Special relativistic Fourier transformation and convolutions*

Eckhard Hitzer
International Christian University
Osawa 3-10-2, 181-8585 Mitaka, Tokyo, Japan
hitzer@icu.ac.jp

April 3, 2019

Abstract

In this paper we use the steerable special relativistic (space-time) Fourier transform (SFT), and relate the classical convolution of the algebra for space-time $Cl(3,1)$-valued signals over the space-time vector space $\mathbb{R}^{3,1}$, with the (equally steerable) Mustard convolution. A Mustard convolution can be expressed in the spectral domain as the point wise product of the SFTs of the factor functions. In full generality do we express the classical convolution of space-time signals in terms of finite linear combinations of Mustard convolutions, and vice versa the Mustard convolution of space-time signals in terms of finite linear combinations of classical convolutions.
Keywords: Convolution, Mustard convolution, space-time Fourier transform, space-time signals, space-time domain, frequency domain
Mathematics Subject Classification: 15A66, 42B10

1 Introduction

Albert Einstein published in his annus mirabilis 1905 the first paper on special relativity entitled *On the Electrodynamics of Moving Bodies* [6]. Since that time special relativity has become the common framework for all physical phenomena in space and time, refining Newton's classical notion of inertial system. For speeds small compared to the speed of light, Newton's inertial systems continue to be used in practice, but any phenomenon relating to the propagation of electromagnetic waves, or high energy particles, the special relativistic space-time theory of Einstein is indispensable. Electromagnetic wave based communication (radio waves, infrared waves, light waves, etc.) networks form the backbone of modern ICT infrastructure, for light switch sensors at home up

* *This paper has been published as:* E. Hitzer, Special relativistic Fourier transformation and convolutions, Mathematical Methods in the Applied Sciences, First published: 04 Mar. 2019, Vol. 42, Issue 7, pp. 2244–2255, 2019, DOI: `10.1002/mma.5502`, URL: `https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.5502`.

to space based satellite data communication. Even for the microscopic description of atoms, only special relativistic quantum mechanics allows to precisely understand atomic spectra.

Therefore it is fundamental to be able to process electromagnetic signals of all kinds with mathematical tools fully adapted to Einstein's special relativity. The most natural mathematical framework for this is the Clifford algebra of space-time, which unifies vector and spinor formalisms [27], with elementary algebraic expressions for the four-dimensional vector space of space-time and all its subspaces. This framework is furthermore ideal for unifying classical and quantum physics [11, 27, 9]. The prevalent tool in optics for all kinds of signal and image processing, and for position to momentum representation changes in quantum mechanics, is the Fourier transform. In this paper we explain how to formulate and apply the Fourier transform in the context of special relativity, described by the Clifford algebra of space-time.

Hamilton's eighteenth century four-dimensional quaternions frequently appear as subalgebras of higher order Clifford geometric algebras [2, 27], invented in the 1870ies. This is also the case for the sixteen dimensional Clifford algebra over the space-time vector space [11, 13, 9], which is of prime importance in physics, and in all applications where time matters as well (motion in time, video sequences, flow fields, ...). P.A.M. Dirac's relativistic equation for the electron has been reformulated in the real framework of space-time algebra [11, 12], and this has been extended to a relativistic multiparticle equation formalism, including a new framework for entanglement [10, 5]. Technically, the quaternion subalgebra structure gives a first hint on how to introduce generalizations of the quaternion Fourier transform (QFT) to functions in these higher order Clifford geometric algebras. For example it allows to non-trivially *generalize* the QFT from four dimensions to a space-time Fourier transform [14, 17] in sixteen dimensions.

Recently it has been shown how the left-sided QFT [4], and the two-sided QFT [24] allow to define convolutions[1] for which in the spectral domain the QFT of the convolution becomes a simple point wise product of the QFTs of the quaternion signal functions. This property is a key for filter based signal and image processing. This paper endeavors to generalize the approach of [24] from four-dimensional quaternions to the sixteen-dimensional Clifford algebra $Cl(3,1)$. Because generalizing results from a lower dimensional non-commutative division algebra to a higher-dimensional non-commutative algebra is non-trivial, i.e. from four to sixteen dimensions, we aspire to work with sufficient algebraic detail[2] to allow all results to be verified directly.

Note, that in Section 2, beginning with Table 1, we also introduce new results for the space-time Fourier transform, which have so far not appeared in the scientific literature. The fundamental results obtained in this work will be of relevance in relativistic physics, relativistic quantum mechanics, propaga-

[1]Sometimes named after [28] *Mustard* convolution.

[2]We also think that for practical applications, it would be of very little help, if we simply appeal to principles of generalizability of Fourier transformations in Clifford algebras, or present results which are valid in all dimensions and all non-Euclidean vector spaces. The necessary ground work for researchers and engineers not specializing in geometric algebra computation, is to present the results as concretely as possible for the specific case of space-time, which we envision to be of great importance in a wide range of applications. An even further abstract generalization of the approach taken in the present work to Clifford algebras of general quadratic spaces $Cl(p,q)$ can now be found in [25].

tion of electromagnetic signals (including satellite communication), communication networks, geographic information systems, navigation, climate observations, global data processing, seismology, geothermal resource engineering, video sequencing and video feature detection, flow field processing, three- and four-dimensional video registration, electromagnetic cavity engineering, synchrotron and gyrotron physics, medical video imaging and many more fields of research and application.

Regarding implementations, standard computer algebra programs like MAPLE have continuously updated well developed free packages like CLIFFORD [1], for Clifford algebra computations. Alternatively numerical packages like MATLAB have free packages like the Clifford Multivector Toolbox [29]. There are also packages for all major modern programing languages [26].

This paper is organized as follows. Section 2 reviews the sixteen dimensional Clifford geometric algebra $Cl(3,1)$ of the four-dimensional space-time vector space $\mathbb{R}^{3,1}$. As an introductory first step, the particular role of a subalgebra isomorphic to quaternions is studied, which is generated by the time-vector and the three-dimensional space volume pseudoscalar. Section 3 then progresses to the full sixteen dimensional algebra $Cl(3,1)$ and reviews the (steerable) space-time Fourier transform (SFT) of [14, 17], and newly introduces related exponential-sine Fourier transforms for sixteen-dimensional Space-time Clifford algebra valued signals over space-time. Section 4 sets out with defining the convolution, and two types of (equally steerable) Mustard convolutions for space-time signals in $Cl(3,1)$ over $\mathbb{R}^{3,1}$. This is followed by the main results of Theorem 4.3 describing the convolution of sixteen dimensional space-time Clifford algebra signals in terms of the two types of Mustard convolutions, Theorem 4.5 expressing the convolution in terms of only four standard Mustard convolutions, and finally vice versa Theorem 4.6 describing the standard Mustard convolution of space-time signals in terms of eight classical convolutions. The paper concludes with Section 5.

2 Algebra for space-time

The algebra for space-time $Cl(3,1) = Cl_{3,1} = \mathcal{G}_{3,1} = \mathbb{R}_{3,1}$ is Clifford's geometric algebra of $\mathbb{R}^{3,1}$. In $\mathbb{R}^{3,1}$ we can introduce the following orthonormal vector basis,

$$\{\boldsymbol{e}_t, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \quad -\boldsymbol{e}_t^2 = \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = 1. \tag{2.1}$$

In the full blade basis of $Cl(3,1)$ we thus get three anti-commuting blades that all square to minus one, they are some of the roots of -1 (compare [21]),

$$\boldsymbol{e}_t^2 = -1, \quad i_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3, \quad i_3^2 = -1, \quad i_{st} = \boldsymbol{e}_t\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3, \quad i_{st}^2 = -1, \tag{2.2}$$

and the commutator

$$[\boldsymbol{e}_t, i_3] = 2\boldsymbol{e}_t i_3 = 2i_{st}. \tag{2.3}$$

The volume-time subalgebra of $Cl(3,1)$ generated by these blades is indeed isomorphic to the quaternion algebra [9].

$$\{1, \boldsymbol{e}_t, i_3, i_{st}\} \longleftrightarrow \{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \tag{2.4}$$

This isomorphism allows us now to transfer the quaternionic $\pm$ split (or orthogonal two-dimensional planes split (OPS)[3]) of [14, 17, 23, 18, 20] to space-time algebra, which turns out to be a very real (physical) space-time split

$$h_{\pm} = \frac{1}{2}(h \pm \boldsymbol{e}_t h \boldsymbol{e}_t^*), \qquad h = h_+ + h_-, \tag{2.5}$$

where $\boldsymbol{e}_t^* = i_3$, is the space-time dual of the unit time direction $\boldsymbol{e}_t$, i.e.,

$$\boldsymbol{e}_t^* = \boldsymbol{e}_t i_{st}^{-1} = -\boldsymbol{e}_t i_{st} = -\boldsymbol{e}_t \boldsymbol{e}_t i_3 = i_3. \tag{2.6}$$

The time direction $\boldsymbol{e}_t$ determines therefore the complementary three-dimensio nal physical Euclidean space with pseudoscalar i_3 as well! Their product $i_{st} = \boldsymbol{e}_t i_3$ is the four-dimensional space-time hypervolume pseudoscalar. Note that

$$\boldsymbol{e}_t h i_3 = h_+ - h_-, \tag{2.7}$$

i.e. under the involution map $\boldsymbol{e}_t()i_3$ the h_+ part is invariant, but the h_- part changes sign, which is related to the Coxeter half-turn [3]. See also Table 1.

