subset $\mathcal{A} \subset L^\rho(G)$ and $f \in L^\rho(G)$, we say that $f \in \overline{\mathcal{A}}$ (that is, f is in the modular closure of $\mathcal{A}$) if there is a sequence $(f_k)_k$ in $\mathcal{A}$, modularly convergent to f with respect to the ordinary convergence.

We recall the following

Proposition 2.6 (see also [36, Theorem 1]) *Let ρ be a monotone, strongly finite and absolutely continuous modular on $L^0(G)$. Then $\overline{\mathcal{C}_c(G)} = L^\rho(G)$ with respect to the modular convergence in the ordinary sense.*

3 The Korovkin theorem

We consider some kinds of rates of approximation associated with the Korovkin theorem in the context of modular convergence. For technical reasons, we sometimes suppose that (G, d) satisfies the following property:

H*) For every $n \in \mathbb{N}$ and s, $t \in G$, with $s \neq t$, there are $n+1$ points x_i, $i = 0, \dots, n+1$, such that $s = x_0$, $t = x_{n+1}$ and $d(x_i, x_{i+1}) \leq \dfrac{1}{n} d(s,t)$ for each $i = 0, \dots, n$.

Some examples of spaces satisfying condition H*) are the Euclidean multidimensional space $\mathbb{R}^N$ endowed with the usual metric and the space $\mathbb{R}^\Lambda$ equipped with the sup-norm, where Λ is any abstract nonempty set (see also [4]).

For every $f \in \mathcal{C}_b(G)$ and $\delta > 0$, let

$$\omega(f;\delta) := \sup\{|f(s) - f(t)| : s, t \in G, d(s,t) \leq \delta\}$$

be the usual modulus of continuity of f. Note that $\omega(f;\delta)$ is an increasing function of δ, $|f(s) - f(t)| \leq \omega(f; d(s,t))$ for each s, $t \in G$, $\omega(f;\delta) \leq 2\,M$ for every δ, where $M = \sup_{t \in G} |f(t)|$, and

$$\omega(f;\gamma\,\delta) \leq (1+\gamma)\,\omega(f;\delta) \tag{2}$$

for every γ, $\delta > 0$ (see also [4]).

Let T be a net of linear operators $T_w : \mathcal{D} \to L^0(G)$, $w \in W$, with $\mathcal{C}_b(G) \subset \mathcal{D} \subset L^0(G)$. Here the set $\mathcal{D}$ is the domain of the operators T_w.

We say that the net T, together with the modular ρ, satisfies *property* (ρ)-$(*)$ iff there exist a subset $X_T \subset \mathcal{D} \cap L^\rho(G)$ with $\mathcal{C}_b(G) \subset X_T$ and an $E > 0$ with $T_w f \in L^\rho(G)$ for any $f \in X_T$ and $w \in W$, and $(\ell) \limsup_w \rho[\tau(T_w f)] \leq E\,\rho[\tau f]$ for every $f \in X_T$ and $\tau > 0$.

Some examples of operators satisfying property (ρ)-$(*)$ can be found in [10].

Let e_r and a_r, $r = 0, \dots, m$, be functions in $\mathcal{C}_b(G)$, with $e_0(t) := 1$ for every $t \in G$. Let us define

$$P_s(t) := \sum_{r=0}^{m} a_r(s) e_r(t), \quad s, t \in G, \tag{3}$$

and assume that

(P1) $P_s(s) = 0$ for all $s \in G$;

(P2) there is a $C_1 > 0$ with $P_s(t) \geq C_1\, d(s,t)$ whenever $s,\, t \in G$.

Examples 3.1

(a) Let $G = I^m$ be endowed with the usual norm $\|\cdot\|_2$, where $I \subset \mathbb{R}$ is a connected set, and $\phi : I \to \mathbb{R}$ be monotone, continuous and such that ϕ^{-1} is Lipschitz on $\phi(I)$. Examples of such functions are $\phi(t) = t$ or $\phi(t) = e^t$, where $I = [a,b] \subset \mathbb{R}$.

For every $t = (t_1, \ldots, t_m) \in G$ set $e_i(t) := \phi(t_i)$, $i = 1, \ldots, m$, and

$$e_{m+1}(t) := \sum_{i=1}^{m} [\phi(t_i)]^2.$$

For each $s = (s_1, \ldots, s_m) \in G$ put $a_0(s) := \sum_{i=1}^{m} [\phi(s_i)]^2$, $a_i(s) = -2\phi(s_i)$, $i = 1, \ldots, m$, and $a_{m+1}(s) \equiv 1$. We get:

$$P_s(t) \;\; := \;\; \sum_{i=0}^{m+1} a_i(s) e_i(t) = \sum_{i=1}^{m} [\phi(s_i) - \phi(t_i)]^2.$$

It is not difficult to see that (P1) and (P2) are satisfied.

(b) Let $G = [a,b]$ with $0 < a < b < \pi/2$, $e_1(t) = \cos t$, $e_2(t) = \sin t$, $t \in G$. Set $a_0(s) \equiv 1$, $a_1(s) = -\cos s$, $a_2(s) = -\sin s$, $s \in G$. For all $s,\, t \in G$ we get:

$$P_s(t) = 1 - \cos s \cos t - \sin s \sin t = 1 - \cos(s - t).$$

It is not difficult to check that (P1) and (P2) are fulfilled (see also [5]).

