

$$*\mathbf{R}* = \kappa\mathbf{T}$$

Physicalization of Curvature

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The Einstein equations are completed.
The electromagnetic nature of energetic phenomena is derived.
Alternative units are introduced.

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1. I,II,III

1.1. I

The Einstein equations of General Relativity

$$\begin{aligned}\kappa T_{ik} &= R_{ik} - \frac{1}{2}g_{ik}R \\ &= [*R*]_{ik} \\ &= g^{jl}[*R*]_{ijkl} \\ &= g^{jl}\frac{1}{4}\eta_{ij}{}^{ab}\eta_{kl}{}^{cd}R_{abcd}\end{aligned}$$

can be written as

$$g^{jl}\frac{[*R*]_{ijkl}}{\kappa} = T_{ik}$$

T_{ik} is the trace of $\frac{[*R*]_{ijkl}}{\kappa}$ and if T_{ik} is an energy momentum tensor so is $\frac{[*R*]_{ijkl}}{\kappa}$ because the metric bears no physical or geometrical unit. Setting

$$\frac{[*R*]_{ijkl}}{\kappa} =: T_{ijkl}$$

implies

$$g^{jl}T_{ijkl} = T_{ik}$$

Details in part I.

1.2. II

The energy momentum tensor of the electromagnetic field in connection with the Einstein equations reads

$$\begin{aligned}R_{ik} - \frac{1}{4}Rg_{ik} &= \kappa[T_{ik} - \frac{1}{4}Tg_{ik}] \\ &= \kappa[F_i{}^b F_{kb} - \frac{1}{4}F^{ab}F_{ab}g_{ik}]\end{aligned}$$

This expression stems from

$$\begin{aligned}[*R*]_{ik} &= \kappa T_{ik} \\ &= \kappa F_i{}^b F_{kb}\end{aligned}$$

which in turn stems from

$$\begin{aligned}[*R*]_{ijkl} &= \kappa T_{ijkl} \\ &= \kappa F_{ij}F_{kl}\end{aligned}$$

Details in part II.

1.3. III

Usually T_{ik} is called 'energy-momentum tensor' and bears the unit

$$[T_{ik}] = \text{kg} \cdot \frac{\text{m}^2}{\text{s}^2} \cdot \frac{1}{\text{m}^3}$$

Regarding T_{ik} as a tensorfield defined on spacetime, a canonical unit for T_{ik} should contain a factor $\frac{1}{\text{m}^4}$ or $\frac{1}{\text{s} \cdot \text{m}^3}$. This means

$$[T_{ik}] = \frac{\text{X}}{\text{m}^4}$$

The unit X has electromagnetic 'roots' because

$$\begin{aligned} \text{X} &= K_{[1]}^2 \text{esu}^2 & K_{[1]} \in \mathbb{R} \\ &\approx 4,817 \cdot 10^{51} \cdot \text{esu}^2 \end{aligned}$$

'esu' being the 'electrostatic unit of charge'. Details in part III.

2. π , τ , $*$, ω , ρ , ∇ , \mathbf{C} , \mathbf{R}

Further literature

$\nabla, g, \mathbf{C}, \mathbf{R}$	[13], pp. 23, 30 – 40 [11], III, §1 – 5
π	[2], pp. 291 – 298 [3]
τ	[2], pp. 298 – 299 [9]
ω	[1], pp. 25 – 28 [4], pp. 374 – 379, 552 – 568
$\omega \circ \alpha$	[10], pp. 120 [11], I, §7

Let M denote a differentiable manifold, $\dim[M] = n$, and let M^L denote a Lorentzian manifold, $\dim[M^L] = 4$, with metric signature $1, 1, 1, -1$.

2.1. π

Let M be given. The *Skew* or *Anti Symmetrizer* $\alpha_{i_1 \dots i_l}^{a_1 \dots a_l}$ and the *Symmetrizer* $\sigma_{i_1 \dots i_l}^{a_1 \dots a_l}$ are defined by

$$\begin{aligned}\alpha_{i_1 \dots i_l}^{a_1 \dots a_l} &:= \sum_p \text{sign}[p] \delta_{i_1}^{a_{p[1]}} \dots \delta_{i_l}^{a_{p[l]}} \\ \sigma_{i_1 \dots i_l}^{a_1 \dots a_l} &:= \sum_p \delta_{i_1}^{a_{p[1]}} \dots \delta_{i_l}^{a_{p[l]}}\end{aligned}$$

$p[1] \dots p[l]$ denotes an array of $1, \dots, l$ which is obtained from $1, \dots, l$ by permutations. The sum is carried out over all possible arrays of $1, \dots, l$. $\text{sign}[p]$, the sign of an array, is positive if $p[1] \dots p[l]$ can be obtained from $1, 2, \dots, l-1, l$ by an even number of permutations, otherwise negative. So the *Permutation Operator* π can be defined as

$$\begin{aligned}\pi_{ij}^{ab} X_{ab} &:= \frac{1}{2} \sigma_{ij}^{ab} X_{ab} + \frac{1}{2} \alpha_{ij}^{ab} X_{ab} \\ &=: X_{ij}^{\boxed{i} \boxed{j}} + X_{ij}^{\boxed{j} \boxed{i}} \\ &=: X_{ij}^{\square\square} + X_{ij}^{\square\Box} \\ &=: [\pi^{\square\square}]_{ij}^{ab} X_{ab} + [\pi^{\square\Box}]_{ij}^{ab} X_{ab}\end{aligned}$$

