Riemann Hypothesis is incorrect (second proof)

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Abstract: A few years ago, I wrote my paper [4]. In the paper [4], I use Nevanlinna's Second Main Theorem of the value distribution theory, denied the Riemann Hypothesis. In this paper, I use the analytic methods, I once again denied the Riemann Hypothesis.
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1. Introduction

Riemann Hypothesis was posed by Riemann in early 50's of the 19 th century in his thesis titled "The Number of Primes less than a Given Number ". It is one of the unsolved "Supper" problems of mathematics. The Riemann Hypothesis is closely related to the well-known Prime Number Theorem. The Riemann Hypothesis states that all the nontrivial zeros of the zeta-function lie on the 'critical line' $\{s : \operatorname{Re} s = \frac{1}{2}\}$.

In this paper, we use the analytical methods, refute the Riemann Hypothesis. For convenience, We will below the Riemann Hypothesis abbreviated to RH.

2. Some theorems in the classic theory

In this paper, $\Gamma(s)$ is the Euler gamma function , $\zeta(s)$ is the Riemann zeta function.

Lemma 2.1. If $\operatorname{Re} w > 0$, then

$$\frac{1}{2\pi i}\int_{(2)} \Gamma(s)\, w^{-s}\, ds = \exp(-w)$$

where Re w is the real part of complex number w.

Let $\eta > 0$ be given, when $|s| \geq \eta$ and $|\arg s| \leq \pi - \eta$, then

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right)$$

If $-4 \leq \sigma \leq 4$, $|t| \geq 1$, then

$$\Gamma(\sigma + it) = \sqrt{2\pi}\, t^{\sigma-\frac{1}{2}} \exp\left(-\tfrac{\pi}{2}|t| + it(\log|t|-1) + i\lambda\tfrac{\pi}{2}(\sigma-\tfrac{1}{2})\right) + O\left(t^{\sigma-\frac{3}{2}} \exp\left(-\tfrac{\pi}{2}|t|\right)\right)$$

where $\lambda = 1$ if $t \geq 1$, $\lambda = -1$ if $t \leq -1$.

See [1] page 523, page 525.

Lemma 2.2. If $\mathrm{Re}\, s > 1$, then

$$\frac{\zeta'}{\zeta}(\mathrm{s}) = -\sum_{n=2}^{\infty} \frac{\Lambda(\mathrm{n})}{n^s}$$

where $\Lambda(\mathrm{n})$ is the Mangoldt function.

Let s is any complex number, we have

$$\frac{\zeta'}{\zeta}(\mathrm{s}) = -\frac{1}{s-1} + c_1 + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}\mathrm{s}+1)$$

where ρ be the nontrivial zeros of $\zeta(\mathrm{s})$, c_1 be the positive constant.

We write $s = \sigma + it$. If $-1 \leq \sigma \leq 2$, $-\pi < \mathrm{Im}\left\{\log(\mathrm{s}-1)\right\} \leq \pi$, $-\pi < \mathrm{Im}\left\{\log(\mathrm{s}-\rho)\right\} \leq \pi$, then

$$\log \zeta(\mathrm{s}) = -\log(\mathrm{s}-1) + \sum_{|\gamma - t| \leq 1} \log(\mathrm{s}-\rho) + \mathrm{O}(\log(|\,\mathrm{t}\,|+2))$$

where Im s is the imaginary part of complex number s.

See [2] page 4, page 31, page 218.

Lemma 2.3. Let $N(\mathrm{T})$ is the number of zeros of $\zeta(\mathrm{s})$ in the rectangle

$0 < \sigma < 1$, $0 < t < T$. then

$$N(\mathrm{T}) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(\mathrm{T}) + \mathrm{O}\left(\frac{1}{T}\right)$$

where $S(\mathrm{T}) = \dfrac{1}{\pi}\arg\zeta(\tfrac{1}{2} + \mathrm{iT})$.

See [3] page 98.

