Representations to the Constant Pi

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Abstract

Some representations to the constant Pi shown:

$$\pi = 4\int_0^1 \sqrt{2x-x^2}\,dx = 4\int_1^2 \sqrt{2x-x^2}\,dx = 3.14159265\ldots$$

Fórmulas

(1) $$\pi = 12\int_0^1 \frac{2(1+x^2)-\sqrt{3}(1-x^2)}{1+14x^2+x^4}\,dx$$

(2) $$\pi = 12\sum_{n=0}^{\infty} \frac{c_n}{n+1}\left(\left(\frac{2}{5}\right)^{n+1} - \left(\frac{\sqrt{3}}{5}\right)^{n+1}\right)$$

donde

$$c_{n+2} = 4c_{n+1} - 5c_n \,, c_0 = 1, c_1 = 4, n \in \mathbb{N} \cup \{0\}$$

(3) $$\pi = \frac{12}{5}\sum_{n=0}^{\infty} \int_{\sqrt{3}}^{2} \left(\frac{4x-x^2}{5}\right)^n dx$$

(4) $$\pi = \frac{12}{5}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{n+k+1}\left(\frac{4}{5}\right)^{n-k}\left(\frac{1}{5}\right)^{k}\left(2^{n+k+1} - \left(\sqrt{3}\right)^{n+k+1}\right)$$

(5) $$\pi = 4\tan^{-1}\left(\frac{Im\, f(a,b)}{Re\, f(a,b)}\right) - 4\tan^{-1}\left(\frac{Im\, g(a,b)}{Re\, g(a,b)}\right)$$

donde

$$f(a,b) = \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2} \,, |ab| < 1$$

$$g(a,b) = (1-ab)\prod_{n=1}^{\infty}(1+a(ab)^n)(1+b(ab)^n)(1-(ab)^{n+1}) \,, |ab| < 1$$

Algunos valores de (a,b) , que verifican la fórmula (5):

$$(a,b)=\left\{\left(\frac{1}{2}+\frac{3i}{4},\frac{1}{2}+\frac{i}{2}\right),\left(\frac{1}{3}+\frac{2i}{3},\frac{1}{2}+\frac{i}{2}\right),\left(\frac{1}{3}+\frac{2i}{3},\frac{1}{3}+\frac{4i}{9}\right),\left(\frac{1}{2}+\frac{3i}{4},\frac{1}{3}+\frac{4i}{9}\right),\left(\frac{1}{4}+\frac{5i}{8},\frac{1}{3}+\frac{4i}{9}\right),\dots\right\}$$

(6) $$\pi=\frac{10}{3}\sum_{n=0}^{\infty}\frac{2^{-2n}c_n}{2n+1}$$

donde

$$c_{n+2}=-\frac{13}{9}c_{n+1}-\frac{4}{9}c_n\ ,c_0=1,c_1=-\frac{7}{9}\ ,n\in\mathbb{N}\cup\{0\}$$

(7) $$\pi=\frac{10}{3}\sum_{n=0}^{\infty}\frac{6^{-2n}c_n}{2n+1}$$

donde

$$c_{n+2}=-13c_{n+1}-36c_n\ ,c_0=1,c_1=-7,n\in\mathbb{N}\cup\{0\}$$

(8) $$\pi=\frac{10}{3}\sum_{n=0}^{\infty}\frac{a_n}{b_n}$$

donde

$$a_{n+2}=-13a_{n+1}-36a_n\ ,a_0=1,a_1=-7\ ,n\in\mathbb{N}\cup\{0\}$$

$$b_{n+2}=72b_{n+1}-1296b_n\ ,b_0=1,b_1=108\ ,n\in\mathbb{N}\cup\{0\}$$

(9) $$\pi=4\sqrt{\frac{b}{2-b}}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(-\frac{b}{2-b}\right)^n+\frac{2}{a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{2n}{n}\binom{n}{k}\binom{k}{m}\frac{(-1)^{k+m}2^{-2n+k-m}(1-b^{k+m+1})}{a^{2k}(k+m+1)}$$

$$a^2>\frac{1}{2}\ ,0<b\le 1$$

(10) $$\pi=4\tan^{-1}\left(\frac{b}{\sqrt{2-b^2}}\right)+4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(1-b^{2k+1})}{2k+1}$$