$\pi_{ij}^{ab} X_{ab}$ leads to

$$\begin{aligned}\pi_{ijk}^{abc} X_{abc} &= X_{ijk}^{\boxed{i} \boxed{j} \boxed{k}} + X_{ijk}^{\boxed{k} \boxed{j} \boxed{i}} + X_{ijk}^{\boxed{j} \boxed{k} \boxed{i}} + X_{ijk}^{\boxed{i} \boxed{k} \boxed{j}} \\ &= X_{ijk}^{\square\square\square} + X_{ijk}^{\square\Box} + X_{ijk}^{\Box\square} \\ &=: [\pi^{\square\square\square}]_{ijk}^{abc} X_{abc} + [\pi^{\square\Box}]_{ijk}^{abc} X_{abc} + [\pi^{\Box\square}]_{ijk}^{abc} X_{abc}\end{aligned}$$

which leads to

$$\pi_{ijkl}^{abcd} X_{abcd} = X_{ijkl}^{\boxed{i} \boxed{j} \boxed{k} \boxed{l}} + X_{ijkl}^{\boxed{l} \boxed{j} \boxed{k} \boxed{i}} + X_{ijkl}^{\boxed{i} \boxed{j} \boxed{l} \boxed{k}} + X_{ijkl}^{\boxed{i} \boxed{k} \boxed{j} \boxed{l}} + X_{ijkl}^{\boxed{i} \boxed{k} \boxed{l} \boxed{j}}$$

$$\begin{aligned}
& + X_{ijk}^{\begin{smallmatrix} i & k & l \\ j \end{smallmatrix}} + X_{ijkl}^{\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}} + X_{ijkl}^{\begin{smallmatrix} i & k \\ j \end{smallmatrix}} + X_{ijk}^{\begin{smallmatrix} i & l \\ j \end{smallmatrix}} + X_{ijkl}^{\begin{smallmatrix} i \\ j & k & l \end{smallmatrix}} \\
= & X_{ijkl}^{\square\square\square\square} + X_{ijkl}^{\square\square\square} + X_{ijkl}^{\square\square\square} + X_{ijkl}^{\square\square\square} + X_{ijkl}^{\square\square\square} \\
=: & [\pi^{\square\square\square\square}]_{ij}^{ab} X_{abcd} + [\pi^{\square\square\square}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\square\square\square}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\square\square\square}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\square\square\square}]_{ijkl}^{abcd} X_{abcd}
\end{aligned}$$

which leads to

$$\begin{aligned}
\pi_{ijklm}^{abcde} X_{abcde} & = X_{ijklm}^{\begin{smallmatrix} i & j & k & l & m \\ m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & k & l \\ m \end{smallmatrix}} \\
& + X_{ijklm}^{\begin{smallmatrix} i & j & k & m \\ l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & k \\ l & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & k \\ l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & l & m \\ k \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & l \\ k & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & l \\ k \end{smallmatrix}} \\
& + X_{ijklm}^{\begin{smallmatrix} i & j & m \\ k & l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j \\ k & l & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j & m \\ k \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j \\ k & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & j \\ k \end{smallmatrix}} \\
& + X_{ijkm}^{\begin{smallmatrix} i & k & l & m \\ j \end{smallmatrix}} + X_{ijkm}^{\begin{smallmatrix} i & k & l \\ j & m \end{smallmatrix}} + X_{ijkm}^{\begin{smallmatrix} i & k & l \\ j \end{smallmatrix}} + X_{ijkm}^{\begin{smallmatrix} i & k & m \\ j & l \end{smallmatrix}} + X_{ijkm}^{\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}} \\
& + X_{ijklm}^{\begin{smallmatrix} i & k & m \\ j & l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & k \\ j & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & l & m \\ j & k \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & l \\ j & m \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i & l \\ j & k \end{smallmatrix}} \\
& + X_{ijklm}^{\begin{smallmatrix} i & m \\ j & k & l \end{smallmatrix}} + X_{ijklm}^{\begin{smallmatrix} i \\ j & k & l & m \end{smallmatrix}} \\
= & X_{ijklm}^{\square\square\square\square\square} + X_{ijkm}^{\square\square\square\square} + X_{ijklm}^{\square\square\square\square} + X_{ijklm}^{\square\square\square\square} + X_{ijklm}^{\square\square\square\square} + X_{ijklm}^{\square\square\square\square} + X_{ijklm}^{\square\square\square\square} \\
=: & [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} \\
& + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\square\square\square\square\square}]_{ijklm}^{abcde} X_{abcde}
\end{aligned}$$