Lemma 2.4. Assume that RH, If $x \geq 2$, then

$$\psi(\mathrm{x}) = \sum_{2 \leq n \leq x} \Lambda(\mathrm{n}) = x + R(x)$$

where $R(x) \ll x^{\frac{1}{2}} \log^2 x$.

See [3] page 113.

3. Some preparation work

Lemma 3.1 Assume that RH, and $0<\delta\le\frac{1}{50}$, then

$$\int_{\frac{1}{2}+\delta}^{2} \left|\log\zeta(\sigma+\mathrm{i}\gamma_0)\right|\mathrm{d}\sigma \ll 1$$

and

$$\int_{-1}^{\frac{1}{2}-\delta} \left|\log\zeta(\sigma+\mathrm{i}\gamma_0)\right|\mathrm{d}\sigma \ll 1$$

where γ_0 is the ordinate of nontrivial first zero of $\zeta(\mathrm{s})$, $\gamma_0\approx 14.134725...$

Proof. By lemma 2.2 and RH, we have

$$\log\zeta(\sigma+\mathrm{i}\gamma_0) \ll \sum_{|\gamma-\gamma_0|\le 1} \left|\log(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)\right| + \mathrm{O}(\log\gamma_0)$$

because

$$\log(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)=\frac{1}{2}\log\left((\sigma-\tfrac{1}{2})^2+(\gamma_0-\gamma)^2\right)+i\,Arg(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)$$

$$\left|\log(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)\right| \ll \left|\log\left((\sigma-\tfrac{1}{2})^2+(\gamma_0-\gamma)^2\right)\right|+1$$

and

$$\log(\sigma-\tfrac{1}{2})^2 \le \log\left((\sigma-\tfrac{1}{2})^2+(\gamma_0-\gamma)^2\right) \le \log\left(\tfrac{9}{4}+1\right)$$

$$\left|\log\left((\sigma-\tfrac{1}{2})^2+(\gamma_0-\gamma)^2\right)\right| \ll \left|\log(\sigma-\tfrac{1}{2})^2\right|+1$$

therefore

$$\left|\log(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)\right| \ll \left|\log(\sigma-\tfrac{1}{2})^2\right|+1$$

And because

$$\int_{\frac{1}{2}+\delta}^{2} \left|\log\left(\sigma-\tfrac{1}{2}\right)^2\right| d\sigma = \int_{\frac{1}{2}+\delta}^{\frac{3}{2}} \left|\log\left(\sigma-\tfrac{1}{2}\right)^2\right| d\sigma + \int_{\frac{3}{2}}^{2} \left|\log\left(\sigma-\tfrac{1}{2}\right)^2\right| d\sigma$$

$$=-2\int_{\frac{1}{2}+\delta}^{\frac{3}{2}} \log\left(\sigma-\tfrac{1}{2}\right)d\sigma+2\int_{\frac{3}{2}}^{2} \log\left(\sigma-\tfrac{1}{2}\right)d\sigma=-2\int_{\delta}^{1} \log\sigma d\sigma+2\int_{1}^{\frac{3}{2}} \log\sigma d\sigma$$

$$=2\delta\log\delta+2\int_{\delta}^{1} d\sigma+O(1)=O(1)$$

therefore

$$\int_{\frac{1}{2}+\delta}^{2} \left|\log\zeta(\sigma+\mathrm{i}\gamma_0)\right|d\sigma \ll \sum_{|\gamma-\gamma_0|\le 1}\int_{\frac{1}{2}+\delta}^{2} \left|\log(\sigma-\tfrac{1}{2}+\mathrm{i}\gamma_0-\mathrm{i}\gamma)\right|d\sigma+1$$

$$\ll \int_{\frac{1}{2}+\delta}^{2} \left|\log(\sigma-\tfrac{1}{2})^2\right|d\sigma+1\ll 1$$

Similarly, we have

$$\int_{-1}^{\frac{1}{2}-\delta} \left|\log\zeta(\sigma+\mathrm{i}\gamma_0)\right|\mathrm{d}\sigma \ll 1$$

This completes the proof of Lemma 3.1.