$$0<b\le 1$$

(11) $$\pi=\frac{4}{a}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{a^2}\right)^k\left(\frac{2-a^2}{a^2}\right)^{n-k}\frac{1}{2k+1}$$

$$a^2 > 1$$

(12) $$\pi^2 = 9 - \frac{21}{2} e^{-1/2} + \sum_{n=0}^{\infty} e^{-\left(\frac{n+1}{2}\right)} \left(\frac{n+3}{(n+1)^2} \right) + 6 \sum_{n=1}^{\infty} (-1)^{n-1} B_n \left(1 - e^{-1/2} \sum_{k=0}^{2n} \frac{2^{-k}}{k!} \right)$$

donde

$$B_n = \left\{ \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots \right\}$$, números de Bernoulli.

(13) $$\pi = 3 + 2 \tan^{-1} \left(\frac{2}{3} - 2 \sum_{n=1}^{\infty} \frac{B_n 3^{2n-1}}{(2n)!} \right)$$

B_n , números de Bernoulli.

(14) $$\frac{1}{\pi \beta} = 2 + 2 \sum_{n=1}^{\infty} \frac{(1 - 2^{-2n+1}) B_n}{(2n)!}$$

donde

$$\beta = \alpha \prod_{n=1}^{\infty} \left(1 - \frac{\alpha^2}{n^2} \right) , \alpha = \frac{1}{6} \prod_{n=1}^{\infty} \left(1 - \frac{1}{36n^2} \right)$$

B_n , números de Bernoulli.

(15) $$\pi = 3 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n}{n} \binom{n}{k} \frac{(-1)^k (1+x^2)^{n-k} x^{2k} 2^{-4n-2k}}{2n+2k+1} - 3 \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{2n-2k}{n-k} \binom{2k}{k} \frac{x^{2n-2k} 2^{-4n}}{2n+1} , 0 < x < 1$$

(16) $$\pi = 2\sqrt{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n}{n} \binom{n}{k} \frac{(-1)^k (1+x^2)^{n-k} x^{2k} 2^{-3n-k}}{2n+2k+1} - 2\sqrt{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{2n-2k}{n-k} \binom{2k}{k} \frac{x^{2n-2k} 2^{-3n}}{2n+1} , 0 < x < 1$$

(17) $$\pi = \frac{3\sqrt{3}}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n}{n} \binom{n}{k} \frac{(-1)^k (1+x^2)^{n-k} x^{2k} 2^{-4n-2k} 3^{n+k}}{2n+2k+1} - \frac{3\sqrt{3}}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{2n-2k}{n-k} \binom{2k}{k} \frac{x^{2n-2k} 2^{-4n} 3^n}{2n+1} , 0 < x < 1$$

En (15),(16),(17) , con $x = 1/2$, se tiene:

(18) $$\pi = 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}\frac{(-1)^k(5/64)^n(1/20)^k}{2n+2k+1} - 3\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{2^{-6n}2^{2k}}{2n+1}$$

(19) $$\pi = 2\sqrt{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}\frac{(-1)^k(5/32)^n(1/10)^k}{2n+2k+1} - 2\sqrt{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{2^{-5n}2^{2k}}{2n+1}$$

(20) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}\frac{(-1)^k(15/64)^n(3/20)^k}{2n+2k+1} - \frac{3\sqrt{3}}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{(3/64)^n 2^{2k}}{2n+1}$$

En las fórmulas (21)-(25) , $\varphi = \frac{1+\sqrt{5}}{2}$

(21) $$\pi^2 = \frac{15}{2}(\ln\varphi)^2 + 15\sum_{n=0}^{\infty}\frac{(-1)^n\varphi^{-n-1}}{(n+1)^2}$$

(22) $$\pi^2 = \frac{15}{2}(\ln\varphi)^2 + 15\sum_{n=0}^{\infty}\left(\frac{1}{a+\varphi}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{n-k}}{(k+1)^2}\ , a > \frac{1-\sqrt{5}}{4}$$

(23) $$\pi^2 = \frac{15}{2}(\ln\varphi)^2 + 15\sum_{n=0}^{\infty}\left(\frac{2}{1+2\varphi}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 2^{-n+k}}{(k+1)^2}$$