From now on, we suppose that $e_r \in L^\rho(G)$, $r = 0, 1, \ldots, m$. Note that this assumption is fulfilled, for example, when G is a space of finite measure μ.

We now state the following theorems, whose proofs are analogous to those of [5, Theorem 4.2] and [5, Theorem 4.3], respectively.

Theorem 3.2 *Let ρ be a strongly finite, monotone and Q-quasi semiconvex modular. Assume that e_r and a_r, $r = 0, \ldots, m$, satisfy* (P1) *and* (P2). *Let T_w, $w \in W$, be a net of positive linear operators having property (ρ)-$(*)$. If $(T_w e_r)_w$ is (ℓ)-modularly convergent to e_r in $L^\rho(G)$ for each $r = 0, \ldots, m$, then $(T_w f)_w$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$.*

If $(T_w e_r)_w$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_w f)_w$ is (ℓ)-strongly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$.

Theorem 3.3 *Let ρ be a monotone, strongly finite, absolutely continuous and Q-quasi semiconvex modular on $L^0(G)$, and T_w, $w \in W$ be a net of positive linear operators satisfying (ρ)-$(*)$. If $(T_w e_r)_w$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_w e_r)_w$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for every $f \in L^\rho(G) \cap \mathcal{D}$ with $f - \mathcal{C}_b(G) \subset X_T$, where $\mathcal{D}$ and X_T are as above.*

Now we present some estimates on rates of approximation for abstract Korovkin-type theorems. Let Ξ be the family of all nets ξ_w, $w \in W$, with $\xi_w \neq 0$ for each $w \in W$ and $(\ell) \lim_w \xi_w = 0$.

Theorem 3.4 *Let $Q \geq 1$, ρ be a monotone, strongly finite and Q-quasi semiconvex modular, T_w, $w \in W$, be a net of positive linear operators and Ξ be as above. For every $w \in W$, let $\xi_w^r \in \Xi$, $r = 0, \dots, m$, and set $\xi_w := \max\{\xi_w^r \colon r = 0, \dots, m\}$. If $\gamma > 0$ is such that $\rho[\gamma(T_w e_r - e_r)] = o(\xi_w^r)$ for each $r = 0, \dots, m$, then for every $f \in \mathcal{C}_c(G) \cap Lip(G)$ there exists a positive real number τ with $\rho[\tau(T_w f - f)] = o(\xi_w)$.*

A similar result holds also when o is replaced by O.

Proof: We now prove only the result concerning o, since the assertion involving O is analogous. Choose $f \in \mathcal{C}_c(G) \cap \mathrm{Lip}(G)$, let $M := 1 + \sup_{t \in G} |f(t)|$ and let C_2 be a strictly positive Lipschitz constant, associated with f. We get

$$|f(s) - f(t)| \leq C_2\, d(s,t) \leq {C_1}^{-1}\, C_2 P_s(t) \quad \text{for every } s, t \in G,$$

namely

$$-{C_1}^{-1} C_2 P_s(t) \leq f(s) - f(t) \leq {C_1}^{-1} C_2 P_s(t) \quad \text{for each } s, t \in G. \tag{4}$$

By applying T_w to (4), since T_w is linear and positive we have

$$-{C_1}^{-1} C_2 (T_w P_s)(s) \leq f(s)(T_w e_0)(s) - (T_w f)(s) \leq {C_1}^{-1} C_2 (T_w P_s)(s),$$

and hence

$$\begin{aligned} |(T_w f)(s) - f(s)| &\leq |(T_w f)(s) - f(s)(T_w e_0)(s)| + |f(s)(T_w e_0)(s) - f(s)| \leq \\ &\leq {C_1}^{-1} C_2 (T_w P_s)(s) + M |(T_w e_0)(s) - e_0(s)| \end{aligned} \tag{5}$$

for each $s \in G$ and $w \in W$.

Let now $\gamma > 0$ be as in the hypotheses and choose a positive real number τ, with

$$\tau \leq \min\Big\{ \frac{\gamma\, C_1}{2\, C_2 (m+1) N\, Q^2}, \frac{\gamma}{2\, M\, Q^2} \Big\}.$$

By applying the modular ρ, from (5) we get

$$\rho[\tau(T_w f - f)] \leq \rho[2\, \tau {C_1}^{-1} C_2 (T_w P_{(\cdot)})(\cdot)] + \rho[2\, \tau\, M (T_w e_0 - e_0)] \tag{6}$$

whenever $w \in W$. By Q-quasi semiconvexity of ρ we have

$$\begin{aligned} \rho[2\, \tau\, M (T_w e_0 - e_0)] &\leq \rho\Big[\frac{\gamma}{Q^2}(T_w e_0 - e_0)\Big] \leq Q\, \frac{1}{Q} \rho\Big[Q\, \frac{\gamma}{Q}(T_w e_0 - e_0)\Big] = \\ &= \rho[\gamma(T_w e_0 - e_0)]. \end{aligned}$$