The generalization of the factor $\frac{1}{2}$ in $\pi_{ij}^{ab} X_{ab}$ reads

$$\frac{A}{B!} \tag{2.1}$$

where A denotes the number of different configurations of a box-array by indices, and B denotes the order of the tensor. So

$$\begin{aligned}
\pi_{ij}^{ab} & = \frac{1}{2} \sigma_{ij}^{ab} + \frac{1}{2} \alpha_{ij}^{ab} \\
& = [\pi^{\square\square}]_{ij}^{ab} + [\pi^{\square}]_{ij}^{ab} \\
\pi_{ijk}^{abc} & = \frac{1}{6} \sigma_{ijk}^{abc} + \frac{1}{3} [\alpha_{ik}^{pc} \sigma_{pj}^{ab} + \alpha_{ij}^{pb} \sigma_{pk}^{ac}] + \frac{1}{6} \alpha_{ijk}^{abc} \\
& = [\pi^{\square\square\square}]_{ijk}^{abc} + [\pi^{\square\square}]_{ijk}^{abc} + [\pi^{\square}]_{ijk}^{abc} \\
\pi_{ijkl}^{abcd} & = \frac{1}{24} \sigma_{ijkl}^{abcd} + \frac{1}{8} [\alpha_{il}^{pd} \sigma_{pj}^{abc} + \alpha_{ik}^{pc} \sigma_{pj}^{abd} + \alpha_{ij}^{pb} \sigma_{pk}^{acd}] + \frac{1}{12} [\alpha_{ik}^{pr} \alpha_{jl}^{qs} \sigma_{pq}^{ab} \sigma_{rs}^{cd} + \alpha_{ij}^{pr} \alpha_{kl}^{qs} \sigma_{pq}^{ac} \sigma_{rs}^{bd}] \\
& + \frac{1}{8} [\alpha_{ikl}^{pcd} \sigma_{pj}^{ab} + \alpha_{ijl}^{pbd} \sigma_{pk}^{ac} + \alpha_{ijk}^{pbc} \sigma_{pl}^{ad}] + \frac{1}{24} \alpha_{ijkl}^{abcd} \\
= & [\pi^{\square\square\square\square}]_{ijkl}^{abcd} + [\pi^{\square\square\square\square}]_{ijkl}^{abcd} + [\pi^{\square\square\square\square}]_{ijkl}^{abcd} + [\pi^{\square\square\square\square}]_{ijkl}^{abcd} + [\pi^{\square\square\square\square}]_{ijkl}^{abcd}
\end{aligned}$$

$$\begin{aligned}
\pi_{ijklm}^{abcde} &= \frac{1}{120}\sigma_{ijklm}^{abcde} + \frac{1}{30}[\alpha_{im}^{pe}\sigma_{pjkl}^{abcd} + \alpha_{il}^{pd}\sigma_{pjkm}^{abce} + \alpha_{ik}^{pc}\sigma_{pjlm}^{abde} + \alpha_{ij}^{pb}\sigma_{pklm}^{acde}] \\
&+ \frac{1}{24}[\alpha_{il}^{pr}\alpha_{jm}^{qs}\sigma_{rs}^{de}\sigma_{pqk}^{abc} + \alpha_{ik}^{pr}\alpha_{jm}^{qs}\sigma_{rs}^{ce}\sigma_{pql}^{abd}] \\
&+ \frac{1}{24}[\alpha_{ik}^{pr}\alpha_{jl}^{qs}\sigma_{rs}^{cd}\sigma_{pqm}^{abe} + \alpha_{ij}^{pr}\alpha_{km}^{qs}\sigma_{rs}^{be}\sigma_{pql}^{acd} + \alpha_{ij}^{pr}\alpha_{kl}^{qs}\sigma_{rs}^{bd}\sigma_{pqm}^{ace}] \\
&+ \frac{1}{20}[\alpha_{ilm}^{pde}\sigma_{pj k}^{abc} + \alpha_{ikm}^{pce}\sigma_{pjl}^{abd} + \alpha_{ikl}^{pcd}\sigma_{pjm}^{abe} + \alpha_{ijm}^{pbe}\sigma_{pkl}^{acd} + \alpha_{ijl}^{pbd}\sigma_{pkm}^{ace} + \alpha_{ijk}^{pbc}\sigma_{plm}^{ade}] \\
&+ \frac{1}{24}[\alpha_{ikm}^{pre}\alpha_{jl}^{qs}\sigma_{rs}^{cd}\sigma_{pq}^{ab} + \alpha_{ikl}^{prd}\alpha_{jm}^{qs}\sigma_{rs}^{ce}\sigma_{pq}^{ab} + \alpha_{ijm}^{pre}\alpha_{kl}^{qs}\sigma_{rs}^{bd}\sigma_{pq}^{ac}] \\
&+ \frac{1}{24}[\alpha_{ijl}^{prd}\alpha_{km}^{qs}\sigma_{rs}^{be}\sigma_{pq}^{ac} + \alpha_{ijk}^{prc}\alpha_{lm}^{qs}\sigma_{rs}^{be}\sigma_{pq}^{ad}] \\
&+ \frac{1}{30}[\alpha_{iklm}^{pcde}\sigma_{pj}^{ab} + \alpha_{ijlm}^{pbde}\sigma_{pk}^{ac} + \alpha_{ijkm}^{pbce}\sigma_{pl}^{ad} + \alpha_{ijkl}^{pbcd}\sigma_{pm}^{ae}] + \frac{1}{120}\alpha_{ijklm}^{abcde} \\
&= [\pi^{\square\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde} + [\pi^{\square\square\square}]_{ijklm}^{abcde}
\end{aligned}$$

The number of independent components per term can be obtained from the following procedure. The dimension of the space, on which the tensor is defined, is inserted into the box in the upper left-hand corner of the array of indices. Going on from this, the other boxes are filled. For every step to the right, the number is raised by 1, and for every step down, the number is decreased by 1. The product of all those numbers of an array is finally multiplied by (2.1).