Throughout the paper, we write

$$z=a+ib \qquad a=\frac{1}{T} \qquad T\ge 50 \qquad b=2\pi$$

It is easy to see that

$$arctg\,\tfrac{b}{a}=\frac{\pi}{2}-h \qquad h=\sum_{k=0}^{\infty} (-1)^k\frac{a^{2k+1}}{(2k+1)b^{2k+1}} \qquad \frac{1}{4\pi T}\le h\le\frac{1}{\pi T}$$

Lemma 3.2. We calculate the three complex numbers.
Because

$$\mathrm{a}+i\,\mathrm{b}=(\mathrm{a}^2+b^2)^{\frac{1}{2}}\exp(\mathrm{i\ arctg}\,\tfrac{b}{a})=(\mathrm{a}^2+b^2)^{\frac{1}{2}}\exp(\mathrm{i}\tfrac{\pi}{2}-\mathrm{ih})$$

therefore when t is the real number, we have

$$z^{\frac{3}{4}-it}=(\mathrm{a}^2+b^2)^{\frac{3}{8}-i\frac{t}{2}}\exp(\mathrm{i}\tfrac{3\pi}{8}-\mathrm{i}\tfrac{3}{4}\mathrm{h}+\tfrac{\pi}{2}t-t\,\mathrm{h})\ll\exp(\tfrac{\pi}{2}t-t\,\mathrm{h})$$

$$z^{-\frac{1}{2}-it}=(\mathrm{a}^2+b^2)^{-\frac{1}{4}-i\frac{t}{2}}\exp(-\mathrm{i}\tfrac{\pi}{4}+i\tfrac{h}{2}+\tfrac{\pi}{2}t-t\,\mathrm{h})\ll\exp(\tfrac{\pi}{2}t-t\,\mathrm{h})$$

$$z^{-\frac{1}{2}+it}=(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}+it}=(\mathrm{a}^2+b^2)^{-\frac{1}{4}+i\frac{t}{2}}\exp(-\mathrm{i}\tfrac{\pi}{4}+\mathrm{i}\tfrac{h}{2}-\tfrac{\pi}{2}t+t\,\mathrm{h})\ll\exp(-\tfrac{\pi}{2}t+t\,\mathrm{h})$$

The three complex numbers required below.

Lemma 3.3

$$\int_{(-\frac{3}{4})} \Gamma(s)\frac{\zeta'}{\zeta}(s)(a+i\,b)^{-s}\,ds \ll 1$$

Proof. By lemma 2.1 and lemma 3.2, we have

$$\int_{(-\frac{3}{4})} \Gamma(s)\frac{\zeta'}{\zeta}(s)(a+i\,b)^{-s}\,ds = i\int_{-\infty}^{+\infty} \Gamma(-\tfrac{3}{4}+it)\frac{\zeta'}{\zeta}(-\tfrac{3}{4}+it)(a+i\,b)^{\frac{3}{4}-it}\,dt$$

$$\ll \int_{-\infty}^{+\infty} \left|\Gamma(-\tfrac{3}{4}+it)\frac{\zeta'}{\zeta}(-\tfrac{3}{4}+it)\right|\left|(a+i\,b)^{\frac{3}{4}-it}\right|dt \ll \int_{-\infty}^{+\infty} (|\,t\,|+2)^{-\frac{5}{4}}\log(|\,t\,|+2)\exp\left(-th\right)dt$$

$$\ll \int_{-\infty}^{+\infty} (|\,t\,|+2)^{-\frac{5}{4}}\log(|\,t\,|+2)\,dt \ll 1$$

This completes the proof of Lemma 3.3.