Hence,

$$(\ell) \lim_w \frac{\rho[2\,\tau\, M(T_w e_0 - e_0)]}{\xi_w^0} = 0.$$

and a fortiori

$$(\ell) \lim_w \frac{\rho[2\,\tau\, M(T_w e_0 - e_0)]}{\xi_w} = 0, \tag{7}$$

taking into account axiom (e) in Definition 2.1. Let $N > 0$ be with $|a_r(s)| \leq N$ for any $r = 0, \ldots, m$ and $s \in G$. By (3) and (P1) we get

$$P_s(t) = P_s(t) - P_s(s) = \sum_{r=0}^{m} a_r(s)(e_r(t) - e_r(s)) \tag{8}$$

for each s, $t \in G$. By applying T_w, from (8) we obtain

$$T_w(P_s)(s) = T_w(P_s)(s) - P_s(s) = \sum_{r=0}^{m} a_r(s)(T_w e_r(s) - e_r(s)),$$

and hence

$$|T_w(P_s)(s)| = |T_w(P_s)(s) - P_s(s)| \leq N \sum_{r=0}^{m} |T_w e_r(s) - e_r(s)|, \tag{9}$$

for every $s \in G$ and $w \in W$. By applying the modular ρ and taking into account Q-quasi semiconvexity, from (9) we have

$$\begin{aligned}
\rho[2\,\tau\, {C_1}^{-1} C_2 (T_w P_{(\cdot)})(\cdot)] \;&\leq\; \sum_{r=0}^{m} \rho[2\,\tau\, {C_1}^{-1} C_2 (m+1) N (T_w e_r - e_r)] \leq \\
&\leq\; \sum_{r=0}^{m} \rho\Big[\frac{\gamma}{Q^2}(T_w e_r - e_r)\Big] \leq \\
&\leq\; Q\,\frac{1}{Q} \sum_{r=0}^{m} \rho\Big[Q\,\frac{\gamma}{Q}(T_w e_r - e_r)\Big] = \sum_{r=0}^{m} \rho[\gamma(T_w e_r - e_r)]
\end{aligned} \tag{10}$$

for each $w \in W$. Since

$$(\ell) \lim_w \frac{\rho[\gamma(T_w e_r - e_r)]}{\xi_w^r} = 0 \quad \text{for every } r = 0, 1, \ldots, m,$$

we get

$$(\ell) \lim_w \frac{\rho[\gamma(T_w e_r - e_r)]}{\xi_w} = 0 \quad \text{for every } r = 0, 1, \ldots, m,$$

and hence

$$(\ell)\lim_w \frac{\rho[2\gamma {C_1}^{-1}C_2(T_w P_{(\cdot)})(\cdot)]}{\xi_w} = 0, \tag{11}$$

by (10) and taking into account axiom (e) of 2.1. From (6), (7) and (11) we obtain

$$(\ell)\lim_w \frac{\rho[\gamma(T_w f - f)]}{\xi_w} = 0,$$

taking into account axioms (a) and (e) in Definition 2.1. This ends the proof. □

Theorem 3.5 *Let Q, $(T_w)_w$, ρ, Ξ be as in Theorem* 3.4*, (G,d) satisfy condition* H*)*, ξ_w^0, $\xi_w^* \in \Xi$, set $\xi_w := \max\{\xi_w^0, \xi_w^*\}$, $w \in W$, and $\psi(s)(t) := d(s,t)$, s, $t \in G$. For every $f \in \mathcal{C}_c(G)$ and $w \in W$ put $\delta_w^f = \|T_w(\psi)\|$, where $\|\cdot\|$ is the sup-norm and the supremum is taken with respect to the support of f. If $\gamma > 0$ satisfies the conditions*

3.5.1) $\rho[\gamma(T_w e_0 - e_0)] = o(\xi_w^0)$ *and*

3.5.2) $\rho[\gamma\,\omega(f;\delta_w^f)] = o(\xi_w^*)$,

then for each $f \in \mathcal{C}_c(G)$ there is a positive real number τ with $\rho[\tau\,(T_w f - f)] = o(\xi_w)$.

Moreover, a similar result holds when the symbol o is replaced by O.

Proof: Let $f \in \mathcal{C}_c(G)$, $M = \sup_{t\in G}|f(t)|$. Observe that $\omega(f;\delta) \le 2\,M$ for each $\delta > 0$. By the properties of the modulus of continuity, we get

$$|f(s) - f(t)| \le \omega(f; d(s,t)) \le \Big(1 + \frac{d(s,t)}{\delta}\Big)\omega(f;\delta) \tag{12}$$

for each $\delta > 0$ and s, $t \in G$. We claim that $\delta_w^f \in \mathbb{R}$ for every $w \in W$. Indeed the support of f is (totally) bounded, and so $\sup_{s,t\in G} d(s,t) < +\infty$. By applying T_w we find an $E_w > 0$ with

$$\sup_{s,t\in G} T_w(d(s,t)) \le E_w\,T_w(e_0),$$

getting the claim. Let $\delta = \delta_w^f$. By applying T_w, keeping fixed s and letting t vary in G, by (12), linearity and monotonicity of T_w we get