2.2. τ

Let g be a metric on M .

$$\begin{aligned}
g_{ij} &= g_{ji} \\
\det[g_{ij}] &=: \mathfrak{g} \neq 0
\end{aligned}$$

With the help of the metric g the *Trace Decomposition Operator* τ and the *Trace Reduction Operator* γ can be defined.

$$\tau_{i_1 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l} := [X_{i_1 \dots i_l} - \gamma_{i_1 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l}] + \gamma_{i_1 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l}$$

with

$$\gamma_{i_1 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l} := g_{i_1 i_2} \tau_{i_3 \dots i_l}^{a_3 \dots a_l} Y_{a_3 \dots a_l}^{[1,2]} + g_{i_1 i_3} \tau_{i_2 i_4 \dots i_l}^{a_2 a_4 \dots a_l} Y_{a_2 a_4 \dots a_l}^{[1,3]} + \dots + g_{i_{l-1} i_l} \tau_{i_1 \dots i_{l-2}}^{a_1 \dots a_{l-2}} Y_{a_1 \dots a_{l-2}}^{[l-1,l]}$$

and

$$g^{p_1 p_2} \gamma_{i_1 \dots p_1 \dots p_2 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l} = g^{p_1 p_2} X_{i_1 \dots p_1 \dots p_2 \dots i_l}$$

2.3. $*$

Let $\mathfrak{g} < 0$.

$$\begin{aligned}
\alpha_{1 \dots n}^{i_1 \dots i_n} &= g^{i_1 a_1} \dots g^{i_n a_n} g_{1 b_1} \dots g_{n b_n} \alpha_{a_1 \dots a_n}^{b_1 \dots b_n} \\
&= g^{i_1 a_1} \dots g^{i_n a_n} g_{1 b_1} \dots g_{n b_n} \alpha_{a_1 \dots a_n}^{1 \dots n} \alpha_{1 \dots n}^{b_1 \dots b_n} \\
&= g^{i_1 a_1} \dots g^{i_n a_n} \alpha_{a_1 \dots a_n}^{1 \dots n} \mathfrak{g} \\
\alpha_{i_1 \dots i_n}^{1 \dots n} &= g_{i_1 a_1} \dots g_{i_n a_n} \alpha_{1 \dots n}^{a_1 \dots a_n} \frac{1}{\mathfrak{g}}
\end{aligned}$$

implies

$$\begin{aligned}
\alpha_{i_1 \dots i_n}^{1 \dots n} \alpha_{a_1 \dots a_n}^{1 \dots n} &= \alpha_{i_1 \dots i_n}^{1 \dots n} g_{a_1 b_1} \dots g_{a_n b_n} \alpha_{1 \dots n}^{b_1 \dots b_n} \frac{1}{\mathfrak{g}} \\
&= \alpha_{i_1 \dots i_n}^{b_1 \dots b_n} g_{a_1 b_1} \dots g_{a_n b_n} \frac{1}{\mathfrak{g}} \\
\alpha_{1 \dots n}^{i_1 \dots i_n} \alpha_{1 \dots n}^{a_1 \dots a_n} &= \alpha_{b_1 \dots b_n}^{i_1 \dots i_n} g^{a_1 b_1} \dots g^{a_n b_n} \mathfrak{g}
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{-\mathfrak{g}} \alpha_{i_1 \dots i_n}^{1 \dots n} \sqrt{-\mathfrak{g}} \alpha_{a_1 \dots a_n}^{1 \dots n} &= -g_{a_1 b_1} \dots g_{a_n b_n} \alpha_{i_1 \dots i_n}^{b_1 \dots b_n} = -\det \begin{bmatrix} g_{i_1 a_1} & \dots & g_{i_1 a_n} \\ \vdots & \ddots & \vdots \\ g_{i_n a_1} & \dots & g_{i_n a_n} \end{bmatrix} \\
\frac{1}{\sqrt{-\mathfrak{g}}} \alpha_{1 \dots n}^{i_1 \dots i_n} \frac{1}{\sqrt{-\mathfrak{g}}} \alpha_{1 \dots n}^{a_1 \dots a_n} &= -g^{a_1 b_1} \dots g^{a_n b_n} \alpha_{b_1 \dots b_n}^{i_1 \dots i_n} = -\det \begin{bmatrix} g^{i_1 a_1} & \dots & g^{i_1 a_n} \\ \vdots & \ddots & \vdots \\ g^{i_n a_1} & \dots & g^{i_n a_n} \end{bmatrix}
\end{aligned}$$

respectively. Set

$$\begin{aligned}
\eta_{i_1 \dots i_n} &:= \sqrt{-\mathfrak{g}} \alpha_{i_1 \dots i_n}^{1 \dots n} \\
\eta^{i_1 \dots i_n} &:= \frac{1}{\sqrt{-\mathfrak{g}}} \alpha_{1 \dots n}^{i_1 \dots i_n}
\end{aligned}$$

These definitions make sense because of

$$\eta_{a_1 \dots a_n} g^{i_1 a_1} \dots g^{i_n a_n} = \eta^{i_1 \dots i_n}$$

so

$$\eta_{i_1 \dots i_n} \eta^{a_1 \dots a_n} = -\alpha_{i_1 \dots i_n}^{a_1 \dots a_n} = -\det \begin{bmatrix} \delta_{i_1}^{a_1} & \dots & \delta_{i_1}^{a_n} \\ \vdots & \ddots & \vdots \\ \delta_{i_n}^{a_1} & \dots & \delta_{i_n}^{a_n} \end{bmatrix}$$