(24) $$\frac{\pi^2}{10\varphi} + \frac{(\ln\varphi)^2}{\varphi} = \sum_{n=0}^{\infty}\left(\frac{1}{1+a}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\varphi^k a^{n-k}}{(k+1)^2}\ , a > \frac{\sqrt{5}-1}{4}$$

(25) $$\frac{\pi^2}{10\varphi} + \frac{(\ln\varphi)^2}{\varphi} = \sum_{n=0}^{\infty}\left(\frac{2}{2+\varphi}\right)^{n+1}\left(\frac{\varphi}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{(k+1)^2}$$

(26) $$\pi = 4ab + 4\int_a^1\sqrt{1-x^2}\,dx + 4\int_b^1\sqrt{1-x^2}\,dx\ , b = \sqrt{1-a^2}, 0 \le a \le 1$$

(27) $$\pi = 6\sqrt{1-x^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}\frac{(-1)^k(1+x^2)^{n-k}x^{2k}2^{-4n-2k-1}}{2n+2k+1}$$
$$+6\sqrt{1-x^2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{2n-2k}{n-k}\binom{2k}{k}2^{-2n}x^{2n-2k}\left(\frac{2^{-2k-1}}{2k+1}-\frac{2^{-2n-1}}{2n+1}\right)$$
$$0 < x < 1$$

(28) $$\pi = 8\int_{1}^{\sqrt{2}}\frac{1}{2-2x+x^2}\,dx = 8\int_{2-\sqrt{2}}^{1}\frac{1}{2-2x+x^2}\,dx = 4\int_{2-\sqrt{2}}^{\sqrt{2}}\frac{1}{2-2x+x^2}\,dx$$

(29) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{2+a}\right)^{n+1}\frac{(-1)^m a^{n-k}2^{k-m}\left(\left(\sqrt{2}\right)^{k+m+1}-1\right)}{k+m+1}$$
$$a > -\sqrt{2}$$

(30) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{2+a}\right)^{n+1}\frac{(-1)^m a^{n-k}2^{k-m}\left(1-\left(2-\sqrt{2}\right)^{k+m+1}\right)}{k+m+1}$$
$$a > -\sqrt{2}$$

(31) π
$$= 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{2+a}\right)^{n+1}\frac{(-1)^m a^{n-k}2^{k-m}\left(\left(\sqrt{2}\right)^{k+m+1}-\left(2-\sqrt{2}\right)^{k+m+1}\right)}{k+m+1}$$
$$a > -\sqrt{2}$$

(32) $$\pi = 8\sum_{n=0}^{\infty}\frac{c_n\left(1-\left(2-\sqrt{2}\right)^{n+1}\right)}{2^{n+1}(n+1)}$$

donde

$$c_{n+2} = 2(c_{n+1}-c_n)\,, c_0 = 1, c_1 = 2$$

(33) $$\pi = 24\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k\left(\left(1/\sqrt{3}\right)^{ak+2m+a+1}-\left(\sqrt{2}-1\right)^{ak+2m+a+1}\right)}{ak+2m+a+1}$$
$$a > 0$$

(34) $$\pi = \frac{3}{a}\ln\left(\frac{4}{3}\right) - \frac{6}{a}\ln\left(1 + \frac{1}{a\sqrt{3}}\right) + 6\left(1 + \frac{1}{a^2}\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{a}\right)^{n-k+m}\frac{(-1)^n\left(\sqrt{3}\right)^{-n-k-m-1}}{n+k+m+1}$$

$$a > \frac{2}{\sqrt{3}}$$

(35) $$\pi = \frac{4}{a}\ln\left(4 - 2\sqrt{2}\right) - \frac{8}{a}\ln\left(1 + \frac{\sqrt{2}-1}{a}\right) + 8\left(1 + \frac{1}{a^2}\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{a}\right)^{n-k+m}\frac{(-1)^n\left(\sqrt{2}-1\right)^{n+k+m+1}}{n+k+m+1}$$

$$a > 2 - \sqrt{2}$$

(36) $$\pi = \frac{6}{a}\ln\left(8 - 4\sqrt{3}\right) - \frac{12}{a}\ln\left(1 + \frac{2-\sqrt{3}}{a}\right) + 12\left(1 + \frac{1}{a^2}\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{1}{a}\right)^{n-k+m}\frac{(-1)^n\left(2-\sqrt{3}\right)^{n+k+m+1}}{n+k+m+1}$$