$$\begin{aligned} & |(T_w f)(s) - f(s)| \le T_w\Big(1 + \frac{d(s,t)}{\delta}\omega(f;\delta)\Big) \le \\ \le\ & \omega(f;\delta)|(T_w e_0)(s) - e_0(s)| + \frac{\omega(f;\delta)}{\delta}T_w(\psi)(s) + \omega(f;\delta)\,e_0(s) + \\ +\ & M|(T_w e_0)(s) - e_0(s)| \le 4\,M(|(T_w e_0)(s) - e_0(s)| + \omega(f;\delta)) \end{aligned} \tag{13}$$

for each $s \in G$. Let now $\gamma > 0$ be as in the hypotheses, and pick $\tau > 0$ with

$$\tau \leq \frac{\gamma}{8\, M\, Q^2}.$$

By applying the modular ρ, taking into account Q-quasi semiconvexity, from (13) we obtain

$$\begin{aligned} \rho[\tau(T_w f - f)] &\leq \rho[8\,\tau\, M(T_w e_0 - e_0)] + \rho[8\,\tau\, M\,\omega(f;\delta)] \leq \\ &\leq \rho\Big[\frac{\gamma}{Q^2}(T_w e_0 - e_0)\Big] + \rho[\gamma\,\omega(f;\delta)]) \leq \\ &\leq Q\,\frac{1}{Q}\rho\Big[Q\,\frac{\gamma}{Q}(T_w e_0 - e_0)\Big] + \rho[\gamma\,\omega(f;\delta)]) = \\ &= \rho[\gamma(T_w e_0 - e_0)] + \rho[\gamma\,\omega(f;\delta)]). \end{aligned} \tag{14}$$

Since

$$(\ell) \lim_w \frac{\rho[\gamma(T_w e_0 - e_0)]}{\xi_w^0} = (\ell) \lim_w \frac{\rho[\gamma\,\omega(f;\delta)]}{\xi_w^*} = 0,$$

then from (14), since

$$\begin{aligned} 0 &\leq \frac{\rho[\tau(T_w f - f)]}{\xi_w} \leq \\ &\leq 8\,M\,Q^2\Big(\frac{\rho[\gamma(T_w e_0 - e_0)]}{\xi_w^0} + \frac{\rho[\gamma\,\omega(f;\delta)]}{\xi_w^*}\Big) \quad \text{for every } w \in W, \end{aligned}$$

and by virtue of the axioms (a), (e) of Definition 2.1, it follows that

$$(\ell) \lim_w \frac{\rho[\tau(T_w f - f)]}{\xi_w} = 0,$$

that is the first assertion. The proof of the last part is analogous. $\square$

4 Examples and applications

We now deal with filter convergence, noting that this kind of convergence satisfies the axioms of Definitions 2.1 (see also [16]) and 2.2 (see also [24, Theorems 3 and 4]).

Definitions 4.1 (a) Let W be any abstract infinite set. A nonempty family $\mathcal{F}$ of subsets of W is a *filter* of W iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for every $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

(b) If $W = (W, \succeq)$ is a directed set, then for each $w \in W$, set $M_w := \{z \in W : z \succeq w\}$. A filter $\mathcal{F}$ of W is said to be *free* iff $M_w \in \mathcal{F}$ for every $w \in W$.

(c) A free filter $\mathcal{F}$ of W is called an *ultrafilter* iff for every set $A \subset W$ we get that either $A \in \mathcal{F}$ or $W \setminus A \in \mathcal{F}$.

(d) Let $\mathcal{F}$ be a free filter of W. A net $(x_w)_w$ is said to be *$\mathcal{F}$-convergent* to a real number x iff

$$\{w \in W : |x_w - x| \le \varepsilon\} \in \mathcal{F} \quad \text{for every } \varepsilon > 0, \tag{15}$$

and in this case we write $(\mathcal{F}) \lim_w x_w = x$.

(e) Let $\underline{\mathbf{x}} = (x_w)_w$ be a net in $\mathbb{R}$, and set

$$A_{\underline{\mathbf{x}}} = \{a \in \mathbb{R} : \{w \in W : x_w \ge a\} \notin \mathcal{F}\},$$

$$B_{\underline{\mathbf{x}}} = \{b \in \mathbb{R} : \{w \in W : x_w \le b\} \notin \mathcal{F}\}.$$

The *$\mathcal{F}$-limit superior* of $(x_w)_w$ is given by

$$(\mathcal{F}) \limsup_w \, x_w = \begin{cases} \sup B_{\underline{\mathbf{x}}}, & \text{if } B_{\underline{\mathbf{x}}} \ne \emptyset, \\ -\infty, & \text{if } B_{\underline{\mathbf{x}}} = \emptyset. \end{cases} \tag{16}$$

The *$\mathcal{F}$-limit inferior* of $(x_w)_w$ is

$$(\mathcal{F}) \liminf_w \, x_w = \begin{cases} \inf A_{\underline{\mathbf{x}}}, & \text{if } A_{\underline{\mathbf{x}}} \ne \emptyset, \\ +\infty, & \text{if } A_{\underline{\mathbf{x}}} = \emptyset \end{cases} \tag{17}$$

(see also [24]).