The following construction is possible

$$\begin{aligned}
\alpha_{i_1 \dots i_k}^{a_1 \dots a_k} X_{a_1 \dots a_k} &= \frac{1}{[n-k]!} \alpha_{i_1 \dots i_k i_{k+1} \dots i_n}^{a_1 \dots a_k i_{k+1} \dots i_n} X_{a_1 \dots a_k} \\
&= -\frac{1}{[n-k]!} \eta_{i_1 \dots i_k i_{k+1} \dots i_n} \eta^{a_1 \dots a_k i_{k+1} \dots i_n} X_{a_1 \dots a_k} \\
&=: -\frac{k!}{[n-k]!} \eta_{i_1 \dots i_k i_{k+1} \dots i_n} [*X]^{i_{k+1} \dots i_n}
\end{aligned}$$

with the *Dual Operator* $*$ and the dual tensor $*X$. Particulary for M^L

$$\begin{aligned}
\alpha X &= \alpha_{abcd}^{abcd} \frac{1}{24} X = -\eta_{abcd} \eta^{abcd} \frac{1}{24} X =: -\eta_{abcd} \frac{1}{24} [*X]^{abcd} \\
\alpha_i^a X_a &= \alpha_{ibcd}^{abcd} \frac{1}{6} X_a = -\eta_{ibcd} \eta^{abcd} \frac{1}{6} X_a =: -\eta_{ibcd} \frac{1}{6} [*X]^{bcd} \\
\alpha_{ij}^{ab} X_{ab} &= \alpha_{ijcd}^{abcd} \frac{1}{2} X_{ab} = -\eta_{ijcd} \eta^{abcd} \frac{1}{2} X_{ab} =: -\eta_{ijcd} [*X]^{cd} \\
\alpha_{ijk}^{abc} X_{abc} &= \alpha_{ijkd}^{abcd} X_{abc} = -\eta_{ijkd} \eta^{abcd} X_{abc} =: -\eta_{ijkd} 6[*X]^d
\end{aligned}$$

$$\alpha_{ijkl}^{abcd} X_{abcd} = \alpha_{ijkl}^{abcd} X_{abcd} = -\eta_{ijkl} \eta^{abcd} X_{abcd} =: -\eta_{ijkl} 24[*X]$$

and

$$\begin{aligned}\pi_{ijkl}^{abcd} \eta_{abcd} &= \eta_{ijkl} \\ \tau_{ijkl}^{abcd} \eta_{abcd} &= \eta_{ijkl} - \gamma_{ijkl}^{abcd} \eta_{abcd}\end{aligned}$$

2.4. ω

Returning to M .

$$X_{i_1 \dots i_l} = \delta_{i_1}^{a_1} \delta_{i_2}^{a_2} \dots \delta_{i_l}^{a_l} X_{a_1 \dots a_l}$$

Because of the property

$$\delta_i^a X_a = X_i$$

with an arbitrary vector X_i , the Kronecker Delta has the eigenvalue 1 and every vector is an eigenvector. Choosing an orthonormal basis $\{x_i^\alpha\}_{\alpha=1}^n$ gives a spectral factorization of the Kronecker Delta

$$\delta_i^a = \sum_{\alpha=1}^n x_i^\alpha x^{\alpha a}$$

This in turn gives

$$\begin{aligned}X_{i_1 \dots i_l} &= \sum_{\alpha_1=1}^n x_{i_1}^{\alpha_1} x^{\alpha_1 a_1} \sum_{\alpha_2=1}^n x_{i_2}^{\alpha_2} x^{\alpha_2 a_2} \dots \sum_{\alpha_l=1}^n x_{i_l}^{\alpha_l} x^{\alpha_l a_l} X_{a_1 \dots a_l} \\ &= \sum_{\alpha_1=1}^n \sum_{\alpha_2=1}^n \dots \sum_{\alpha_l=1}^n x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_l}^{\alpha_l} [x^{\alpha_1 a_1} x^{\alpha_2 a_2} \dots x^{\alpha_l a_l} X_{a_1 \dots a_l}]\end{aligned}$$

a decomposition of a tensor into a sum of products of vectors. For given $X_{i_1 \dots i_l}$ only $l-1$ sums are linear independent. The *Factorization Operator* ω for a tensor $X_{i_1 \dots i_l}$ reads

$$\omega_{i_1 \dots i_l}^{a_1 \dots a_l} X_{a_1 \dots a_l} := \sum_{\alpha=1}^{n^{l-1}} Y_{i_1}^{[1, \alpha]} \dots Y_{i_l}^{[l, \alpha]}$$

For $l=2$ this becomes

$$\omega_{ij}^{ab} X_{ab} = \sum_{\alpha=1}^n Y_i^\alpha Z_j^\alpha$$

If there is a positive definite scalar product available, a factorization of X_{ij} is delivered by the *Singular Value Decomposition*

$$X_{ij} = Y_i^a S_{ab} Z_j^b$$

with two orthogonal matrices Y and Z and a diagonal matrix S whose non zero diagonal elements s^α are positive and called the singular values of X .