We further note, that with respect to $f \in \{\boldsymbol{e}_t, i_3, i_{st}\} \subset \mathbb{R}^{3,1}$, every multivector $A \in Cl(3,1)$ can be split into *commuting* and *anticommuting* parts [21].

Lemma 2.1 (Commuting and anticommuting with $f \in \{\boldsymbol{e}_t, i_3, i_{st}\} \subset \mathbb{R}^{3,1}$ [21])**.** *Every multivector $A \in Cl(3,1)$ has, with respect to every $f \in \{\boldsymbol{e}_t, i_3, i_{st}\} \subset \mathbb{R}^{3,1}$, where we note that $f^{-1} = -f$, the unique decomposition denoted by*

$$A_{+f} = \frac{1}{2}(A + f^{-1}Af), \qquad A_{-f} = \frac{1}{2}(A - f^{-1}Af)$$
$$A = A_{+f} + A_{-f}, \qquad A_{+f}\, f = f A_{+f}, \qquad A_{-f}\, f = -f A_{-f}. \tag{2.8}$$

Notation 2.2 (Argument reflection)**.** *For a function $h : \mathbb{R}^{3,1} \to Cl(3,1)$ and a multi-index $\phi = (\phi_1, \phi_2)$ with $\phi_1, \phi_2 \in \{0,1\}$ we set*

$$h^{\phi} = h^{(\phi_1,\phi_2)}(\boldsymbol{x}) := h((-1)^{\phi_1} t, (-1)^{\phi_2} \vec{x}). \tag{2.9}$$

In (2.5) the involution $\boldsymbol{e}_t()\boldsymbol{e}_t^* = \boldsymbol{e}_t()i_3$, plays an important role. For the quaternion algebra $\mathbb{H}$ this has been studied in [14, 17, 23, 18, 20]. The isomorphism (2.4) provides the algebraic means to transfer these results[4] to the space-time subalgebra $\{1, \boldsymbol{e}_t, i_3, i_{st}\}$. Involutions like $\boldsymbol{e}_t()i_3$, and decompositions like (2.5) and Lemma 2.1, provided the key to the geometric interpretation of the two-sided QFT [23, 18, 20], and are significant and efficient in establishing several types of quaternion signal convolutions both in the spatial, as well as in the spectral domain [24].

We therefore begin our investigation of the convolutions of space-time signals, $h : \mathbb{R}^{3,1} \to Cl(3,1)$, also with the study of involutions of $Cl(3,1)$, using the three square roots of -1: $\{\boldsymbol{e}_t, i_3, i_{st}\}$.

[3] As explained in further detail in [18, 20], the geometry encoded in the OPS is not a simple result of a rotation in the three-dimensional space spanned by the imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The intricate geometry of this split is fully investigated in [23].

[4] The full development of the sixteen dimensional Clifford Fourier transform for $Cl(3,1)$ valued signals over space time is left to Section 3.

	space-time algebra basis elements															
inv.	1	e_{23}	e_{31}	e_{12}	e_1	e_2	e_3	i_3	e_t	e_{t23}	e_{t31}	e_{t12}	e_{t1}	e_{t2}	e_{t3}	i_{st}
$\boldsymbol{e}_t()\boldsymbol{e}_t$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
$i_3()i_3$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$i_{st}()i_{st}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\boldsymbol{e}_t()i_3$	i_{st}	$-e_{t1}$	$-e_{t2}$	$-e_{t3}$	e_{t23}	e_{t31}	e_{t12}	$-e_t$	$-i_3$	e_1	e_2	e_3	$-e_{23}$	$-e_{31}$	$-e_{12}$	1
$\boldsymbol{e}_t()i_{st}$	$-i_3$	e_1	e_2	e_3	e_{23}	e_{31}	e_{12}	-1	$-i_{st}$	e_{t1}	e_{t2}	e_{t3}	e_{t23}	e_{t31}	e_{t12}	$-e_t$
$i_3()i_{st}$	e_t	e_{t23}	e_{313}	e_{t12}	$-e_{t1}$	$-e_{t2}$	$-e_{t3}$	$-i_{st}$	1	e_{23}	e_{31}	e_{12}	$-e_1$	$-e_2$	$-e_3$	$-i_3$

Table 1: Involutions of space-time algebra $Cl(3,1)$. If the involution only changes the sign of the element, only the sign is given. Abbreviations: $i_3 = e_{123}$, $i_{st} = e_{t123}$, $e_{t1} = e_t e_1$, $e_{12} = e_1 e_2$, etc.

Following Table 1, and giving the 16 element basis set of the algebra for space-time $Cl(3,1)$ in the first line the name B, we find the following important basis subsets, spanning eight-dimensional subspaces

$$\begin{aligned}
B_+ &= \{e_t - i_3, (e_t - i_3)e_1, (e_t - i_3)e_2, (e_t - i_3)e_3, \\
&\qquad 1 + i_{st}, (1 + i_{st})e_1, (1 + i_{st})e_2, (1 + i_{st})e_3\}, \\
B_- &= \{e_t + i_3, (e_t + i_3)e_1, (e_t + i_3)e_2, (e_t + i_3)e_3, \\
&\qquad 1 - i_{st}, (1 - i_{st})e_1, (1 - i_{st})e_2, (1 - i_{st})e_3\}, \\
B_{+e_t} &= \{1, e_{23}, e_{31}, e_{12}, e_t, e_{t23}, e_{t31}, e_{t12}\}, \\
B_{-e_t} &= \{\, e_1, e_2, e_3, i_3, e_{t1}, e_{t2}, e_{t3}, i_{st}\}, \\
B_{+i_3} &= \{1, e_{23}, e_{31}, e_{12}, e_1, e_2, e_3, i_3\}, \\
B_{-i_3} &= \{e_t, e_{t23}, e_{t31}, e_{t12}, e_{t1}, e_{t2}, e_{t3}, i_{st}\},
\end{aligned} \tag{2.10}$$

where $B_\pm$ is defined by (2.5), and $B_{\pm e_t}$, $B_{\pm i_3}$ according to Lemma 2.1. The eight-dimensional plus and minus parts of the algebra $Cl(3,1)$ arising from the split with (2.5) can also be specified as

$$\begin{aligned}
Cl(3,1)_+ &= \mathrm{span}[e_t - i_3, (e_t - i_3)\vec{x}, 1 + i_{st}, (1 + i_{st})\vec{y};\ \forall \vec{x}, \vec{y} \in \mathbb{R}^3], \\
Cl(3,1)_- &= \mathrm{span}[e_t + i_3, (e_t + i_3)\vec{x}, 1 - i_{st}, (1 - i_{st})\vec{y};\ \forall \vec{x}, \vec{y} \in \mathbb{R}^3].
\end{aligned} \tag{2.11}$$

The following *identities* hold for $m_\pm \in Cl(3,1)_\pm$,

$$e^{\alpha \boldsymbol{e}_t} m_\pm e^{\beta i_3} = m_\pm e^{(\beta \mp \alpha) i_3} = e^{(\alpha \mp \beta)\boldsymbol{e}_t} m_\pm. \tag{2.12}$$