Some examples frequently used in the literature are $(W, \succeq) = (\mathbb{N}, \ge)$, $W \subset [a, w_0[\subset \mathbb{R}$ with the usual order, where $w_0 \in \mathbb{R} \cup \{+\infty\}$ is a limit point of W, or $(W, \succeq) = (\mathbb{N}^2, \ge) = (\mathbb{N} \times \mathbb{N}, \ge)$, where in $\mathbb{N}^2$ the symbol $\ge$ denotes the usual componentwise order (see also [14]).

(f) The *Fréchet filter* is the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite. Observe that the limit, limit superior and limit inferior with respect to $\mathcal{F}_{\text{cofin}}$ coincide with the usual ones (see also [24]).

Remark 4.2 It is not difficult to check that, if $\mathcal{F}$ is any fixed free filter of W, in the $\mathcal{F}$-convergence setting, given $(x_w)_w, (y_w)_w \in \Xi$, we get $x_w = o(y_w)$ if and only if $\{w \in W\colon\ x_w \le \varepsilon\, y_w\} \in \mathcal{F}$ for every $\varepsilon > 0$ and $x_w = O(y_w)$ if and only if there is a positive real number C with $\{w \in W\colon\ x_w \le C\, y_w\} \in \mathcal{F}$. So, our Korovkin-type theorems about convergence and rates of approximations with respect to the axiomatic convergence in Definitions 2.1 and 2.2 contain [28, Lemma 2.4, Theorem 2.5 and Corollaries 2.6-2.8].

In the filter convergence context, it is possible also to relax the positivity condition on the involved linear operators. For instance, let I be a bounded interval of $\mathbb{R}$, $\mathcal{C}^2(I)$ (resp. $\mathcal{C}^2_b(I)$) be the space of all functions defined on I, (resp. bounded and) continuous together with their first and second derivatives, $\mathcal{C}_+ := \{f \in \mathcal{C}^2_b(I) : f \ge 0\}$, $\mathcal{C}^2_+ := \{f \in \mathcal{C}^2_b(I) : f'' \ge 0\}$.

Let e_r, $r = 1, \ldots, m$ and a_r, $r = 0, \ldots, m$ be functions in $\mathcal{C}^2_b(I)$, $P_s(t)$, $s, t \in I$, be as in (3), and suppose that $P_s(t)$ satisfies the above conditions (P1), (P2) and

(P3) there is a positive real number C_0 with $P_s''(t) \geq C_0$ whenever s, $t \in I$, where the second derivative is taken with respect to t.

Some examples in which all properties (P1), (P2) and (P3) are satisfied can be found in [6].

We now state the following Korovkin-type theorem for not necessarily positive linear operators, in the setting of modular filter convergence, whose proof is analogous to that of [5, Theorem 5.2].

Theorem 4.3 *Let $\mathcal{F}$ be any free filter of W, ρ be as in Theorem 3.2, e_r, a_r, $r = 0, \ldots, m$ and $P_s(t)$, s, $t \in I$, satisfy properties* (P1), (P2) *and* (P3). *Assume that T_w, $w \in W$ is a net of linear operators which fulfil property (ρ)-$(*)$, and that $\{w \in W : T_w(\mathcal{C}_+ \cap \mathcal{C}^2_+) \subset \mathcal{C}_+\} \in \mathcal{F}$. If $(T_w e_r)_w$ is (ℓ)-modularly $\mathcal{F}$-convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(I)$, then $(T_w f)_w$ is (ℓ)-modularly $\mathcal{F}$-convergent to f in $L^\rho(I)$ for each $f \in \mathcal{C}^2_b(I)$.*

If $(T_w e_r)_w$ is (ℓ)-strongly $\mathcal{F}$-convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(I)$, then $(T_w f)_w$ is (ℓ)-strongly $\mathcal{F}$-convergent to f in $L^\rho(I)$ for every $f \in \mathcal{C}^2_b(I)$.

Furthermore, if ρ is absolutely continuous and $(T_w e_r)_w$ is (ℓ)-strongly $\mathcal{F}$-convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(I)$, then $(T_w f)_w$ is (ℓ)-modularly $\mathcal{F}$-convergent to f in $L^\rho(I)$ for every $f \in L^\rho(I) \cap \mathcal{D}$ with $f - \mathcal{C}_b(I) \subset X_T$.

Other examples of convergences, satisfying axioms (a)-(j) in Definitions 2.1 and 2.2, are the *single convergence* and the *almost convergence* (see also [22]).

Let $W = \mathbb{N}$. A sequence $(x_n)_n$ is said to *singly converge* (resp. *almost converge*) to $x \in \mathbb{R}$ iff

$$\lim_n \frac{x_{m+1} + x_{m+2} + \ldots + x_{m+n}}{n} = x$$

for every $m \geq 0$ (resp. uniformly with respect to m), where the involved limit is the usual one. It is not difficult to check that single and almost convergence satisfy axioms (a)-(j) in Definitions 2.1 and 2.2. In [22] it is proved that almost convergence is strictly stronger than single convergence, and that for every filter $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ of $\mathbb{N}$ there are sequences, $\mathcal{F}$-convergent to a point x_0 but not singly convergent, and a fortiori not almost convergent, to x_0. Moreover an example of a sequence, almost convergent (and a fortiori singly convergent) to 0 but not $\mathcal{F}$-convergent to 0 for any free filter of $\mathbb{N}$, is given. Thus, in general, single and almost convergence are not generated by any free filter.