Lemma 3.4.

$$\int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+it)(a+ib)^{-\frac{1}{2}-it}\log\frac{t}{2\pi}\,dt \ll \log^2 T$$

Proof. By lemma 2.1 and lemma 3.2, we have

$$\int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+it)(a+ib)^{-\frac{1}{2}-it}\log\frac{t}{2\pi}\,dt$$

$$=\sqrt{2\pi}(a^2+b^2)^{-\frac{1}{4}}\exp(-i\tfrac{\pi}{4}+i\tfrac{h}{2})\int_{\gamma_0}^{+\infty} \exp\left(-t\,h+it\,(\log t-1)\right)(a^2+b^2)^{-i\frac{t}{2}}\log\left(\frac{t}{2\pi}\right)dt$$

$$+O\left((a^2+b^2)^{-\frac{1}{4}}\int_{\gamma_0}^{\infty} t^{-1}\exp\left(-t\,h\right)\log t\,dt\right) = I_1\left(\sqrt{2\pi}(a^2+b^2)^{-\frac{1}{4}}\exp(-i\tfrac{\pi}{4}+i\tfrac{h}{2})\right)+I_2$$

we write

$$r=(a^2+b^2)^{\frac{1}{2}} \qquad 2\pi \le r \le 2\pi+1$$

$$I_1=\int_{\gamma_0}^{+\infty} \exp\left(-t\,h+it(\log t-\log r-1)\right)\log\left(\frac{t}{2\pi}\right)dt$$

$$=\int_{\gamma_0}^{+\infty} \frac{\exp\left(-t\,h\right)}{i\log t-i\log r}\log\left(\frac{t}{2\pi}\right)d\exp\left(it(\log t-\log r-1)\right)$$

$$=-i\int_{\gamma_0}^{+\infty} \frac{\exp\left(-t\,h\right)}{\log t-\log r}\left(\log t-\log 2\pi\right)d\exp\left(it\log\tfrac{t}{re}\right)$$

$$= -i\int_{\gamma_0}^{+\infty} \left(\exp(-t\,\mathrm{h}) + \frac{\exp(-t\,\mathrm{h})}{\log t - \log r}(\log r - \log 2\pi)\right) d\exp\left(\mathrm{it}\,\log\tfrac{t}{re}\right)$$

$$= O(1) + i\int_{\gamma_0}^{+\infty} \left(-\mathrm{h}\exp(-t\,\mathrm{h}) + \left(-h\frac{\exp(-t\,\mathrm{h})}{\log t - \log r} - \frac{\exp(-t\,\mathrm{h})}{\mathrm{t}\,(\log t - \log r)^2}\right)\log\tfrac{r}{2\pi}\right)\exp\left(\mathrm{it}\,\log\tfrac{t}{re}\right)dt$$

$$\ll \int_{\gamma_0}^{+\infty} \left(\mathrm{h}\exp(-t\,\mathrm{h}) + \frac{1}{\mathrm{t}\,(\log t - \mathrm{logr})^2}\right)dt \ll 1$$

$$I_2 \ll \int_{\gamma_0}^{h^{-2}} \mathrm{t}^{-1}\exp(-t\,\mathrm{h})\log\mathrm{t}\,dt + \int_{h^{-2}}^{\infty} \mathrm{t}^{-1}\exp(-t\,\mathrm{h})\log\mathrm{t}\,dt$$

$$\ll \int_{\gamma_0}^{h^{-2}} \mathrm{t}^{-1}\log\mathrm{t}\,dt + h^2\log h^{-1}\int_{h^{-2}}^{\infty} \exp(-t\,\mathrm{h})dt \ll (\mathrm{logh})^2 \ll \log^2 T$$

This completes the proof of Lemma 3.4.