$$a > \frac{2}{3}\left(2\sqrt{3} - 3\right)$$

(37) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(-1)^n\left(\sqrt{2} - 1\right)^{2n+(a-2)k+2m+1}\left(\frac{1}{2n+(a-2)k+2m+1} + \frac{\left(\sqrt{2}-1\right)^a}{2n+(a-2)k+2m+a+1}\right)$$

$$a > \frac{\ln\left(1/\sqrt{2}\right)}{\ln\left(\sqrt{2}-1\right)}$$

(38) $$\pi = \frac{12\sqrt{ab}}{b-a}\ln\left(\frac{(1+a)b}{(1+b)a}\right) + \frac{12}{\sqrt{ab}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n}{n+k+2}\left(\frac{a+b}{ab}\right)^{n+1}\left(\frac{1}{a+b}\right)^{k}$$

$$a < b, ab = \left(2 + \sqrt{3}\right)^2, a + b < 1 + ab$$

(39) $$\pi = \frac{18}{11}\left(8+5\sqrt{3}\right)\ln\left(\frac{22+12\sqrt{3}}{39}\right) + 12\left(2-\sqrt{3}\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n}{n+k+2}\left(\frac{64-36\sqrt{3}}{3}\right)^{n+1}\left(\frac{12-3\sqrt{3}}{52}\right)^{k}$$

(40) $$\pi = 6\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{3}\right)^{-an-1}}{an+1} + 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}\left(\sqrt{3}\right)^{-an-(2-a)(k+m)-a-1}}{an+(2-a)(k+m)+a+1}$$

$$a > 0$$

(41) $$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n\left(1+\sqrt{2}\right)^{-an-1}}{an+1} + 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}\left(1+\sqrt{2}\right)^{-an-(2-a)(k+m)-a-1}}{an+(2-a)(k+m)+a+1}$$

$$a > 0$$

(42) $$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n\left(2+\sqrt{3}\right)^{-an-1}}{an+1} + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}\left(2+\sqrt{3}\right)^{-an-(2-a)(k+m)-a-1}}{an+(2-a)(k+m)+a+1}$$

$$a > 0$$

(43) $$\pi = 4a\sum_{n=0}^{\infty}\frac{1}{(2an+1)(2an+a+1)} + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\sum_{s=0}^{k}\binom{n+1}{k}\binom{n-k+1}{m}\binom{k}{s}\frac{(-1)^{n+m+s}(2/3)^{n-s+2}(1/3)^{s}}{an+(2-a)(k+m)-2s+a+1}$$

$$a > 0$$

(44) $$\pi = 6\sum_{n=0}^{\infty}\frac{c_n\left(1/\sqrt{3}\right)^{n+1}}{n+1} + 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(1/\sqrt{3}\right)^{3n-k+4}}{3n-k+4}$$

$$(45)\qquad \pi = 8\sum_{n=0}^{\infty}\frac{c_n\left(\sqrt{2}-1\right)^{n+1}}{n+1} + 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(\sqrt{2}-1\right)^{3n-k+4}}{3n-k+4}$$

$$(46)\qquad \pi = 12\sum_{n=0}^{\infty}\frac{c_n\left(2-\sqrt{3}\right)^{n+1}}{n+1} + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(2-\sqrt{3}\right)^{3n-k+4}}{3n-k+4}$$

En las fórmulas (44),(45),(46) , se tiene:

$$c_{n+3} = -(c_{n+1} + c_n)\ , c_0 = 1, c_1 = 0, c_2 = -1$$

$$(47)\qquad \pi = 16\ln 2 + 8\ln\left(\frac{5-2\sqrt{3}}{8-2\sqrt{3}}\right)$$

$$+ 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n(5/2)^{n-k+1}\left(2-\sqrt{3}\right)^{n+k+2}}{n+k+2}$$

$$(48)\qquad \pi = 6\sum_{n=0}^{\infty}\frac{c_n\left(1/\sqrt{3}\right)^{n+1}}{n+1} - 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k\left(1/\sqrt{3}\right)^{3n-k+4}}{3n-k+4}$$

$$(49)\qquad \pi = 8\sum_{n=0}^{\infty}\frac{c_n\left(\sqrt{2}-1\right)^{n+1}}{n+1} - 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k\left(\sqrt{2}-1\right)^{3n-k+4}}{3n-k+4}$$

$$(50)\qquad \pi = 12\sum_{n=0}^{\infty}\frac{c_n\left(2-\sqrt{3}\right)^{n+1}}{n+1} - 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k\left(2-\sqrt{3}\right)^{3n-k+4}}{3n-k+4}$$