$$\begin{aligned}\omega_{ij}^{ab} X_{ab} &= \sum_{\alpha=1}^n s^\alpha y_i^\alpha z_j^\alpha \\ &= \sum_{\alpha=1}^n [\sqrt{s^\alpha} y_i^\alpha] [\sqrt{s^\alpha} z_j^\alpha] \\ &= \sum_{\alpha=1}^n Y_i^\alpha Z_j^\alpha\end{aligned}$$

2.4.1. $\omega \circ \sigma$

Again, if there is a positive definite scalar product available, a factorization of X_{ij} is delivered by the *Eigenvalue Decomposition*

$$X_{ij} = Y_i^a \Lambda_{ab} Y_j^b$$

for a square, symmetric matrix X with an orthogonal matrix Y and a diagonal matrix Λ whose diagonal elements λ^α are the eigenvalues of X .

$$\begin{aligned} [\omega \circ \sigma]_{ij}^{ab} X_{ab} &:= \sum_{\alpha=1}^n \lambda^\alpha y_i^\alpha y_j^\alpha \\ &= \sum_{\alpha=1}^n [\sqrt{\lambda^\alpha} y_i^\alpha] [\sqrt{\lambda^\alpha} y_j^\alpha] \\ &= \sum_{\alpha=1}^n Y_i^\alpha Y_j^\alpha \end{aligned}$$

or

$$\begin{aligned} [\omega \circ \sigma]_{ij}^{ab} X_{ab} &= \sum_{\alpha=1}^n \text{sign}[\lambda^\alpha] \sqrt{|\lambda^\alpha|} y_i^\alpha [\sqrt{|\lambda^\alpha|} y_j^\alpha] \\ &= \sum_{\alpha=1}^n \text{sign}[\lambda^\alpha] Z_i^\alpha Z_j^\alpha \end{aligned}$$

2.4.2. $\omega \circ \alpha$

$$\begin{aligned} [\omega \circ \alpha]_{i_1 \dots i_n}^{a_1 \dots a_n} X_{a_1 \dots a_n} &= [\omega \circ \alpha]_{i_1 \dots i_n}^{a_1 \dots a_n} g_{a_1 b_1} \dots g_{a_n b_n} X^{b_1 \dots b_n} \\ &= \alpha_{i_1 \dots i_n}^{a_1 \dots a_n} \sum_{\alpha_1}^n x_{a_1}^{\alpha_1} x_{b_1}^{\alpha_1} \dots \sum_{\alpha_n}^n x_{a_n}^{\alpha_n} x_{b_n}^{\alpha_n} X^{b_1 \dots b_n} \\ &= \alpha_{i_1 \dots i_n}^{a_1 \dots a_n} \sum_p x_{a_1}^{p[1]} x_{b_1}^{p[1]} \dots x_{a_n}^{p[n]} x_{b_n}^{p[n]} X^{b_1 \dots b_n} \\ &= \alpha_{i_1 \dots i_n}^{a_1 \dots a_n} x_{a_1}^{[1]} \dots x_{a_n}^{[n]} \sum_p \text{sign}[p] x_{b_1}^{p[1]} \dots x_{b_n}^{p[n]} X^{b_1 \dots b_n} \\ &=: \alpha_{i_1 \dots i_n}^{a_1 \dots a_n} Y_{a_1}^{[1]} \dots Y_{a_n}^{[n]} \end{aligned}$$

implies

$$[\omega \circ \alpha]_{i_1 \dots i_n}^{a_1 \dots a_n} X_{a_1 \dots a_{n-1}} [*X]_{a_n} = \alpha_{i_1 \dots i_n}^{a_1 \dots a_n} Y_{a_1}^{[1]} \dots Y_{i_{k-1}}^{[k-1]} [*Y]_{a_k}^{[k]} Y_{i_{k+1}}^{[k+1]} \dots Y_{a_n}^{[n]}$$

$*Y_{a_k}^{[k]}$ can be chosen

$$*Y_{a_k}^{[k]} = *X_{a_k}$$

so with suitable relabeling

$$[\omega \circ \alpha]_{i_1 \dots i_{n-1}}^{a_1 \dots a_{n-1}} X_{a_1 \dots a_{n-1}} = \alpha_{i_1 \dots i_{n-1}}^{a_1 \dots a_{n-1}} Y_{a_1}^{[1]} \dots Y_{i_{n-1}}^{[n-1]}$$

For $n = 3$

$$[\omega \circ \alpha]_{ij}^{ab} X_{ab} = \alpha_{ij}^{ab} Y_a Z_b$$

2.5. ρ

On M^L there is the possible decomposition

$$\omega_{ik}^{ac} g_{ac} = \sum_{\alpha=1}^3 x_{\alpha i} x_{\alpha k} - u_i u_k$$

for the metric, the $x_{\alpha i}$ being normed and spacelike, and u_i being normed and timelike.