Lemma 3.5.

$$\int_{\gamma_0}^{+\infty} \left|\Gamma'(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it} - \Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\log z\right| t^{-1}dt \ll \log^2 T$$

Proof. When $t \geq \gamma_0$, by lemma 2.1, we have

$$\Gamma'(\tfrac{1}{2}+\mathrm{it}) \ll \left|\Gamma(\tfrac{1}{2}+\mathrm{it})\log(\tfrac{1}{2}+\mathrm{it})\right| + \left|\frac{\Gamma(\tfrac{1}{2}+\mathrm{it})}{\tfrac{1}{2}+\mathrm{it}}\right|$$

$$\ll \exp\left(-\frac{\pi}{2}t\right)\log\mathrm{t} + \mathrm{t}^{-1}\exp\left(-\frac{\pi}{2}t\right) \ll \exp\left(-\frac{\pi}{2}t\right)\log\mathrm{t}$$

By lemma 2.1 and lemma 3.2, we have

$$\int_{\gamma_0}^{+\infty} \left|\Gamma'(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it} - \Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\log z\right| t^{-1}dt$$

$$\ll \int_{\gamma_0}^{+\infty} \mathrm{t}^{-1}\exp(-th)\log t dt \ll \int_{\gamma_0}^{h^{-2}} \mathrm{t}^{-1}\exp(-th)\log t dt + \int_{h^{-2}}^{+\infty} \mathrm{t}^{-1}\exp(-th)\log t dt$$

$$\ll \int_{\gamma_0}^{h^{-2}} \mathrm{t}^{-1}\log t dt + \mathrm{h}^2\mathrm{logh}^{-2}\int_{h^{-2}}^{+\infty} \exp(-th)dt \ll \log^2 T$$

This completes the proof of Lemma 3.5.

Lemma 3.6. Assume that RH, then

$$\int_{\gamma_0}^{+\infty} \Gamma'(\tfrac{1}{2}+\mathrm{it})\, z^{-\frac{1}{2}-it}\, S(\mathrm{t})dt \ll 1$$

and

$$\int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+\mathrm{it})\, z^{-\frac{1}{2}-it}\, S(\mathrm{t})dt \ll 1$$

where $S(\mathrm{T}) = \dfrac{1}{\pi}\arg\zeta(\tfrac{1}{2}+\mathrm{iT})$.

Proof. By lemma 3.2, it is easy to see that

$$\Gamma'(\tfrac{1}{2}+\mathrm{it})\, z^{-\frac{1}{2}-it} = \Gamma'(\tfrac{1}{2}+\mathrm{it})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}-i\frac{t}{2}} \exp\left(-\mathrm{i}\tfrac{\pi}{4}+i\tfrac{h}{2}+\tfrac{\pi}{2}t-t\,\mathrm{h}\right)$$

$$= \Gamma'(\tfrac{1}{2}+\mathrm{it})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}-i\frac{t}{2}} \exp\left(\tfrac{\pi}{2}t-t\,\mathrm{h}\right)\left(\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)+i\sin\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)$$

We write

$$H(\mathrm{s}) = \Gamma'(\mathrm{s})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{s}{2}}$$

$$G_1(\mathrm{s}) = H(\mathrm{s})\left(\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)s\right)+\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$\mathrm{G}_2(\mathrm{s}) = H(\mathrm{s})\left(\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)s\right)-\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$G_3(\mathrm{s}) = H(1-\mathrm{s})\left(\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)s\right)+\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$\mathrm{G}_4(\mathrm{s}) = H(1-\mathrm{s})\left(\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)s\right)-\exp\left(\left(-i\tfrac{\pi}{2}+ih\right)(s-1)\right)\right)$$