Particularly useful cases of (2.12) are $(\alpha, \beta) = (\pi/2, 0)$ and $(0, \pi/2)$:

$$e_t m_\pm = \mp m_\pm i_3, \qquad m_\pm i_3 = \mp e_t m_\pm. \tag{2.13}$$

Because of (2.3), we have for the product of exponentials

$$\begin{aligned}
e^{\alpha \boldsymbol{e}_t} e^{\beta i_3} &= e^{\beta i_3} e^{\alpha \boldsymbol{e}_t} + [\boldsymbol{e}_t, i_3] \sin(\alpha) \sin(\beta) \\
&= e^{\beta i_3} e^{\alpha \boldsymbol{e}_t} + 2 i_{st} \sin(\alpha) \sin(\beta),
\end{aligned} \tag{2.14}$$

which (in the same form for general multivector square roots of -1) has been used in [22] in order to derive a general convolution theorem for Clifford Fourier transformations. Moreover, note that

$$\boldsymbol{e}_t[\boldsymbol{e}_t, i_3] = 2(\boldsymbol{e}_t)^2 i_3 = -2 i_3. \tag{2.15}$$

We furthermore note the useful anticommutation relationships

$$\boldsymbol{e}_t[\boldsymbol{e}_t, i_3] = -[\boldsymbol{e}_t, i_3]\boldsymbol{e}_t, \qquad i_3[\boldsymbol{e}_t, i_3] = -[\boldsymbol{e}_t, i_3]i_3, \tag{2.16}$$

and therefore

$$e^{\alpha \boldsymbol{e}_t}[\boldsymbol{e}_t, i_3] = [\boldsymbol{e}_t, i_3]e^{-\alpha \boldsymbol{e}_t}, \qquad e^{\beta i_3}[\boldsymbol{e}_t, i_3] = [\boldsymbol{e}_t, i_3]e^{-\beta i_3}. \tag{2.17}$$

And because of the fundamental anticommutation

$$\boldsymbol{e}_t i_3 = -i_3 \boldsymbol{e}_t \;\Rightarrow\; e^{\alpha \boldsymbol{e}_t} i_3 = i_3 e^{-\alpha \boldsymbol{e}_t}. \tag{2.18}$$

3 The steerable space-time Fourier transform (SFT)

The steerable space-time Fourier transform maps 16-dimensional space-time algebra functions $h : \mathbb{R}^{3,1} \to Cl_{3,1}$ to 16-dimensional space-time spectrum functions $\mathcal{F}\{h\} : \mathbb{R}^{3,1} \to Cl_{3,1}$. It is *defined* for $h \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$ in the following way[5]

$$h \to \mathcal{F}^{\boldsymbol{e}_t, i_3}\{h\}(\boldsymbol{\omega}) = \mathcal{F}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t\, t\omega_t} h(\boldsymbol{x})\, e^{-i_3 \vec{x}\cdot\vec{\omega}} d^4\boldsymbol{x}\,, \tag{3.1}$$

with

- space-time vectors $\boldsymbol{x} = t\boldsymbol{e}_t + \vec{x} \in \mathbb{R}^{3,1}$, $\vec{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 \in \mathbb{R}^3$
- space-time volume $d^4\boldsymbol{x} = dtdxdydz$
- space-time frequency vectors $\boldsymbol{\omega} = \omega_t \boldsymbol{e}_t + \vec{\omega} \in \mathbb{R}^{3,1}$, $\vec{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3 \in \mathbb{R}^3$

Note, that we usually omit the upper indexes showing the special square roots of -1 selected for the transform, as in $\mathcal{F}^{\boldsymbol{e}_t, i_3}\{h\} = \mathcal{F}\{h\}$.

Remark 3.1. *The above SFT is a steerable operator*[6] *depending on the choice of unit time direction* $\boldsymbol{e}_t$ *in the forward light cone of* $\mathbb{R}^{3,1}$. *In the case of a local inertial frame of reference, the vector* $\boldsymbol{e}_t$ *in* $\mathbb{R}^{3,1}$ *specifies the velocity of the observer.*

Remark 3.2. *The three-dimensional integration part*

$$\int h(\boldsymbol{x})\, e^{-i_3 \vec{x}\cdot\vec{\omega}} d^3\vec{x}$$

in (3.1) *fully corresponds to the Clifford algebra Fourier transform (CFT) in* $Cl(3,0)$, *compare [15, 16]. For this CFT in* $Cl(3,0)$ *fast Clifford Fourier transform algorithms exist, based on the split of the eight-dimensional algebra* $Cl(3,0)$ *(see e.g. Section 3.2 of [15]) into a quadruple of conventional fast Fourier transforms.*

[5] Alternatively one can assume the signal functions h to be Schwartz functions. For square integrable functions the integral in (3.1) may not converge absolutely, then a definition in terms of a L^2 norm density argument may become necessary.

[6] Note that the *steerability* of the closely related general two-sided QFT has been discussed at length in [18, 20].

For $h \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$, and assuming that in any finite interval h and the partial coordinate derivatives of h are piecewise continuous, and have at most a finite number of extrema and discontinuities, h being continuous at $\boldsymbol{x} \in \mathbb{R}^{3,1}$, and assuming that $\mathcal{F}\{h\} \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$, then the *inverse* $\mathcal{F}^{-1}$ of the SFT (3.1) is given by

$$h(\boldsymbol{x}) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{3,1}} e^{\boldsymbol{e}_t\, t\omega_t}\, \mathcal{F}\{h\}(\boldsymbol{\omega})\; e^{i_3 \vec{x}\cdot\vec{\omega}} d^4\boldsymbol{\omega}\ . \tag{3.2}$$

The $\pm$ split of the QFT can now, via the isomorphism (2.4) of quaternions to the volume-time subalgebra of the space-time algebra, be *extended* to splitting general space-time algebra multivector functions over $\mathbb{R}^{3,1}$. This leads to the following interesting result [14],

$$\begin{aligned}
\mathcal{F}\{h\} &= \mathcal{F}\{h\}_+ + \mathcal{F}\{h\}_- \\
&= \int_{\mathbb{R}^{3,1}} h_+\, e^{-i_3(\,\vec{x}\cdot\vec{\omega} -\, t\omega_t\,)} d^4\boldsymbol{x} + \int_{\mathbb{R}^{3,1}} h_-\, e^{-i_3(\,\vec{x}\cdot\vec{\omega} +\, t\omega_t\,)} d^4\boldsymbol{x} \\
&= \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t(\,t\omega_t - \vec{x}\cdot\vec{\omega}\,)} h_+\, d^4\boldsymbol{x} + \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t(\,t\omega_t + \vec{x}\cdot\vec{\omega}\,)} h_-\, d^4\boldsymbol{x}\,.
\end{aligned} \tag{3.3}$$

This result shows us that the SFT is identical to a sum of right and left propagating multivector wave packets. We therefore see that these physically important wave packets arise absolutely naturally from elementary purely algebraic considerations.

Remark 3.3. *Equation* (3.3) *shows best how to compute a SFT. It is simply the consecutive computation of a three-dimensional CFT of Remark 3.2, followed by a standard one-dimensional time-frequency Fourier transform on the resulting components. For the* h_+ *part in* (3.3) *this can, e.g., be written as*

$$\begin{aligned}
\mathcal{F}\{h\}_+ &= \int_{\mathbb{R}^{3,1}} h_+\, e^{-i_3(\,\vec{x}\cdot\vec{\omega} -\, t\omega_t\,)} d^4\boldsymbol{x} \\
&= \int_{-\infty}^{\infty} \left\{ \int_{\mathbb{R}^3} h_+(t,\vec{x}) e^{-i_3\,\vec{x}\cdot\vec{\omega}} d^3\vec{x} \right\} e^{i_3\, t\omega_t} dt,
\end{aligned} \tag{3.4}$$

and similarly for the h_- *part.*

We further define for later use the following two mixed *exponential-sine* Fourier transforms

$$\mathcal{F}^{\boldsymbol{e}_t,\pm s}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t t\omega_t} h(\boldsymbol{x})(\pm 1)\sin(-\vec{x}\cdot\vec{\omega}\,) d^4\boldsymbol{x}, \tag{3.5}$$

$$\mathcal{F}^{\pm s, i_3}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3,1}} (\pm 1)\sin(-t\omega_t) h(\boldsymbol{x}) e^{-i_3\vec{x}\cdot\vec{\omega}}\, d^4\boldsymbol{x}. \tag{3.6}$$