En las fórmulas (48),(49),(50) , se tiene:

$$c_{n+3} = -(c_{n+1} - c_n)\ , c_0 = 1, c_1 = 0, c_2 = -1$$

$$(51)\qquad \pi = 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\left(1/\sqrt{3}\right)^{an+(2-a)k+1}}{an+(2-a)k+1}$$

$$+ 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(1/\sqrt{3}\right)^{an+(2-a)k+a+1}}{an+(2-a)k+a+1}$$

$$a > \frac{\ln(2/3)}{\ln\left(1/\sqrt{3}\right)}$$

$$(52)\qquad \pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\left(\sqrt{2}-1\right)^{an+(2-a)k+1}}{an+(2-a)k+1}$$

$$+ 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(\sqrt{2}-1\right)^{an+(2-a)k+a+1}}{an+(2-a)k+a+1}$$

$$a > \frac{\ln\left(2\sqrt{2}-2\right)}{\ln\left(\sqrt{2}-1\right)}$$

(53) $$\pi = 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\left(2-\sqrt{3}\right)^{an+(2-a)k+1}}{an+(2-a)k+1} + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^n\left(2-\sqrt{3}\right)^{an+(2-a)k+a+1}}{an+(2-a)k+a+1}$$

$$a > \frac{\ln\left(4\sqrt{3}-6\right)}{\ln\left(2-\sqrt{3}\right)}$$

(54) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty} a_n\,(r^n + r^{3n+1})$$

donde

$$a_n = \binom{2n}{n}\frac{2^{-2n}}{2n+1}\ , n \in \mathbb{N}\cup\{0\}$$

$$r = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{31}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{31}{3}}}$$

La fórmula (54) , se puede escribir de varias formas alternativas:

(55) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty}(a_{3n} + (a_n + a_{3n+1})r + a_{3n+2}r^2)\,r^{3n}$$

(56) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty} b_n\, r^n$$

donde

$$b_n = \begin{cases} a_n\ , n = 3k\ , 3k+2\ , k \in \mathbb{N}\cup\{0\} \\ a_n + a_{\frac{n-1}{3}}\ , n = 3k+1\ , k \in \mathbb{N}\cup\{0\} \end{cases}$$

(57) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty} b_n\,(\alpha_n + \beta_n r + \gamma_n r^2)$$

donde

$$\alpha_{n+1} = \gamma_n, \beta_{n+1} = \alpha_n - \gamma_n, \gamma_{n+1} = \beta_n, \alpha_0 = 1, \beta_0 = 0, \gamma_0 = 0\ , n \in \mathbb{N}\cup\{0\}$$

(58) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty}(a_{3n} + (a_n + a_{3n+1})r + a_{3n+2}r^2)(\delta_n + \varepsilon_n r + \rho_n r^2)$$

donde

$$\delta_{n+1} = \delta_n - \rho_n, \varepsilon_{n+1} = -\delta_n + \varepsilon_n + \rho_n, \rho_{n+1} = -\varepsilon_n + \rho_n, \delta_0 = 1, \varepsilon_0 = 0, \rho_0 = 0$$

(59) $$\pi = 2\sqrt{r}\sum_{n=0}^{\infty} a_n \left((\alpha_n + \alpha_{3n+1}) + (\beta_n + \beta_{3n+1})r + (\gamma_n + \gamma_{3n+1})r^2\right)$$