It is easy to see that

$$G_1(\tfrac{1}{2}+\mathrm{it}) = 2\Gamma'(\tfrac{1}{2}+\mathrm{it})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}-\mathrm{i}\frac{t}{2}} \exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)$$

$$\mathrm{G}_2(\tfrac{1}{2}+\mathrm{it}) = 2i\Gamma'(\tfrac{1}{2}+\mathrm{it})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}-i\frac{t}{2}} \exp\left(\tfrac{\pi}{2}t-ht\right)\sin\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)$$

$$\mathrm{G}_3(\tfrac{1}{2}+it) = 2\Gamma'(\tfrac{1}{2}-it)(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}+i\frac{t}{2}} \exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)$$

$$\mathrm{G}_4(\tfrac{1}{2}+\mathrm{it}) = 2i\Gamma'(\tfrac{1}{2}-\mathrm{it})(\mathrm{a}^2+\mathrm{b}^2)^{-\frac{1}{4}+i\frac{t}{2}} \exp\left(\tfrac{\pi}{2}t-ht\right)\sin\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)$$

Assume that RH and $0<\delta\le\frac{1}{50}$, by the contour integration method, we have

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}+\int_{-1+i\infty}^{-1+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}+\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}=0$$

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}=-\int_{-1+i\infty}^{-1+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}-\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}=J_1+J_2$$

By lemma 2.1 and lemma 3.2,

$$J_1=-\int_{-1+i\infty}^{-1+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}\ll\int_{\gamma_0}^{\infty}\left|G_1(-1+it)\right|\left|\log\zeta(-1+\mathrm{it})\right|\mathrm{dt}$$

$$\ll\int_{\gamma_0}^{\infty}\left|\Gamma'(-1+it)\right|\exp\left(\tfrac{\pi}{2}t-th\right)(\log t)\,dt\ll\int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}(\log t)^2\exp(-th)\,dt\ll\int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}(\log t)^2\,dt\ll 1$$

By lemma 2.1, lemma 3.1 and lemma 3.2, we have

$$J_2=-\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G_1(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}\ll\int_{-1}^{\frac{1}{2}-\delta_0}\left|G_1(\sigma+\mathrm{i}\,\gamma_0)\right|\left|\log\zeta(\sigma+\mathrm{i}\,\gamma_0)\right|\mathrm{d}\sigma$$

$$\ll\int_{-1}^{\frac{1}{2}-\delta_0}\left|\log\zeta(\sigma+\mathrm{i}\,\gamma_0)\right|\mathrm{d}\sigma\ll 1$$

When $\delta\to 0$, we have

$$\int_{\gamma_0}^{\infty} G_1(\tfrac{1}{2}+\mathrm{it})\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt}\ll 1$$

Similarly,

$$\int_{\gamma_0}^{\infty} G_2(\tfrac{1}{2}+\mathrm{it})\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt}\ll 1$$

Assume that RH and $0<\delta\le\frac{1}{50}$, by the contour integration method, we have

$$\int_{2+i\gamma_0}^{2+i\infty} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}+\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}+\int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}=0$$

$$\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds} = -\int_{2+i\gamma_0}^{2+i\infty} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds} - \int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds}$$

same as above

$$\int_{\frac{1}{2}+\delta+i\gamma_0}^{\frac{1}{2}+\delta+i\infty} G_3(\mathrm{s})\log\zeta(\mathrm{s})\,\mathrm{ds} \ll 1$$

When $\delta \to 0$, we have

$$\int_{\gamma_0}^{\infty} G_3(\tfrac{1}{2}+it)\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt} \ll 1$$

Similarly,

$$\int_{\gamma_0}^{\infty} G_4(\tfrac{1}{2}+it)\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt} \ll 1$$

Synthesizethe above conclusion, we have

$$\int_{\gamma_0}^{\infty} \left(G_1(\tfrac{1}{2}+it)+G_3(\tfrac{1}{2}+it)\right)\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt}$$

$$=\int_{\gamma_0}^{\infty} \left(H(\tfrac{1}{2}+it)+H(\tfrac{1}{2}-it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt}$$

$$=2\int_{\gamma_0}^{\infty} \left(\mathrm{Re}\,H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)\left(\log|\zeta(\tfrac{1}{2}+\mathrm{it})|+\mathrm{i}\arg\zeta(\tfrac{1}{2}+\mathrm{it})\right)\mathrm{dt}$$

therefore

$$\int_{\gamma_0}^{\infty} \left(\mathrm{Re}\,H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)S(\mathrm{t})\,\mathrm{dt} \ll 1$$