With the help of

$$\begin{aligned}
\sin(-t\omega_t) &= \frac{\boldsymbol{e}_t}{2}(e^{-\boldsymbol{e}_t t\omega_t} - e^{\boldsymbol{e}_t t\omega_t}), \\
\sin(-\vec{x}\cdot\vec{\omega}) &= \frac{i_3}{2}(e^{-i_3\vec{x}\cdot\vec{\omega}} - e^{i_3\vec{x}\cdot\vec{\omega}}),
\end{aligned} \tag{3.7}$$

we can rewrite the above mixed exponential-sine Fourier transforms in terms of the SFT of (3.1) as

$$\mathcal{F}^{\boldsymbol{e}_t,\pm s}\{h\} = \pm\tfrac{1}{2}(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{hi_3\} - \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{hi_3\}), \tag{3.8}$$

$$\mathcal{F}^{\pm s,i_3}\{h\} = \pm\tfrac{1}{2}(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{\boldsymbol{e}_t h\} - \mathcal{F}^{-\boldsymbol{e}_t,i_3}\{\boldsymbol{e}_t h\}). \tag{3.9}$$

We further note the following useful relationships using the argument reflection of Notation 2.2

$$\mathcal{F}^{-\boldsymbol{e}_t,i_3}\{h\} = \mathcal{F}^{\boldsymbol{e}_t,i_3}\{h^{(1,0)}\} = \mathcal{F}\{h^{(1,0)}\}, \quad \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{h\} = \mathcal{F}\{h^{(0,1)}\}, \tag{3.10}$$

and similarly

$$\mathcal{F}^{\boldsymbol{e}_t,-s}\{h\} = \mathcal{F}^{\boldsymbol{e}_t,s}\{h^{(0,1)}\}, \quad \mathcal{F}^{-s,i_3}\{h\} = \mathcal{F}^{s,i_3}\{h^{(1,0)}\}. \tag{3.11}$$

4 Convolution and Mustard convolution

We define the *convolution* of two space-time signals $a, b \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^{3,1}} a(\boldsymbol{y})b(\boldsymbol{x}-\boldsymbol{y})d^4\boldsymbol{y}, \tag{4.1}$$

provided that the integral exists[7].

The *Mustard* convolution [28] of two space-time signals $a, b \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$ is defined as

$$(a \star_M b)(\boldsymbol{x}) = \mathcal{F}^{-1}(\mathcal{F}\{a\}\mathcal{F}\{b\}). \tag{4.2}$$

provided that the integral exists.

Remark 4.1. *The Mustard convolution has the conceptual and computational advantage to simply yield as spectrum in the SFT Fourier domain the point wise product of the SFTs of the two signals, just as for the classical complex Fourier transform. On the other hand, by its very definition, the Mustard convolution depends on the choice of the pair $\boldsymbol{e}_t, i_3$, of square roots of -1 used in the definition* (3.1) *of the SFT. The Mustard convolution* (4.2) *is therefore a steerable operator, depending on the choice of unit time direction $\boldsymbol{e}_t$ in the space-time forward light cone of $\mathbb{R}^{3,1}$. This may be of advantage in applications to special relativistic physics, electromagnetic signal processing, optics, and aero-space navigation.*

We additionally define a further type of (steerable) *exponential-sine* Mustard convolution as

$$(a \star_{Ms} b)(\boldsymbol{x}) = \mathcal{F}^{-1}(\mathcal{F}^{\boldsymbol{e}_t,s}\{a\}\mathcal{F}^{s,i_3}\{b\}). \tag{4.3}$$

In the following two Subsections we will first express the convolution (4.1) in terms of the Mustard convolution (4.2) and vice versa.

[7] The integrals in (4.1), (4.2) and (4.3) exist, e.g. for compactly supported functions, or for functions $a, b \in L^1(\mathbb{R}^{3,1}, Cl_{3,1})$, etc.

4.1 Expressing the convolution in terms of the Mustard convolution

In [4] Theorem 4.1 on page 584 expresses the classical convolution of two quaternion functions with the help of the general left-sided QFT as a sum of 40 Mustard convolutions. Similar results have been established for the general two-sided QFT in [24]. Based on the isomorphism (2.4) and on the splits of (2.5) and of Lemma 2.1, we generalize these results now to the SFT. Moreover, we use Theorem 5.12 on page 327 of [22], which expresses the convolution of two Clifford signal functions (higher dimensional generalizations of quaternion or space-time functions) in the Clifford Fourier domain with the help of the general two-sided Clifford Fourier transform (CFT), the latter is in turn a generalization of the QFT and SFT to general Clifford algebras with non-degenerate quadratic forms. We restate this theorem here again, specialized for space-time functions and the SFT of (3.1).

Theorem 4.2 (SFT of convolution)**.** *The SFT of the convolution* (4.1) *of two functions* $a, b \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$ *can be expressed as*

$$
\begin{aligned}
&\mathcal{F}\{a \star b\} = \\
&\quad \mathcal{F}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}\{b_{+i_3}\} + \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}^{\boldsymbol{e}_t,i_3}\{b_{-i_3}\} \\
&+ \mathcal{F}^{\boldsymbol{e}_t,i_3}\{a_{-\boldsymbol{e}_t}\}\mathcal{F}^{-\boldsymbol{e}_t,i_3}\{b_{+i_3}\} + \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{a_{-\boldsymbol{e}_t}\}\mathcal{F}^{-\boldsymbol{e}_t,i_3}\{b_{-i_3}\} \\
&+ 2\mathcal{F}^{\boldsymbol{e}_t,s}\{a_{+\boldsymbol{e}_t}\}i_{st}\mathcal{F}^{s,i_3}\{b_{+i_3}\} + 2\mathcal{F}^{\boldsymbol{e}_t,-s}\{a_{+\boldsymbol{e}_t}\}i_{st}\mathcal{F}^{s,i_3}\{b_{-i_3}\} \\
&+ 2\mathcal{F}^{\boldsymbol{e}_t,s}\{a_{-\boldsymbol{e}_t}\}i_{st}\mathcal{F}^{-s,i_3}\{b_{+i_3}\} + 2\mathcal{F}^{\boldsymbol{e}_t,-s}\{a_{-\boldsymbol{e}_t}\}i_{st}\mathcal{F}^{-s,i_3}\{b_{-i_3}\}.
\end{aligned} \tag{4.4}
$$

Note that due to the commutation properties of (3.5) and (3.6) we can place the pseudosalar i_{st} also inside the exponential-sine transform terms as e.g. in

$$
\begin{aligned}
\mathcal{F}^{\boldsymbol{e}_t,s}\{a_{+\boldsymbol{e}_t}\}i_{st}\mathcal{F}^{s,i_3}\{b_{+i_3}\} &= \mathcal{F}^{\boldsymbol{e}_t,s}\{a_{+\boldsymbol{e}_t}i_{st}\}\mathcal{F}^{s,i_3}\{b_{+i_3}\} \\
&= \mathcal{F}^{\boldsymbol{e}_t,s}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}^{s,i_3}\{i_{st}b_{+i_3}\}.
\end{aligned} \tag{4.5}
$$

By applying the inverse SFT to (4.4), we can now easily express the convolution of two space-time signals $a \star b$ in terms of only eight Mustard convolutions (4.2) and (4.3).

