Beukers Integral , Formulas

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July 14 , 2008

Abstract

In this note we show some formulas related to an integral of beukers:

$$\zeta(2)=\frac{\pi^2}{6}=\int_0^1\int_0^1 \frac{1}{1-x\,y}\,dx\,dy$$

Introduction

Beukers [2] used some double integrals to give an elegant proof to Apéry result , which states that $\zeta(3)$ is irrational.

Fórmulas

(1) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}+\int_0^1\int_1^a \frac{1}{x(x-y)}\,dx\,dy\ ,\ a\geq 1$$

(2) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}+\sum_{n=1}^{\infty}\frac{b^n(1-a^{-n})}{n^2}+\int_b^1\int_1^a \frac{1}{x(x-y)}\,dx\,dy\ ,\ a\geq 1,0\leq b\leq 1$$

(3) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}+\sum_{n=0}^{\infty}\int_0^{a-1}(-x)^n\ln\left(\frac{1+x}{x}\right)dx\ ,\ 1\leq a<2$$

(4) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^{n+m}b^{m+1}(a-1)^{n-m+1}}{(m+1)(n-m+1)}$$
$$+\int_b^1\int_1^a \frac{1}{x(x-y)}\,dx\,dy\ ,\ 1\leq a<2,0\leq b<1$$

(5) $$\zeta(2) = \sum_{m=0}^{\infty} \sum_{k=0}^{2^n-1} (2^n m + k + 1)^{-2} = \sum_{k=0}^{2^n-1} \sum_{m=0}^{\infty} (2^n m + k + 1)^{-2} \;, n \in \mathbb{N} \cup \{0\}$$

(6) $$\zeta(2) = \sum_{n=0}^{\infty} \binom{n+m}{n} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(n+k+1)^2} \;, m \in \mathbb{N}$$

En (6) , con $m = 1$, se tiene:

$$\zeta(2) = \sum_{n=0}^{\infty} \binom{n+1}{n} \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right) = \sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)^2}$$

(7) $$\zeta(2) = \int_0^1 \int_{1-y}^1 \frac{1}{1-x\,y}\, dx\, dy - \int_0^1 \frac{\ln(1-x+x^2)}{x}\, dx$$

(8) $$\frac{\pi^2}{18} = -\int_0^1 \frac{\ln(1-x+x^2)}{x}\, dx$$

(9) $$\zeta(2) = \int_0^1 \int_{\sqrt{1-y^2}}^1 \frac{1}{1-x\,y}\, dx\, dy - \int_0^1 \frac{\ln\left(1-x\sqrt{1-x^2}\right)}{x}\, dx$$

(10) $$\frac{7\pi^2}{72} = -\int_0^1 \frac{\ln\left(1-x\sqrt{1-x^2}\right)}{x}\, dx$$

(11) $$\zeta(2) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{a^{k+1}}{(k+1)^2} + \int_0^1 \int_a^1 \frac{1}{1-x\,y}\, dx\, dy \;, 0 < a < 1$$

(12) $$\zeta(2) = \sum_{n=1}^{\infty} \frac{(a\,b)^n}{n^2} + \iint \frac{1}{1-x\,y}\, dx\, dy \;, 0 \le a \le 1 \;, 0 \le b \le 1$$

La integral doble es sobre la región: $R = [0,1]^2 - [0,a] \times [0,b]$.

(13) $$\zeta(2)=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k-m}\sum_{s=0}^{m}\binom{m}{s}\frac{(-1)^k a^{-k-2}}{(n+m+1-s)(k-m-n+1+s)}$$
$$+\int_0^{\infty}\int_0^{1}\frac{a}{(1+a\,x)(1+y)(a\,x+y+a\,x\,y)}dx\,dy$$
$$+\int_0^{1}\int_1^{\infty}\frac{a}{(1+a\,x)(1+a\,y)(x+y+a\,x\,y)}dx\,dy\,,a>2$$

(14) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{a^n}{n^2}+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^k(1+a)^{n-k+1}(1-a^{n+k+1})}{(n+1)(n+k+1)}$$
$$+\iint\frac{1}{1-x\,y}dx\,dy\,,\frac{1}{2}\le a\le 1$$

La integral doble es sobre la región: $R=\{(x,y)|a\le x\le 1,1+a-x\le y\le 1\}$

(15) $$\zeta(2)=\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^k a^{4n+4}}{(n+1)(n+k+1)}+4\iint\frac{x\,y}{1-x^2y^2}dx\,dy\,,0\le a\le 1$$

La integral doble es sobre la región: $R=\{(x,y)|x^2+y^2\ge a,0\le x\le 1,0\le y\le 1\}$