Similarly,

$$\int_{\gamma_0}^{\infty} \left(G_1(\tfrac{1}{2}+it)-G_3(\tfrac{1}{2}+it)\right)\log\zeta(\tfrac{1}{2}+\mathrm{it})\,\mathrm{dt}$$

$$=2i\int_{\gamma_0}^{\infty} \left(\mathrm{Im}\,H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)\left(\log|\zeta(\tfrac{1}{2}+\mathrm{it})|+\mathrm{i}\arg\zeta(\tfrac{1}{2}+\mathrm{it})\right)\mathrm{dt}$$

therefore

$$\int_{\gamma_0}^{\infty} \left(\mathrm{Im}\,H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\cos\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)S(\mathrm{t})\,\mathrm{dt} \ll 1$$

Similarly,

$$\int_{\gamma_0}^{\infty} \left(\operatorname{Re} H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\sin\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)S(\mathrm{t})\,\mathrm{dt} \ll 1$$

$$\int_{\gamma_0}^{\infty} \left(\operatorname{Im} H(\tfrac{1}{2}+it)\right)\left(\exp\left(\tfrac{\pi}{2}t-ht\right)\sin\left(-\tfrac{\pi}{4}+\tfrac{h}{2}\right)\right)S(\mathrm{t})\,\mathrm{dt} \ll 1$$

Therefore

$$\int_{\gamma_0}^{+\infty} \Gamma'(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\, S(\mathrm{t})dt \ll 1$$

We use the same process, we can get

$$\int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\, S(\mathrm{t})dt \ll 1$$

This completes the proof of Lemma 3.6.

Lemma 3.7 Assume that RH, we have

$$\sum_{-\infty<\gamma<+\infty} \Gamma(\tfrac{1}{2}+\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}-i\gamma} \ll \log^2\mathrm{T}$$

Where γ be the ordinates of the nontrivial zeros of $\zeta(\mathrm{s})$.

Proof.

$$\sum_{-\infty<\gamma<+\infty} \Gamma(\tfrac{1}{2}+\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}-i\gamma} = \sum_{\gamma_0<\gamma<+\infty} \Gamma(\tfrac{1}{2}+\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}-i\gamma} + \sum_{\gamma_0<\gamma<+\infty} \Gamma(\tfrac{1}{2}-\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}+i\gamma}$$

$$= A_1 + A_2$$

$$A_1 = \sum_{\gamma_0\le\gamma<+\infty} \Gamma(\tfrac{1}{2}+\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}-i\gamma} = \int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\, d\ N(\mathrm{t})$$

by lemma 2.3, the above formula

$$= \int_{\gamma_0}^{+\infty} \Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\, d\left(\frac{t}{2\pi}\log\frac{t}{2\pi}-\frac{t}{2\pi}+\frac{7}{8}+S(\mathrm{t})+\mathrm{O}\left(\frac{1}{t}\right)\right)$$

$$=\frac{1}{2\pi}\int_{\gamma_0}^{+\infty}\Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\left(\log\frac{t}{2\pi}\right)dt+\int_{\gamma_0}^{+\infty}\Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\,d\left(S(\mathrm{t})+\mathrm{O}\left(\frac{1}{t}\right)\right)$$

By lemma 3.4, the above formula

$$=-\int_{\gamma_0}^{+\infty}\left(i\,\Gamma'(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}-i\,\Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\,\mathrm{logz}\right)S(\mathrm{t})dt$$

$$+O\left(\int_{\gamma_0}^{+\infty}\left|\Gamma'(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}+\Gamma(\tfrac{1}{2}+\mathrm{it})\,\mathrm{z}^{-\frac{1}{2}-it}\,\mathrm{logz}\right|\left(\frac{1}{t}\right)dt\right)+O\left(\log^2 T\right)$$

by lemma 3.5 and lemma 3.6, above formulas $\ll \log^2 T$.