Theorem 4.3 (Convolution in terms of two types of Mustard convolution)**.** *The convolution* (4.1) *of two space-time functions* $a, b \in L^1(\mathbb{R}^{3,1}; Cl_{3,1})$ *can be expressed in terms of four Mustard convolutions* (4.2) *and four exponential-sine Mustard convolutions* (4.3) *as*

$$
\begin{aligned}
a \star b = {}& a_{+\boldsymbol{e}_t} \star_M b_{+i_3} + a^{(0,1)}_{+\boldsymbol{e}_t} \star_M b_{-i_3} + a_{-\boldsymbol{e}_t} \star_M b^{(1,0)}_{+i_3} + a^{(0,1)}_{-\boldsymbol{e}_t} \star_M b^{(1,0)}_{-i_3} \\
&+ 2a_{+\boldsymbol{e}_t} \star_{Ms} i_{st}b_{+i_3} + 2a^{(0,1)}_{+\boldsymbol{e}_t} \star_{Ms} i_{st}b_{-i_3} \\
&+ 2a_{-\boldsymbol{e}_t} \star_{Ms} i_{st}b^{(1,0)}_{+i_3} + 2a^{(0,1)}_{-\boldsymbol{e}_t} \star_{Ms} i_{st}b^{(1,0)}_{-i_3}.
\end{aligned} \tag{4.6}
$$

Remark 4.4. *We use the convention, that terms such as* $a_{+\boldsymbol{e}_t} \star_{Ms} i_{st}b_{+i_3}$*, should be understood with brackets* $a_{+\boldsymbol{e}_t} \star_{Ms} (i_{st}b_{+i_3})$*, which are omitted to avoid clutter.*

Furthermore, applying (3.8) and (3.9), we can expand the terms in (4.4) with exponential-sine transforms into sums of products of SFTs. For example, the first term gives, using $\boldsymbol{e}_t i_{st} = -i_3$,

$$
\begin{aligned}
&\mathcal{F}^{\boldsymbol{e}_t,s}\{a_{+\boldsymbol{e}_t}\} i_{st} \mathcal{F}^{s,i_3}\{b_{+i_3}\} \\
&= \tfrac{1}{4}\big(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{a_{+\boldsymbol{e}_t} i_3\} - \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{a_{+\boldsymbol{e}_t} i_3\}\big)\big(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{\boldsymbol{e}_t i_{st} b_{+i_3}\} - \mathcal{F}^{-\boldsymbol{e}_t,i_3}\{\boldsymbol{e}_t i_{st} b_{+i_3}\}\big) \\
&= \tfrac{1}{4}\big(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{a_{+\boldsymbol{e}_t} i_3\} - \mathcal{F}^{\boldsymbol{e}_t,-i_3}\{a_{+\boldsymbol{e}_t} i_3\}\big)\big(\mathcal{F}^{\boldsymbol{e}_t,i_3}\{-i_3 b_{+i_3}\} - \mathcal{F}^{-\boldsymbol{e}_t,i_3}\{-i_3 b_{+i_3}\}\big) \\
&= \tfrac{1}{4}\Big(\mathcal{F}\{a_{+\boldsymbol{e}_t} i_3\}\mathcal{F}\{-i_3 b_{+i_3}\} - \mathcal{F}\{a_{+\boldsymbol{e}_t} i_3\}\mathcal{F}\{-i_3 b_{+i_3}^{(1,0)}\} \\
&\qquad -\mathcal{F}\{a_{+\boldsymbol{e}_t}^{(0,1)} i_3\}\mathcal{F}\{-i_3 b_{+i_3}\} + \mathcal{F}\{a_{+\boldsymbol{e}_t}^{(0,1)} i_3\}\mathcal{F}\{-i_3 b_{+i_3}^{(1,0)}\}\Big) \\
&= \tfrac{1}{4}\Big(\mathcal{F}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}\{b_{+i_3}^{(1,0)}\} - \mathcal{F}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}\{b_{+i_3}\} \\
&\qquad -\mathcal{F}\{a_{+\boldsymbol{e}_t}^{(0,1)}\}\mathcal{F}\{b_{+i_3}^{(1,0)}\} + \mathcal{F}\{a_{+\boldsymbol{e}_t}^{(0,1)}\}\mathcal{F}\{b_{+i_3}\}\Big)\,, \qquad (4.7)
\end{aligned}
$$

because

$$
\mathcal{F}\{a_{+\boldsymbol{e}_t} i_3\}\mathcal{F}\{-i_3 b_{+i_3}\} = \mathcal{F}\{a_{+\boldsymbol{e}_t}\} i_3(-i_3)\mathcal{F}\{b_{+i_3}^{(1,0)}\} = \mathcal{F}\{a_{+\boldsymbol{e}_t}\}\mathcal{F}\{b_{+i_3}^{(1,0)}\}, \quad \text{etc.} \qquad (4.8)
$$

where we applied (2.18) for the first equality.

By taking the inverse SFT of (4.7) we obtain an identity for expressing a mixed exponential-sine Mustard convolution (4.3) in terms of four standard Mustard convolutions (4.2),

$$
\begin{aligned}
&a_{+\boldsymbol{e}_t} \star_{Ms} i_{st} b_{+i_3} \qquad (4.9)\\
&= a_{+\boldsymbol{e}_t} \star_M b_{+i_3}^{(1,0)} - a_{+\boldsymbol{e}_t} \star_M b_{+i_3} - a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3}^{(1,0)} + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3}.
\end{aligned}
$$

This now allows us in turn to express the space-time signal convolution purely in terms of standard Mustard convolutions,

$$
\begin{aligned}
&a \star b = \qquad (4.10)\\
&\tfrac{1}{2}\big(a_{+\boldsymbol{e}_t} \star_M b_{+i_3}^{(1,0)} + a_{+\boldsymbol{e}_t} \star_M b_{+i_3} - a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3}^{(1,0)} + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3} \\
&\ + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3}^{(1,0)} + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3} - a_{+\boldsymbol{e}_t} \star_M b_{-i_3}^{(1,0)} + a_{+\boldsymbol{e}_t} \star_M b_{-i_3} \\
&\ + a_{-\boldsymbol{e}_t} \star_M b_{+i_3} + a_{-\boldsymbol{e}_t} \star_M b_{+i_3}^{(1,0)} - a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3} + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3}^{(1,0)} \\
&\ + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3} + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3}^{(1,0)} - a_{-\boldsymbol{e}_t} \star_M b_{-i_3} + a_{-\boldsymbol{e}_t} \star_M b_{-i_3}^{(1,0)} \big)\,.
\end{aligned}
$$

Furthermore, we can combine four pairs of space-time split terms, e.g.,

$$
a_{+\boldsymbol{e}_t} \star_M b_{+i_3} + a_{+\boldsymbol{e}_t} \star_M b_{-i_3} = a_{+\boldsymbol{e}_t} \star_M b, \ \text{ etc.} \qquad (4.11)
$$

This leaves only twelve terms for expressing a classical convolution in terms of a Mustard convolution,

$$
\begin{aligned}
&a \star b = \\
&\tfrac{1}{2}\big(a_{+\boldsymbol{e}_t} \star_M b_{+i_3}^{(1,0)} + a_{+\boldsymbol{e}_t} \star_M b - a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3}^{(1,0)} + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b \\
&\ + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3}^{(1,0)} - a_{+\boldsymbol{e}_t} \star_M b_{-i_3}^{(1,0)} \\
&\ + a_{-\boldsymbol{e}_t} \star_M b_{+i_3} + a_{-\boldsymbol{e}_t} \star_M b^{(1,0)} - a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{+i_3} + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b^{(1,0)} \\
&\ + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b_{-i_3} - a_{-\boldsymbol{e}_t} \star_M b_{-i_3} \big)\,. \qquad (4.12)
\end{aligned}
$$

Moreover, we can combine with the help of the involution $\boldsymbol{e}_t()i_3$ of (2.7) four pairs of terms like

$$\begin{aligned} a_{+\boldsymbol{e}_t} \star_M b_{+i_3}^{(1,0)} - a_{+\boldsymbol{e}_t} \star_M b_{-i_3}^{(1,0)} &= a_{+\boldsymbol{e}_t} \star_M [b_{+i_3}^{(1,0)} - b_{-i_3}^{(1,0)}] \\ = a_{+\boldsymbol{e}_t} \star_M (\boldsymbol{e}_t[b_{+i_3}^{(1,0)} + b_{-i_3}^{(1,0)}]i_3) &= a_{+\boldsymbol{e}_t} \star_M \boldsymbol{e}_t b^{(1,0)} i_3, \end{aligned} \tag{4.13}$$

where in the final result we omit the round brackets, i.e. we understand $a_{+\boldsymbol{e}_t} \star_M \boldsymbol{e}_t b^{(1,0)} g = a_{+\boldsymbol{e}_t} \star_M (\boldsymbol{e}_t b^{(1,0)} i_3)$. This in turn leaves only eight terms for expressing a classical convolution in terms of Mustard convolutions,

$$\begin{aligned} a \star b = \tfrac{1}{2}(\, & a_{+\boldsymbol{e}_t} \star_M \boldsymbol{e}_t b^{(1,0)} i_3 + a_{+\boldsymbol{e}_t} \star_M b - a_{+\boldsymbol{e}_t}^{(0,1)} \star_M \boldsymbol{e}_t b^{(1,0)} i_3 + a_{+\boldsymbol{e}_t}^{(0,1)} \star_M b \\ & + a_{-\boldsymbol{e}_t} \star_M \boldsymbol{e}_t b i_3 + a_{-\boldsymbol{e}_t} \star_M b^{(1,0)} - a_{-\boldsymbol{e}_t}^{(0,1)} \star_M \boldsymbol{e}_t b i_3 + a_{-\boldsymbol{e}_t}^{(0,1)} \star_M b^{(1,0)}\,). \end{aligned} \tag{4.14}$$