We now consider a kind of "triangular statistical convergence" investigated in [4, 25]. Let $A = (a_{i,j})_{i,j}$ be a non-negative two-dimensional infinite matrix and $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a fixed function. We say that A is a *summability matrix* iff it satisfies the following conditions:

$(A1)$ $\displaystyle\sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} \leq 1$ for each $i \in \mathbb{N}$,

$(A2)$ $\lim_i \sum_{j\in\mathbb{N},\Psi(i,j)\geq 0} a_{i,j} > 0$,
$(A3)$ $\lim_i a_{i,j} = 0$ for every $j \in \mathbb{N}$
(see also [30]).

For every $K \subset \mathbb{N}^2$, set $K_i := \{j \in \mathbb{N}\colon\ (i,j) \in K,\ \Psi(i,j) \geq 0\}$. The *$\Psi$-$A$-density* of K is given by

$$\delta_A^\Psi(K) := \lim_i \sum_{j\in K_i} a_{i,j}, \tag{18}$$

provided that the limit on the right hand exists in $\mathbb{R}$.

It is not difficult to see that the Ψ-A-density satisfies the following properties:
$(D1)$ $\delta_A^\Psi(\mathbb{N}^2) > 0$.
$(D2)$ If $K \subset H$, then $\delta_A^\Psi(K) \leq \delta_A^\Psi(H)$.
$(D3)$ If $\delta_A^\Psi(K) = \delta_A^\Psi(H) = 0$, then $\delta_A^\Psi(K \cup H) = 0$
(see also [4]). Observe that from $(D1)$-$(D3)$ it follows that the family

$$\mathcal{F}_A^\Psi := \{K \subset \mathbb{N}^2 : \delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0\} \tag{19}$$

is a filter of $\mathbb{N}^2$. In order to show that $\mathcal{F}_A^\Psi$ is free, it will be enough to check that, if $p,\, q \in \mathbb{N}$ and $K := \{(i,j) \in \mathbb{N}^2\colon\ i \geq p,\ j \geq q\}$, then $\delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0$. Indeed, we get $\mathbb{N}^2 \setminus K \subset K^{(1)} \cup K^{(2)}$, where

$$K^{(1)} = \bigcup_{j=1}^{q-1} (\mathbb{N} \times \{j\}), \quad K^{(2)} = \bigcup_{i=1}^{p-1} (\{i\} \times \mathbb{N}).$$

From $(A3)$ it follows that

$$\delta_A^\Psi(K^{(1)}) = \lim_i \sum_{j\in[1,q-1],\Psi(i,j)=0} a_{i,j} \leq \lim_i \sum_{j=1}^{q-1} a_{i,j} = 0.$$

Furthermore, observe that $\delta_A^\Psi(K^{(2)}) = 0$, since $K_i^{(2)} = \emptyset$ for every $i \geq p$. Hence, $\delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0$, that is the claim.

Definition 4.4 Let $A = (a_{i,j})_{i,j}$ be a summability matrix. The double sequence $(x_w)_w$ is said to *Ψ-A-statistically converge* to a real number x iff $(\mathcal{F}_A^\Psi) \lim_w x_w = x$, that is iff for every $\varepsilon > 0$ we get

$$\lim_i \sum_{j\in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N}\colon\ \Psi(i,j) \geq 0,\ |x_w - x| \geq \varepsilon\}$, and we write st_A^Ψ-$\lim_i x_w = x$.

By st_A^Ψ-$\limsup_i x_w$ and st_A^Ψ-$\liminf_i x_w$ we denote the quantities $(\mathcal{F}_A^\Psi) \limsup_w x_w$ and $(\mathcal{F}_A^\Psi) \liminf_w x_w$, defined as in (16) and (17), respectively.

Let $(a_{i,j})_{i,j}$ be defined by

$$a_{i,j} := \begin{cases} \frac{1}{i^2} & \text{if } j \leq i^2, \\ \\ 0 & \text{otherwise,} \end{cases}$$

put $\Psi(i,j) = i - j$, i, $j \in \mathbb{N}$, and pick any double sequence $(x_{i,j})_{i,j}$ in $\mathbb{R}$. For every $\varepsilon > 0$ we get $K_i(\varepsilon) := \{j \in \mathbb{N}\colon\ j \leq i,\ x_{i,j} \geq \varepsilon\} \subset \{j \in \mathbb{N}\colon\ j \leq i\}$. Thus we obtain

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} \leq \lim_i \sum_{j \leq i} \frac{1}{i^2} = \lim_i \frac{1}{i} = 0, \tag{20}$$

and thus $(x_{i,j})_{i,j}$ Ψ-A-statistically converges to 0. We get $\lim\limits_i \sum\limits_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} = 0$, and so condition $(A2)$ is not fulfilled. Note that, in this case, the class $\mathcal{F}_A^\Psi$ defined as in (19) is not a filter, because it coincides with the family of all subsets of $\mathbb{N}^2$.