$$u_i u^i = -1$$

Alternatively,

$$v_i = c u_i$$

will be used, $c \in \mathbb{R}$, defined in (9.1). So

$$\omega_{ic}^{ka} \delta_a^c = \sum_{\alpha=1}^3 x_i^\alpha x^{\alpha k} - u_i u^k =: n_i^k - u_i u^k$$

implying the *Spatial Temporal Decomposition Operator* ρ

$$\rho_{i_1 \dots i_l}^{k_1 \dots k_l} := [n_{i_1}^{k_1} - u_{i_1} u^{k_1}] \dots [n_{i_l}^{k_l} - u_{i_l} u^{k_l}]$$

and

$$\rho_{ijkl}^{abcd} \eta_{abcd} = \eta_{jkl} u_i - \eta_{ikl} u_j + \eta_{ijl} u_k - \eta_{ijk} u_l$$

with

$$\eta_{ijk} := \eta_{ijkl} u^d$$

2.6. ∇

Covariant Derivative ∇ on M : tensorial, linear, product rule, commutes with contractions. For two covariant derivatives, ∇ and $\tilde{\nabla}$,

$$\nabla_i X_j - \tilde{\nabla}_i X_j = -G_{ij}{}^a X_a$$

holds. G is a tensorfield. There are special covariant derivatives Y_{ij} of X_j : Y is a tensor which has the components $Y_{ij} = \partial_i X_j$ in one basis. Set

$$\nabla_i X_j = \partial_i X_j - \Gamma_{ij}{}^a X_a$$

for the difference $\nabla_i X_j - \partial_i X_j$.

$$\begin{aligned} Y_{ij} &= \frac{\partial x'^a}{\partial x^i} \frac{\partial x'^b}{\partial x^j} Y'_{ab} \\ \partial_i X_j &= \frac{\partial}{\partial x^i} [X'_b \frac{\partial x'^b}{\partial x^j}] \\ &= \frac{\partial x'^a}{\partial x^i} \frac{\partial x'^b}{\partial x^j} \partial'_a X'_b + X'_b \frac{\partial^2 x'^b}{\partial x^i \partial x^j} \\ \Gamma_{ij}^k &= \frac{\partial x'^a}{\partial x^i} \frac{\partial x'^b}{\partial x^j} \frac{\partial x^k}{\partial x'^c} \Gamma'_{ab}{}^c + \frac{\partial^2 x'^a}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x'^a} \end{aligned}$$

Equating

$$\begin{aligned}
\nabla_i[X^a Y_a] &= \partial_i[X^a Y_a] \\
&= \partial_i X^a Y_a + X^a \partial_i Y_a \\
\nabla_i[X^a Y_a] &= [\partial_i Y_a - \Gamma_{ia}^b Y_b] X^a + \nabla_i X^a Y_a \\
&= \partial_i Y_a X^a - \Gamma_{ib}^a X^b Y_a + \nabla_i X^a Y_a
\end{aligned}$$

implies

$$\nabla_i X^j = \frac{\partial X^j}{\partial x^i} + \Gamma_{ia}^j X^a$$

By the same reasoning

$$\nabla_k[X_{j_1 \dots j_m}^{i_1 \dots i_n} Y_{i_1}^{[1]} \dots Y_{i_n}^{[n]} Z^{[1]j_1} \dots Z^{[m]j_m}] = \partial_k[X_{j_1 \dots j_m}^{i_1 \dots i_n} Y_{i_1}^{[1]} \dots Y_{i_n}^{[n]} Z^{[1]j_1} \dots Z^{[m]j_m}]$$

implies the formula

$$\nabla_k X_{j_1 \dots j_m}^{i_1 \dots i_n} := \partial_k X_{j_1 \dots j_m}^{i_1 \dots i_n} + \sum_{\alpha=1}^n X_{j_1 \dots j_m}^{i_1 \dots i_{\alpha-1} a i_{\alpha+1} \dots i_n} \Gamma_{ka}^{i_\alpha} - \sum_{\beta=1}^m X_{j_1 \dots j_{\beta-1} b j_{\beta+1} \dots j_m} \Gamma_{kj\beta}^b$$

for the covariant derivative of a tensorfield.

2.7. C, R

$$\begin{aligned}
\nabla_k \partial_l &= -\Gamma_{kl}^a \partial_a \\
\nabla_j \nabla_k \partial_l &= -\nabla_j [\Gamma_{kl}^a \partial_a] \\
&= -\partial_j \Gamma_{kl}^a \partial_a + \Gamma_{jk}^b \Gamma_{bl}^a \partial_a + \Gamma_{jl}^b \Gamma_{kb}^a \partial_a \\
\nabla_i \nabla_j \nabla_k \partial_l &= -\nabla_i \partial_j \Gamma_{kl}^a \partial_a - \partial_j \Gamma_{kl}^a \nabla_i \partial_a \\
&\quad + \nabla_i \Gamma_{jk}^b \Gamma_{bl}^a \partial_a + \Gamma_{jk}^b \nabla_i \Gamma_{bl}^a \partial_a + \Gamma_{jk}^b \Gamma_{bl}^a \nabla_i \partial_a \\
&\quad + \nabla_i \Gamma_{jl}^b \Gamma_{kb}^a \partial_a + \Gamma_{jl}^b \nabla_i \Gamma_{kb}^a \partial_a + \Gamma_{jl}^b \Gamma_{kb}^a \nabla_i \partial_a \\
&= -\partial_i \partial_j \Gamma_{kl}^a \partial_a \\
&\quad + [\partial_b \Gamma_{kl}^a \Gamma_{ij}^b + \partial_j \Gamma_{bl}^a \Gamma_{ik}^b + \partial_j \Gamma_{kb}^a \Gamma_{il}^b + \partial_i [\Gamma_{jk}^b \Gamma_{bl}^a] + \partial_i [\Gamma_{jl}^b \Gamma_{kb}^a]] \partial_a \\
&\quad - \Gamma_{ij}^c [\Gamma_{ck}^b \Gamma_{bl}^a + \Gamma_{cl}^b \Gamma_{kb}^a] \partial_a - \Gamma_{ik}^c [\Gamma_{jc}^b \Gamma_{bl}^a + \Gamma_{jl}^b \Gamma_{cb}^a] \partial_a - \Gamma_{il}^c [\Gamma_{jk}^b \Gamma_{bc}^a + \Gamma_{jc}^b \Gamma_{kb}^a] \partial_a
\end{aligned}$$