By lemma 2.1 and lemma 3.2, we have

$$A_2=\sum_{\gamma_0\le\gamma<+\infty}\Gamma(\tfrac{1}{2}-\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}+i\gamma}\ll\sum_{\gamma_0\le\gamma<+\infty}\exp\left(-\pi\gamma+\gamma h\right)\ll 1$$

This completes the proof of Lemma 3.7

Lemma 3.8. Assume that RH, if $T\ge 2,$ then

$$\sum_{n=2}^{\infty}\Lambda(\mathrm{n})\exp\left(-\frac{n}{T}\right)=T+O\left(\mathrm{T}^{\frac{1}{2}}\log^2 T\right)$$

Proof. By lemma 2.4, we have

$$\sum_{n=2}^{\infty}\Lambda(\mathrm{n})\exp\left(-\frac{n}{T}\right)==\int_{2}^{\infty}\exp\left(-\frac{x}{T}\right)d\psi(\mathrm{x})$$

$$=\int_{2}^{\infty}\exp\left(-\frac{x}{T}\right)d\left(x+R(x)\right)=\int_{2}^{\infty}\exp\left(-\frac{x}{T}\right)dx+\frac{1}{T}\int_{2}^{\infty}\exp\left(-\frac{x}{T}\right)R(x)\,\mathrm{dx}+\mathrm{O}(1)$$

$$=T\exp\left(-\frac{2}{T}\right)+O\left(\frac{1}{T}\int_{2}^{\infty}x^{\frac{1}{2}}(\mathrm{logx})^2\exp\left(-\frac{x}{T}\right)\mathrm{dx}\right)+\mathrm{O}(1)$$

$$=T+O\left(T^{\frac{1}{2}}\int_{0}^{\infty}x^{\frac{1}{2}}(\mathrm{logx}+\mathrm{logT})^2\exp\left(-x\right)\mathrm{dx}\right)+\mathrm{O}(1)=\mathrm{T}+\mathrm{O}\left(T^{\frac{1}{2}}\log^2 T\right)$$

This completes the proof of Lemma 3.8.

4. Conclusion

When $a=\dfrac{1}{T}$, $T\ge 50$, $b=2\pi$, n is the positive integer, by lemma 2.1, we have

$$\frac{1}{2\pi i}\int_{(2)} \Gamma(\mathrm{s})(\mathrm{a}+i\,\mathrm{b})^{-s} n^{-s} ds = \exp\left(-an-ibn\right)=\exp\left(-\frac{n}{T}\right)$$

By lemma 2.2, we have

$$-\sum_{n=2}^{\infty} \Lambda(\mathrm{n})\exp\left(-\frac{n}{T}\right)=\frac{1}{2\pi i}\int_{(2)} \Gamma(\mathrm{s})\frac{\zeta'}{\zeta}(\mathrm{s})(\mathrm{a}+i\,\mathrm{b})^{-s} ds$$

by lemma 2.2 and RH, the above formula

$$=-(\mathrm{a}+i\,\mathrm{b})^{-1}+\sum_{-\infty<\gamma<+\infty} \Gamma(\tfrac{1}{2}+\mathrm{i}\,\gamma)(\mathrm{a}+i\,\mathrm{b})^{-\frac{1}{2}-i\gamma}+\frac{\zeta'}{\zeta}(0)+\frac{1}{2\pi i}\int_{(-\frac{3}{4})} \Gamma(\mathrm{s})\frac{\zeta'}{\zeta}(\mathrm{s})(\mathrm{a}+i\,\mathrm{b})^{-s} ds$$

by lemma 3.3 and lemma 3.7, the above formula $\ll \log^2 T$

By lemma 3.8, we get a contradiction, therefore the RH is incorrect.

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