Finally, we note, that (4.14) contains pairs of functions $a_{\pm\boldsymbol{e}_t}$ with unreflected and reflected second three-dimensional space vector argument. Adding these pairs leads to even $\oplus$ or odd $\ominus$ symmetry in the second three-dimensional space vector argument. That is, we combine

$$a_{+\boldsymbol{e}_t}^{\oplus} = \tfrac{1}{2}(a_{+\boldsymbol{e}_t} + a_{+\boldsymbol{e}_t}^{(0,1)}), \qquad a_{+\boldsymbol{e}_t}^{\ominus} = \tfrac{1}{2}(a_{+\boldsymbol{e}_t} - a_{+\boldsymbol{e}_t}^{(0,1)}). \tag{4.15}$$

Remembering the Notation 2.2, the space-time function $a_{+\boldsymbol{e}_t}^{\oplus}$, mapping $\mathbb{R}^{3,1} \to Cl(3,1)$, is therefore symmetrized in its three-dimensional space vector argument $\vec{x}$, whereas $a_{+\boldsymbol{e}_t}^{\ominus}$ is antisymmetrized in its three-dimensional space vector argument $\vec{x}$.

This finally allows us to write the classical convolution in terms of just four Mustard convolutions.

Theorem 4.5 (Convolution in terms of Mustard convolution)**.** *The convolution* (4.1) *of two space-time functions $a, b \in L^1(R^{3,1}; Cl(3,1))$ can be expressed in terms of four standard Mustard convolutions* (4.2) *as*

$$a \star b = \tfrac{1}{2}(\, a_{+\boldsymbol{e}_t}^{\ominus} \star_M \boldsymbol{e}_t b^{(1,0)} i_3 + a_{+\boldsymbol{e}_t}^{\oplus} \star_M b + a_{-\boldsymbol{e}_t}^{\ominus} \star_M \boldsymbol{e}_t b i_3 + a_{-\boldsymbol{e}_t}^{\oplus} \star_M b^{(1,0)}\,). \tag{4.16}$$

4.2 Expressing the Mustard convolution in terms of the convolution

Now we will simply write out the Mustard convolution (4.2) and simplify it until only standard convolutions (4.1) remain. In this Subsection we will use the general space-time split of equation (2.5).

We begin by writing the Mustard convolution (4.2) of two space-time functions $a, b \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, with space-time vector arguments $\boldsymbol{x} = t\boldsymbol{e}_t + \vec{x}$, $\boldsymbol{y} = t'\boldsymbol{e}_t + \vec{y}$, and $\boldsymbol{z} = t''\boldsymbol{e}_t + \vec{z}$, all in $\mathbb{R}^{3,1}$,

$$\begin{aligned} a \star_M b(\boldsymbol{x}) &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{3,1}} e^{\boldsymbol{e}_t t_1 \omega_t} \mathcal{F}\{a\}(\boldsymbol{\omega}) \mathcal{F}\{b\}(\boldsymbol{\omega}) e^{i_3 \vec{x}\cdot\vec{\omega}} d^4\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{3,1}} e^{\boldsymbol{e}_t t_1 \omega_t} \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t t_1' \omega_t} a(\boldsymbol{y}) e^{-i_3 \vec{y}\cdot\vec{\omega}} d^4\boldsymbol{y} \\ &\qquad \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t t_1'' \omega_t} b(\boldsymbol{z}) e^{-i_3 \vec{z}\cdot\vec{\omega}} d^4\boldsymbol{z} e^{i_3 \vec{x}\cdot\vec{\omega}} d^4\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{3,1}} \int_{\mathbb{R}^{3,1}} \int_{\mathbb{R}^{3,1}} e^{\boldsymbol{e}_t (t_1 - t_1')\omega_t} (a_+(\boldsymbol{y}) + a_-(\boldsymbol{y})) e^{-i_3 \vec{y}\cdot\vec{\omega}} \\ &\qquad e^{-\boldsymbol{e}_t t_1'' \omega_t} (b_+(\boldsymbol{z}) + b_-(\boldsymbol{z})) e^{i_3 (\vec{x}-\vec{z})\cdot\vec{\omega}} d^4\boldsymbol{y} d^4\boldsymbol{z} d^4\boldsymbol{\omega}. \end{aligned} \tag{4.17}$$

Next, we use the identities (2.12) in order to shift the inner factor $e^{-i_3\vec{y}\cdot\vec{\omega}}$ to the left and $e^{-\boldsymbol{e}_t t_1''\omega_t}$ to the right, respectively. We abbreviate $\int_{\mathbb{R}^{3,1}}\int_{\mathbb{R}^{3,1}}\int_{\mathbb{R}^{3,1}}$ to $\iiint$.

$$
\begin{aligned}
&a \star_M b(\boldsymbol{x}) = \qquad (4.18)\\
&= \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1')\omega_t}e^{\boldsymbol{e}_t\vec{y}\cdot\vec{\omega}}a_+(\boldsymbol{y})b_+(\boldsymbol{z})e^{i_3 t_1''\omega_t}e^{i_3(\vec{x}-\vec{z})\cdot\vec{\omega}}d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1')\omega_t}e^{\boldsymbol{e}_t\vec{y}\cdot\vec{\omega}}a_+(\boldsymbol{y})b_-(\boldsymbol{z})e^{-i_3 t_1''\omega_t}e^{i_3(\vec{x}-\vec{z})\cdot\vec{\omega}}d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1')\omega_t}e^{-\boldsymbol{e}_t\vec{y}\cdot\vec{\omega}}a_-(\boldsymbol{y})b_+(\boldsymbol{z})e^{i_3 t_1''\omega_t}e^{i_3(\vec{x}-\vec{z})\cdot\vec{\omega}}d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1')\omega_t}e^{-\boldsymbol{e}_t\vec{y}\cdot\vec{\omega}}a_-(\boldsymbol{y})b_-(\boldsymbol{z})e^{-i_3 t_1''\omega_t}e^{i_3(\vec{x}-\vec{z})\cdot\vec{\omega}}d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}.
\end{aligned}
$$

Furthermore, we abbreviate the inner function products as $ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z}) := a_\pm(\boldsymbol{y})b_\pm(\boldsymbol{z})$, and apply the space-time split of equation (2.5) once again to obtain $ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z}) = [ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z})]_+ + [ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z})]_- = ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z})_+ + ab_{\pm\pm}(\boldsymbol{y},\boldsymbol{z})_-$. We omit the square brackets and use the convention that the final space-time split indicated by the final $\pm$ index should be performed last. This allows to further apply (2.12) again in order to shift the factors $e^{\pm i_3 t_1''\omega_t}\; e^{i_3(\vec{x}-\vec{z})\cdot\vec{\omega}}$ to the left. We end up with the following eight terms

$$
\begin{aligned}
&a \star_M b(\boldsymbol{x}) = \qquad (4.19)\\
&= \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'-t_1'')\omega_t}e^{\boldsymbol{e}_t(\vec{y}-(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{++}(\boldsymbol{y},\boldsymbol{z})_+d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}e^{\boldsymbol{e}_t(\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{++}(\boldsymbol{y},\boldsymbol{z})_-d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}e^{\boldsymbol{e}_t(\vec{y}-(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{+-}(\boldsymbol{y},\boldsymbol{z})_+d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'-t_1'')\omega_t}e^{\boldsymbol{e}_t(\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{+-}(\boldsymbol{y},\boldsymbol{z})_-d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'-t_1'')\omega_t}e^{\boldsymbol{e}_t(-\vec{y}-(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{-+}(\boldsymbol{y},\boldsymbol{z})_+d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}e^{\boldsymbol{e}_t(-\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{-+}(\boldsymbol{y},\boldsymbol{z})_-d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}e^{\boldsymbol{e}_t(-\vec{y}-(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{--}(\boldsymbol{y},\boldsymbol{z})_+d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'-t_1'')\omega_t}e^{\boldsymbol{e}_t(-\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}ab_{--}(\boldsymbol{y},\boldsymbol{z})_-d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}.
\end{aligned}
$$