(60) $$\pi \frac{n\, 2^{-2n-1}}{(2n-1)}\binom{2n}{n} = \frac{n}{2n-1}\sum_{k=1}^{n-1}\frac{2^{-k}}{(1+n^2)^{n-k}}\prod_{j=1}^{k}\frac{2n-2j+1}{n-j} + 2^{-2n+2}\binom{2n-2}{n-1}\tan^{-1}\left(\frac{n-1}{n+1}\right) + \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(-1)^k n^{-2k-2n+1}}{2k+2n-1}$$

$$n \in \mathbb{N}$$

(61) $$\pi \frac{n\, 2^{-2n}}{2n-1}\binom{2n}{n} = \sum_{j=1}^{r} f\left(n, x_{j-1}, x_j\right), n \in \mathbb{N}$$

donde

$$f(n,x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{n+k-1}{k}\binom{k}{m}(-1)^m\left(\frac{x^2+y^2}{2+x^2+y^2}\right)^{k-m}\left(\frac{2}{2+x^2+y^2}\right)^{m+n}\left(\frac{y^{2m+1}-x^{2m+1}}{2m+1} + \frac{y^{2m+2n-1}-x^{2m+2n-1}}{2m+2n-1}\right), 0 \le x < y \le 1, n \in \mathbb{N}$$

$$0 = x_0 < x_1 < x_2 < \cdots < x_r = 1, r \in \mathbb{N}$$

Ejemplo : $n = 1,2,3$

$$\pi = 2\left(f\left(1,0,\frac{1}{2}\right) + f\left(1,\frac{1}{2},1\right)\right)$$

$$\pi = 4\left(f\left(2,0,\frac{1}{2}\right) + f\left(2,\frac{1}{2},1\right)\right)$$

$$\pi = 4\left(f\left(2,0,\frac{1}{3}\right) + f\left(2,\frac{1}{3},\frac{2}{3}\right) + f\left(2,\frac{2}{3},1\right)\right)$$

$$\pi = \frac{16}{3}\left(f\left(3,0,\frac{1}{2}\right) + f\left(3,\frac{1}{2},1\right)\right)$$

$$\pi = \frac{16}{3}\left(f\left(3,0,\frac{1}{3}\right) + f\left(3,\frac{1}{3},\frac{2}{3}\right) + f\left(3,\frac{2}{3},1\right)\right)$$

(62) $$\pi = \frac{2^{2n}(2n-1)}{n}\binom{2n}{n}^{-1}\sum_{k=0}^{\infty} 2^{-k}\sum_{m=0}^{k}\binom{k}{m}\binom{n+m-1}{m}\frac{(-1)^m(2m+n)}{(2m+1)(2m+2n-1)}$$

$$n \in \mathbb{N}$$

(63) $$\pi\ \frac{n}{2^{2n}(2n-1)}\binom{2n}{n} = a\,b + \int_a^{\infty}(1+x^2)^{-n}\,dx + \int_b^{1}\sqrt{x^{-1/n}-1}\,dx$$

$$a > 0, b(1+a^2)^n = 1, n \in \mathbb{N}$$

(64) $$\pi = \frac{100}{13}\sum_{n=0}^{\infty}\frac{c_n}{(n+1)26^n}\left(\left(\sqrt{2}-\frac{6}{5}\right)^{n+1} - \left(-\frac{1}{5}\right)^{n+1}\right)$$

donde

$$c_{n+2} = -10c_{n+1} - 650c_n\ , c_0 = 1, c_1 = -10, n \in \mathbb{N}\cup\{0\}$$

(65) $$\pi = \frac{1}{\sqrt{2-\sqrt{2}}\ \cos(\theta/2)}\left(4 - 8\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\left(\frac{n\pi}{4}\right)\cos(n\theta)\right)\ , |\theta| \le \frac{\pi}{8}$$

(66) $$\pi = \frac{4}{\theta}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi}{8}\right)\sin\big((2n-1)\theta\big)\ , 0 < \theta \le \frac{\pi}{8}$$

(67) $$\pi = 2a\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{a^2+b^2}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{b^{n-k}\left((a-b)^{n+k+1} - (-b)^{n+k+1}\right)}{(n+k+1)2^k}$$

$$a > 0, b > 0$$

(68) $$\pi = 16\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{n}\left(\frac{\sqrt{2-\sqrt{2}}}{8}\right)^n c_n$$

donde

$$c_{n+2} = 2\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}\ c_{n+1} - \frac{4c_n}{2+\sqrt{2+\sqrt{2}}}\ , c_1 = 1, c_2 = 2\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}$$

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