Remarks 4.5 (a) If we take $\Psi(i,j) = i - j$, i, $j \in \mathbb{N}$, then we obtain the notion of *triangular A-statistical convergence.*

(b) If $A = C_1$ is the Cesàro matrix, defined by setting

$$a_{i,j} := \begin{cases} \frac{1}{i} & \text{if } j \leq i, \\ \\ 0 & \text{otherwise,} \end{cases}$$

then the Ψ-A-statistical convergence can be viewed as an extension of the classical statistical convergence (see also [4]).

We now claim that, in general, filter convergence in $\mathbb{N}^2$ is not equal to Ψ-A-statistical convergence, that is there exists some filter $\mathcal{F}$ of $\mathbb{N}^2$ such that, for every summability matrix A, there is a set $K \in \mathcal{F} \setminus \mathcal{F}_A^\Psi$. Fix arbitrarily a summability matrix $A = (a_{i,j})_{i,j}$. By $(A1)$ and $(A2)$ there exist a real number $B_0 \in (0,1]$ and an infinite subset $S \subset \mathbb{N}$ with $\sum\limits_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} \geq B_0$ whenever $i \in S$. Let S_1 and S_2 be two disjoint infinite subsets of S, with $S = S_1 \cup S_2$. Let $K := \{(i,1)\colon\ i \in \mathbb{N}\} \cup \{(i,j) : i \in S_1,\ \Psi(i,j) \geq 0\}$. Taking into account $(A3)$, we get

$$\begin{aligned} 0 &\leq \liminf_{i \in \mathbb{N} \setminus S_1} \sum_{j \in \mathbb{N}: (i,j) \in K, \Psi(i,j) \geq 0} a_{i,j} \leq \limsup_{i \in \mathbb{N} \setminus S_1} \sum_{j \in \mathbb{N}: (i,j) \in K, \Psi(i,j) \geq 0} a_{i,j} \leq \\ &\leq \limsup_i a_{i,1} = \lim_i a_{i,1} = 0. \end{aligned} \tag{21}$$

Thus all inequalities in (21) are equalities, and in particular we have

$$\lim_{i \in \mathbb{N} \setminus S_1} \sum_{j \in \mathbb{N}: (i,j) \in K, \Psi(i,j) \geq 0} a_{i,j} = 0. \tag{22}$$

Since $\limsup_{i\in S_1} \sum_{j\in\mathbb{N}:(i,j)\in K,\Psi(i,j)\geq 0} a_{i,j} \geq B_0$, from this and (22) it follows that $\mathbb{N}^2 \setminus K \notin \mathcal{F}_A^\Psi$. From this and $(A2)$ it follows that $K \notin \mathcal{F}_A^\Psi$. Thus we get that any ultrafilter $\mathcal{F}$ of $\mathbb{N}$ contains at least a set not belonging to $\mathcal{F}_A^\Psi$, because it contains either K or $\mathbb{N}^2 \setminus K$. This proves the claim. So, our results extend [4, Theorems 1-4] and [25, Theorems 3-5].

We now give some applications to Mellin-type convolution operators and Kantorovich-type discrete operators.

Examples 4.6 (a) We deal with a direct extension to the multivariate case of the classical one-dimensional moment kernel (see also [9, 17]).

Let $W = [1, +\infty[$ or $W = \mathbb{N}$, $G = [0, 1]^N$ equipped with the usual topology, ρ be as in Theorem 3.4. Let $\mathcal{F}$ be a free filter of W, containing a subset $F \subset W$ such that $W \setminus F$ is infinite. An example is the filter $\mathcal{F}_{\rm st}$ of all subsets of $\mathbb{N}$ having asymptotic density one, and $\mathbb{N} \setminus F$ is the set of all perfect squares or the set of all prime numbers (see also [20]). For every $w \in W$ and $\mathbf{t} =(t_1, t_2, \ldots, t_N) \in G$, let $K_w(\mathbf{t}) = (w+1)^N t_1^w \cdot \ldots \cdot t_N^w$ if $w \in F$ and $K_w(\mathbf{t}) = (w+1)^{N+1} t_1^w \cdot \ldots \cdot t_N^w$ if $w \in W \setminus F$. For $f \in \mathcal{C}(G)$ and $\mathbf{s} = (s_1, s_2, \ldots, s_N) \in G$ set

$$(M_w f)(\mathbf{s}) = \int_G K_w(\mathbf{t}) f(\mathbf{st})\, d\mathbf{t},$$

where $\mathbf{st} = (s_1 t_1, s_2 t_2, \ldots, s_N t_N)$ and $d\mathbf{t} = dt_1\, dt_2 \ldots dt_N$. For each $\mathbf{t} \in G$, set $e_0(\mathbf{t}) = 1$, $e_r(\mathbf{t}) = t_r$, $r = 1, \ldots, N$, and $e_{N+1}(\mathbf{t}) = \sum_{r=1}^N t_r^2$. It is easy to see that (P1), (P2) and (P3) are satisfied. We get

$$\int_G K_w(\mathbf{t}) d\mathbf{t} = (w+1)^N \left(\int_0^1 t_1^w\, dt_1 \right) \cdot \ldots \cdot \left(\int_0^1 t_N^w\, dt_N \right) = 1$$

if $w \in F$, and

$$\int_G K_w(\mathbf{t}) d\mathbf{t} = (w+1)^{N+1} \left(\int_0^1 t_1^w\, dt_1 \right) \cdot \ldots \cdot \left(\int_0^1 t_N^w\, dt_N \right) = w+1$$