We now only show explicitly how to simplify the second triple integral, the others follow the same pattern.

$$
\begin{aligned}
&\tfrac{1}{(2\pi)^4}\iiint e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}e^{\boldsymbol{e}_t(\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}[a_+(\boldsymbol{y})b_+(\boldsymbol{z})]_-d^4\boldsymbol{y}d^4\boldsymbol{z}d^4\boldsymbol{\omega}\\
&=\tfrac{1}{(2\pi)^4}\iint\int_{\mathbb{R}}e^{\boldsymbol{e}_t(t_1-t_1'+t_1'')\omega_t}d\omega_t\int_{\mathbb{R}^3}e^{\boldsymbol{e}_t(\vec{y}+(\vec{x}-\vec{z}))\cdot\vec{\omega}}d\vec{\omega}[a_+(\boldsymbol{y})b_+(\boldsymbol{z})]_-d^4\boldsymbol{y}d^4\boldsymbol{z}\\
&=\iint\delta(t_1-t_1'+t_1'')\delta(\vec{y}+(\vec{x}-\vec{z}))[a_+(\boldsymbol{y})b_+(t_1'',\vec{z})]_-d^4\boldsymbol{y}d^4\boldsymbol{z}\\
&=\int_{\mathbb{R}^2}[a_+(\boldsymbol{y})b_+(-(t_1-t_1'),\vec{x}+\vec{y})]_-d^4\boldsymbol{y}\\
&=\int_{\mathbb{R}^2}[a_+(\boldsymbol{y})b_+(-(t_1-t_1'),-(-\vec{x}-\vec{y}))]_-d^4\boldsymbol{y}\\
&=\int_{\mathbb{R}^2}[a_+(\boldsymbol{y})b_+^{(1,1)}(t_1-t_1',-\vec{x}-\vec{y})]_-d^4\boldsymbol{y}\\
&=[a_+\star b_+^{(1,1)}(t_1,-\vec{x})]_-.
\end{aligned}
\tag{4.20}
$$

Note that $a_+\star b_+^{(1,1)}(t_1,-\vec{x})$ means to first apply the convolution to the pair of functions a_+ and $b_+^{(1,1)}$, and only then to evaluate them with the argument $(t_1,-\vec{x})$. So in general $a_+\star b_+^{(1,1)}(t_1,-\vec{x})\neq a_+\star b_+(-t_1,\vec{x})$. Simplifying the other seven triple integrals similarly we finally obtain the desired decomposition of the Mustard convolution (4.2) in terms of the classical convolution.

Theorem 4.6 (Mustard convolution in terms of standard convolution)**.** *The Mustard convolution* (4.2) *of two space-time functions* $a,b\in L^1(R^{3,1};Cl(3,1))$ *can be expressed in terms of eight standard convolutions* (4.1) *as*

$$
\begin{aligned}
&a\star_M b(\boldsymbol{x})=\\
&=[a_+\star b_+(\boldsymbol{x})]_+ + [a_+\star b_+^{(1,1)}(t_1,-\vec{x})]_-\\
&\quad+[a_+\star b_-^{(1,0)}(\boldsymbol{x})]_+ + [a_+\star b_-^{(0,1)}(t_1,-\vec{x})]_-\\
&\quad+[a_-\star b_+^{(0,1)}(t_1,-\vec{x})]_+ + [a_-\star b_+^{(1,0)}(\boldsymbol{x})]_-\\
&\quad+[a_-\star b_-^{(1,1)}(t_1,-\vec{x})]_+ + [a_-\star b_-(\boldsymbol{x})]_-.
\end{aligned}
\tag{4.21}
$$

Remark 4.7. *If we would explicitly insert according to* (2.12) $a_\pm=\frac{1}{2}(a\pm\boldsymbol{e}_t a i_3)$ *and* $b_\pm=\frac{1}{2}(b\pm\boldsymbol{e}_t b i_3)$*, and similarly explicitly insert the second level space-time split* $[\ldots]_\pm$*, we would obtain up to a maximum of 64 terms. It is therefore obvious how significant and efficient the use of the space-time split* (2.5) *is in this context.*

Remark 4.8. *The computation of a typical term in Theorem 4.6 can be illustrated by writing out, e.g., the third term in full detail*

$$
\begin{aligned}
&[a_+\star b_-^{(1,0)}(\boldsymbol{x})]_+\\
&=\frac{1}{2}[\int_{\mathbb{R}^{3,1}}\tfrac{1}{2}(a(\boldsymbol{y})+e_t a(\boldsymbol{y})i_3)\\
&\qquad\tfrac{1}{2}(b(-(t_x-t_y),\vec{x}-\vec{y})-e_t b(-(t_x-t_y),\vec{x}-\vec{y})i_3)d^4\boldsymbol{y}\\
&\quad+e_t\int_{\mathbb{R}^{3,1}}\tfrac{1}{2}(a(\boldsymbol{y})+e_t a(\boldsymbol{y})i_3)\\
&\qquad\tfrac{1}{2}(b(-(t_x-t_y),\vec{x}-\vec{y})-e_t b(-(t_x-t_y),\vec{x}-\vec{y})i_3)d^4\boldsymbol{y}\ i_3].
\end{aligned}
\tag{4.22}
$$

Remark 4.9. *The steerability (compare Remark 3.1) of the Mustard convolution* (4.2) *is seen in Theorem 4.6 in the explicit occurrence of the algebra of space-time split* (2.5)*.*

5 Conclusion

We have introduced the Clifford algebra $Cl(3,1)$ for space-time $\mathbb{R}^{3,1}$ together with the space-time split, based on the time vector $\boldsymbol{e}_t$ and its dual three-dimensional space volume pseudoscalar $\boldsymbol{e}_t^* = i_3$. In this context we looked in detail at a number of involutions in $Cl(3,1)$ connected with $\boldsymbol{e}_t$, i_3 and their product, the space-time hypervolume pseudoscalar $i_{st} = \boldsymbol{e}_t i_3$. Next, we briefly reviewed for space-time Clifford algebra $Cl(3,1)$ valued signals over $\mathbb{R}^{3,1}$ the steerable space-time Fourier transform, and defined a pair of related exponential-sine type Fourier transforms. This was followed by definitions of the (classical) convolution for space-time signals and two types of steerable Mustard convolutions (with point wise products in the spectral domain). Finally we expressed the convolution in terms of Mustard convolutions (Theorems 4.3 and 4.5), and vice versa the Mustard convolution in terms of classical convolutions in Theorem 4.6.

We expect our results to be relevant for applied mathematics, physics, engineering, navigation, geographic information systems (GIS), in particular for special relativistic quantum mechanics, optics, electro-dynamics and aero-space navigation. Furthermore, we expect applications in electromagnetic signal transmission and processing. In the convolutions one signal function could be an electromagnetic signal, the other a filter function, window function, continuous mother wavelet, etc.

Acknowledgments

The author wishes to thank God: *In the beginning God created the heavens and the earth*[8] [8]. He further thanks his family for their kind support, Hiroshi Suzuki for his gift of friendship.

[8] I add a brief reflection about the *interdisciplinary context* of mathematical research, including life sciences and theology: *Natural selection has shown insidious imperialistic tendencies. The offering of post-hoc explanations of phenotypic traits by reference to their hypothetical effects on fitness in their hypothetical environments of selection has spread from evolutionary theory to a host of other traditional disciplines: philosophy, psychology, anthropology, sociology, and even to aesthetics and theology. Some people really do seem to think that natural selection is a universal acid, and that nothing can resist its powers of dissolution. However, the internal evidence to back this imperialistic selectionism strikes us as very thin. Its credibility depends largely on the reflected glamour of natural selection which biology proper is said to legitimise. Accordingly, if natural selection disappears from biology, its offshoots in other fields seem likely to disappear as well. This is an outcome much to be desired since, more often than not, these offshoots have proved to be not just post hoc but ad hoc, crude, reductionist, scientistic rather than scientific, shamelessly self-congratulatory, and so wanting in detail that they are bound to accommodate the data, however that data may turn out. So it really does matter whether natural selection is true.*[7]

References

[1] R. Abłamowicz, B. Fauser, *Mathematics of Clifford - a Maple package for Clifford and Graßmann algebras*, Adv. in Appl. Cliff. Als., Vol. 15, Is. 2, pp. 157—181 (2005).