(16) $$\zeta(2)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{s=0}^{n-k-m}\binom{n-k-m}{s}\frac{2^{-m-s-1}a^{n-m-s+1}}{(m+s+1)(n-m-s+1)}$$
$$+\int_0^{1/2}\int_a^{1/2}\frac{1}{(1-x-y)(1-x)(1-y)}dx\,dy\,,0\le a<\frac{1}{2}$$

(17) $$\zeta(2)=\sum_{n=0}^{\infty}\left(\frac{2^{n+1}-1}{n+1}\right)\left(\ln(1+a)+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k(1-(1+a)^{-k})}{k}\right)$$
$$+\int_0^{1}\int_a^{1}\frac{1}{1-x\,y}dx\,dy\,,0\le a<1$$

(18) $$\zeta(2)=\sum_{n=0}^{\infty}\left(\frac{(1+a)^{n+1}-1}{n+1}\right)\left(\ln 2+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k(1-2^{-k})}{k}\right)$$
$$+\int_a^{1}\int_0^{1}\frac{1}{1-x\,y}dx\,dy\,,0\le a<1$$

(19) $$\zeta(2)=\sum_{n=0}^{\infty}3^{-n-1}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{f(a,b,n,k,m)}{(n-k+m+1)(k+1)}$$
$$+\int_{b}^{1}\int_{a}^{1}\frac{1}{3-x-y-x\,y}\,dx\,dy\;,-1\le a<1,-1\le b<1$$

$$f(a,b,n,k,m)=a^{n-k+m+1}(1-b^{k+1})+b^{k+1}+(-1)^{n+m}(1+(-1)^{k})+(-1)^{k}$$

(20) $$\zeta(2)=\sum_{n=0}^{\infty}3^{-n-1}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{f(a,b,n,k,m)}{(n-k+m+1)(k+1)}$$
$$+\iint\frac{1}{3-x-y-x\,y}\,dx\,dy\;,0\le a<1,0\le b<1$$

La integral doble es sobre la región: $R=[-1,1]^2-[-a,a]\times[-b,b]$, y

$$f(a,b,n,k,m)=a^{n-k+m+1}b^{k+1}(1+(-1)^{n+k+m})(1+(-1)^{k})$$

(21) $$\zeta(2)=2a\ln 2-\frac{a^2}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^{2n+1}(2^{2n}-1)B_n}{n(2n+1)!}+\int_{a}^{\infty}\int_{0}^{\infty}\frac{1}{e^{x+y}-1}\,dx\,dy$$
$$+\int_{0}^{a}\int_{a}^{\infty}\frac{1}{e^{x+y}-1}\,dx\,dy\;,0\le a<\pi$$

$B_n=\left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\frac{691}{2730},\frac{7}{6},\ldots\right\}$, números de Bernoulli.

(22) $$\zeta(2)=\sum_{n=1}^{\infty}\frac{e^{-na}(2-e^{-na})}{n^2}+\int_{0}^{a}\int_{0}^{a}\frac{1}{e^{x+y}-1}\,dx\,dy\;,a\ge 0$$

(23) $$\zeta(2)=\sum_{n=0}^{\infty}\frac{2^{-2n-1}}{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}+4\int_{1/2}^{1}\int_{0}^{1-x}\frac{1}{1-x^2+y^2}\,dy\,dx$$

(24) $$\zeta(2)=\int_{0}^{1}\int_{1}^{\infty}\frac{1}{x(x-y)}\,dx\,dy$$

(25) $$\zeta(2)=\int_{1}^{\infty}\int_{1}^{\infty}\frac{1}{x\,y(x\,y-1)}\,dx\,dy$$

(26) $$\zeta(2) = \int_0^\infty \int_0^\infty \frac{1}{(1+x)(1+y)(x+y+x\,y)}\,dx\,dy$$

(27) $$\zeta(2) = \int_0^{1/2} \int_0^{1/2} \frac{1}{(1-x)(1-y)(1-x-y)}\,dx\,dy$$

(28) $$\zeta(2) = \int_0^\infty \int_0^\infty \frac{1}{e^{x+y}-1}\,dx\,dy$$

(29) $$\zeta(2) = \int_1^\infty \int_1^\infty \frac{1}{x\,y\,(x+y-1)}\,dx\,dy$$

(30) $$\zeta(2) = \int_0^\infty \int_0^\infty \frac{1}{(1+x)(1+y)(1+x+y)}\,dx\,dy$$

(31) $$\zeta(2) = \int_0^1 \int_0^1 \frac{1}{1-x+x\,y}\,dx\,dy$$

(32) $$\zeta(2) = \int_{-1}^1 \int_{-1}^1 \frac{1}{3-x-y-x\,y}\,dx\,dy$$

(33) $$\zeta(2) = \int_0^\infty \int_0^\infty \frac{1}{e^x+e^y-1}\,dx\,dy$$

Observación. Todas las fórmulas se han tomado de la referencia (5).

Referencias

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