if $w \in W \setminus F$. Hence for every $\mathbf{s} \in G$ we get $(M_w e_0)(\mathbf{s}) = e_0(\mathbf{s}) = 1$ if $w \in F$ and $(M_w e_0)(\mathbf{s}) - e_0(\mathbf{s}) = w$ if $w \in W \setminus F$. From this it follows that the operators M_w, $w \in W$, does not satisfy the classical Korovkin theorem. However we have

$$\begin{aligned} |(M_w e_r)(\mathbf{s}) - e_r(\mathbf{s})| &\leq \int_G K_w(\mathbf{t})(1 - t_r)\, d\mathbf{t} = \\ &= (w+1)\Big(\int_0^1 t_1^w (1 - t_1)\, dt_1 \Big) \cdot \ldots \cdot \Big(\int_0^1 t_N^w (1 - t_N)\, dt_N \Big) = \\ &= (w+1)\Big(\int_0^1 t_r^w (1 - t_N)\, dt_r \Big) = 1 - \frac{w+1}{w+2} = \frac{1}{w+2} = O\Big(\frac{1}{w}\Big) \end{aligned}$$

for every $r = 1, 2, \ldots, N$, $w \in F$ and $\mathbf{s} \in G$. Analogously it is possible to see that

$$|(M_w e_r^2)(\mathbf{s}) - e_r^2(\mathbf{s})| \le \frac{2}{w+3} = O\Big(\frac{1}{w}\Big)$$

whenever $r = 1, 2, \ldots, N$, $w \in F$ and $\mathbf{s} \in G$. From this it follows that

$$|(M_w e_{N+1})(\mathbf{s}) - e_{N+1}(\mathbf{s})| = O\Big(\frac{1}{w}\Big)$$

for $w \in W$ and $\mathbf{s} \in G$. Thus there exists a positive real number C_* with

$$|(M_w e_r^2)(\mathbf{s}) - e_r^2(\mathbf{s})| \le \frac{C_*}{w} \tag{23}$$

for any $r = 0, 1, 2, \ldots, N+1$, $w \in F$ and $\mathbf{s} \in G$. Let $Q \ge 1$ be a constant related with Q-quasi semiconvexity of ρ. By applying the modular ρ, we get

$$\rho\Big[\frac{1}{Q^2}(M_w e_r - e_r)\Big] \le \rho\Big[\frac{C_*}{w\, Q^2}\Big] \le \frac{C_*}{w}\, \rho[1] \tag{24}$$

for each $r = 0, 1, 2, \ldots, N+1$ and $w \in F$. Thus, thanks to strong finiteness of ρ, all the hypotheses of Theorem 3.4 are satisfied. So for every $f \in \mathcal{C}_c(G) \cap$ Lip(G) there is $\tau > 0$ with $\rho[\tau(M_w f - f)] = O\Big(\frac{1}{w}\Big)$.

(b) We consider bivariate Kantorovich-type operators. Let $W = \mathbb{N}$, $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ and H be an infinite set with $\mathbb{N} \setminus H \in \mathcal{F}$. Note that H does exist, since $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$. Let $W = \mathbb{N}$, $G = [0,1]^2$ be endowed with the usual topology, $\widetilde{\Phi}$ be as in Example 2.5, $\varphi \in \widetilde{\Phi}$ be with $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$, $\rho = \rho^\varphi$ be as in (1). Proceeding as in [15] (see also [5]), for every locally integrable function $f \in L^0(G)$, $n \in \mathbb{N}$ and x, $y \in [0,1]$ set

$$P_n(f)(x,y) = (n+1)^2 \sum_{k,j=0,\ldots,n,k+j\le n} p_{n,k,j}(x,y) \int_{k/(n+1)}^{(k+1)/(n+1)} \int_{j/(n+1)}^{(j+1)/(n+1)} f(u,v)\, du\, dv,$$

where

$$p_{n,k,j} = \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j}, \quad k, j \ge 0, x, y \ge 0, x+y \le 1.$$

Let $(s_n)_n$ be the sequence defined by setting

$$s_n = \begin{cases} 1 & \text{if } n \in \mathbb{N} \setminus H, \\ 0 & \text{if } n \in H. \end{cases}$$

For every $n \in \mathbb{N}$ and x, $y \ge 0$ with $x + y \le 1$, set

$$P_n^*(f)(x,y) = s_n\, P_n(f)(x,y).$$

For each t_1, $t_2 \in [0,1]$, set $e_0(t_1,t_2) = 1$, $e_1(t_1,t_2) = t_1$, $e_2(t_1,t_2) = t_2$, $e_3(t_1,t_2) = t_1^2 + t_2^2$ (see also [5, Example 4.5(b)]). By proceeding analogously as in [15], it is possible to find a $\gamma > 0$ with $\rho[\gamma\,(P_n^* e_i - e_i)] = O\Big(\frac{1}{n}\Big)$. By Theorem 3.4, for every $f \in \mathcal{C}_c(G) \cap \text{Lip}(G)$ there is a positive real number τ with $\rho[\tau\,(P_n^* f - f)] = O\Big(\frac{1}{n}\Big)$.

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