[2] W.K. Clifford, *Applications of Grassmann's Extensive Algebra*, American Journal of Mathematics Pure and Applied 1 (1878), pp. 350–358.

[3] H.S.M. Coxeter, *Quaternions and Reflections*, The American Mathematical Monthly, **53**(3) (Mar., 1946), pp. 136–146.

[4] H. De Bie, N. De Schepper, T.A. Ell, K. Rubrecht and S.J. Sangwine, *Connecting spatial and frequency domains for the quaternion Fourier transform*, Applied Mathematics and Computation, Vol. 271, (2015) pp. 581–593.

[5] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge (UK), 2003.

[6] A. Einstein, *Zur Elektrodynamik bewegter Körper*. Annalen der Physik. 17 (10): 891—921, (1905–06–30).

[7] J. Fodor and M. Piattelli-Palmarini, *Survival of the fittest theory: Darwinism's limits*, New Scientist, Vol. 205, Iss. 2746, 3 February 2010, pp. 28–31. DOI: 10.1016/S0262-4079(10)60303-X , preprint: `http://dingo.sbs.arizona.edu/~massimo/publications/PDF/JF_MPP_darwinisms_limits.pdf`

[8] *Genesis chapter 1 verse 1*, in The Holy Bible, English Standard Version, Crossway Bibles, Good News Publishers, Wheaton, Illinois, 2001.

[9] P. Girard, *Quaternions, algb̀re de Clifford et physique relativiste*. Presses polytechniques, et universitaires romandes, 2004 (French); *Quaternions, Clifford Algebras and Relativistic Physics*. Birkhäuser, Basel, 2007 (English).

[10] T.F. Havel and C.J.L. Doran, *Interaction and Entanglement in the Multiparticle Spacetime Algebra*, In C. Doran, L. Dorst and J. Lasenby eds., Applied Geometrical Algebras in computer Science and Engineering AGACSE 2001, p. 227 (Chapter 21), Birkhäuser (2002).

[11] D. Hestenes, *Space-Time Algebra*, 2nd ed., Birkhäuser, Basel, 2015.

[12] E. Hitzer, *Relativistic Physics as Application of Geometric Algebra*, in K. Adhav (ed.), Proc. of the Int. Conf. on Relativity 2005 (ICR2005), University of Amravati, India, January 2005, pp. 71–90 (2005). Preprint: `http://vixra.org/abs/1306.0121` .

[13] E. Hitzer and G. Utama, *The GeometricAlgebra Java Package – Novel Structure Implementation of 5D Geometric Algebra* $\mathbb{R}_{4,1}$ *for Object Oriented Euclidean Geometry, Space-Time Physics and Object Oriented Computer Algebra*, Mem. Fac. Eng. Univ. Fukui **53**(1), pp. 47–59 (2005). Preprint: `http://vixra.org/abs/1306.0120` .

[14] E. Hitzer, *Quaternion Fourier Transform on Quaternion Fields and Generalizations*, Adv. in App. Cliff. Alg., **17** (2007), pp. 497–517. DOI: 10.1007/s00006-007-0037-8 , preprint: http://arxiv.org/abs/1306.1023 .

[15] B. Mawardi and E. Hitzer, *Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra* $Cl(3,0)$, Advances in Applied Clifford Algebras, **16**(1), pp. 41–61 (2006). DOI: 10.1007/s00006-006-0003-x, Preprint: http://vixra.org/abs/1306.0089 .

[16] E. Hitzer and B. Mawardi, *Clifford Fourier Transform on Multivector Fields and Uncertainty Principles for Dimensions* $n = 2(mod\,4)$ *and* $n = 3(mod\,4)$, in P. Angles (ed.), Adv. App. Cliff. Alg. Vol. **18**, S3,4, pp. 715–736 (2008). DOI: 10.1007/s00006-008-0098-3, Preprint: http://vixra.org/abs/1306.0127 .

[17] E. Hitzer, *Directional Uncertainty Principle for Quaternion Fourier Transforms*, Adv. in App. Cliff. Alg., **20**(2) (2010), pp. 271–284. DOI: 10.1007/s00006-009-0175-2 , preprint: http://arxiv.org/abs/1306.1276 .

[18] E. Hitzer, *OPS-QFTs: A new type of quaternion Fourier transforms based on the orthogonal planes split with one or two general pure quaternions.* In Numerical Analysis and Applied Mathematics ICNAAM 2011, AIP Conf. Proc. 1389, pp. 280–283 (2011); DOI: 10.1063/1.3636721 , preprint: http://arxiv.org/abs/1306.1650 .

[19] E. Hitzer, *Creative Peace License*, http://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/

[20] E. Hitzer and S.J. Sangwine, *The orthogonal 2D planes split of quaternions and steerable quaternion Fourier transformations*, in: E. Hitzer and S.J. Sangwine (Eds.), Quaternion and Clifford Fourier Transforms and Wavelets, Trends in Mathematics, Vol. **27**, Birkhäuser, 2013, pp. 15–40. DOI: 10.1007/978-3-0348-0603-9_2 , preprint: http://arxiv.org/abs/1306.2157 .

[21] E. Hitzer, J. Helmstetter and R. Abłamowicz, *Square Roots of* -1 *in Real Clifford Algebras*, in: E. Hitzer and S.J. Sangwine (Eds.), Quaternion and Clifford Fourier Transforms and Wavelets, Trends in Mathematics, Vol. **27**, Birkhäuser, 2013, pp. 123–153, DOI: 10.1007/978-3-0348-0603-9_7, preprint: http://arxiv.org/abs/1204.4576 .

[22] E. Hitzer, *Two-Sided Clifford Fourier Transform with Two Square Roots of* -1 *in* $Cl(p,q)$ Adv. Appl. Clifford Algebras, Vol. **24**(2) (2014), pp. 313–332, DOI: 10.1007/s00006-014-0441-9 , preprint: http://arxiv.org/abs/1306.2092 .

[23] E. Hitzer, *The orthogonal planes split of quaternions and its relation to quaternion geometry of rotations*, IOP Jour. of Phys.: Conf. Ser. (JPCS) Vol. **597** (2015) 012042. DOI: 10.1088/1742-6596/597/1/012042. Open Acc. URL: http://iopscience.iop.org/1742-6596/597/1/012042/pdf/1742-6596_597_1_012042.pdf, preprint: http://viXra.org/abs/1411.0362 .

[24] E. Hitzer, *General two-sided quaternion Fourier transform, convolution and Mustard convolution*, Adv. Appl. Clifford Algebras, Vol. **27**(1) (2017), pp. 381–395, DOI: 10.1007/s00006-016-0684-8. Preprint: http://vixra.org/abs/1601.0165

[25] E. Hitzer, *General Steerable Two-sided Clifford Fourier Transform, Convolution and Mustard Convolution*, Adv. App. Clifford Algebras, Vol. **27**(3) (2017), pp. 2215–2234, DOI: 10.1007/s00006-016-0687-5.

[26] For up to date information about the latest Clifford algebra software see: https://gaupdate.wordpress.com/category/software/

[27] P. Lounesto, *Clifford Algebras and Spinors*, London Mathematical Society Lecture Note Series (Book 286), 2nd ed., Cambridge University Press, Cambridge (UK), 2001.

[28] D. Mustard, *Fractional convolution*, J. Aust. Math. Soc. Ser. B Vol. **40** (1998), pp. 257–265.

[29] S. Sangwine, E. Hitzer, *Clifford Multivector Toolbox (for MATLAB)*, Adv. Appl. Clifford Algebras, Vol. **27**(1) (2017), pp. 539–558. DOI: 10.1007/s00006-016-0666-x.

This paper has been published as: E. Hitzer, Special relativistic Fourier transformation and convolutions, Mathematical Methods in the Applied Sciences, First published: 04 Mar. 2019, Vol. 42, Issue 7, pp. 2244–2255, 2019, DOI: 10.1002/mma.5502, URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.5502