Antisymmetric parts

$$\begin{aligned}
\alpha_{kl}^{cd} \nabla_c \partial_d &= -\alpha_{kl}^{cd} \Gamma_{cd}^p \partial_p \\
&=: -S_{kl}^p \partial_p \\
\alpha_{kl}^{cd} \nabla_j \nabla_c \partial_d &= -\nabla_j S_{kl}^p \partial_p - S_{kl}^p \nabla_j \partial_p \\
\alpha_{jk}^{bc} \nabla_b \nabla_c \partial_l &= -\alpha_{jk}^{bc} [\partial_b \Gamma_{cl}^p + \Gamma_{cl}^q \Gamma_{bq}^p] \partial_p - S_{jk}^p \nabla_p \partial_l \\
&=: -K_{jkl}^p \partial_p - S_{jk}^p \nabla_p \partial_l \\
\alpha_{ij}^{ab} \nabla_a \nabla_b \nabla_k \partial_l &= -K_{ijk}^p \nabla_p \partial_l - K_{ijl}^p \nabla_k \partial_p - S_{ij}^p \nabla_p \nabla_k \partial_l \\
\alpha_{jk}^{bc} \nabla_i \nabla_b \nabla_c \partial_l &= -\nabla_i K_{jkl}^p \partial_p - K_{jkl}^p \nabla_i \partial_p - \nabla_i S_{jk}^p \nabla_p \partial_l - S_{jk}^p \nabla_i \nabla_p \partial_l
\end{aligned}$$

with the *Torsion Tensor* S_{kl}^p and the *Curvature Tensor* K_{jkl}^p . Equating

$$\alpha_{jkl}^{bcd} \alpha_{cd}^{pq} \nabla_b \nabla_p \partial_q = \alpha_{jkl}^{bcd} \alpha_{bc}^{pq} \nabla_p \nabla_q \partial_d$$

implies the *First* or *Algebraic Bianchi Identity*

$$-\alpha_{jkl}^{bcd}[\nabla_b S_{cd}{}^p \partial_p + S_{cd}{}^q \nabla_b \partial_q] = -\alpha_{jkl}^{bcd}[K_{bcd}{}^p \partial_p - S_{bc}{}^q \nabla_q \partial_d]$$

equivalently

$$\alpha_{jkl}^{bcd} K_{bcd}{}^p = \alpha_{jkl}^{bcd} [\nabla_b S_{cd}{}^p + S_{cd}{}^q S_{qb}{}^p]$$

and equating

$$\alpha_{ijk}^{abc} \alpha_{ab}{}^{pq} \nabla_p \nabla_q \nabla_c \partial_l = \alpha_{ijk}^{abc} \alpha_{bc}{}^{pq} \nabla_a \nabla_p \nabla_q \partial_l$$

implies the *Second* or *Differential Bianchi Identity*

$$\alpha_{ijk}^{abc} [K_{abc}{}^p \nabla_p \partial_l + S_{ab}{}^p \nabla_p \nabla_c \partial_l] = \alpha_{ijk}^{abc} [\nabla_a K_{bcl}{}^p \partial_p + \nabla_a S_{bc}{}^p \nabla_p \partial_l + S_{bc}{}^p \nabla_a \nabla_p \partial_l]$$

equivalently

$$0 = \alpha_{ijk}^{abc} [\nabla_a K_{bcl}{}^p - S_{bc}{}^q K_{aql}{}^p]$$

Next

$$\alpha_{ij}^{ab} \nabla_a \nabla_b X_{k_1 \dots k_n} = -K_{ijk_1}{}^p X_{pk_2 \dots k_n} - \dots - K_{ijk_n}{}^p X_{k_1 \dots k_{n-1} p} - S_{ij}{}^p \nabla_p X_{k_1 \dots k_n}$$

implies

$$\begin{aligned} \alpha_{ij}^{ab} \nabla_a \nabla_b g_{kl} &= -K_{ijk}{}^p g_{pl} - K_{ijl}{}^p g_{pk} - S_{ij}{}^p \nabla_p g_{kl} \\ &= -\sigma_{kl}^{cd} K_{ijcd} - S_{ij}{}^p \nabla_p g_{kl} \end{aligned}$$

Finally

$$\begin{aligned} \nabla_i g_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^a g_{ak} - \Gamma_{ik}^a g_{aj} \\ &=: \partial_i g_{jk} - \Gamma_{ijk} - \Gamma_{ikj} \end{aligned}$$

implies

$$-\nabla_i g_{jk} + \nabla_k g_{ij} + \nabla_j g_{ik} = -\partial_i g_{jk} + \partial_k g_{ij} + \partial_j g_{ik} + S_{ikj} + S_{ijk} - \sigma_{jk}^{bc} \Gamma_{bci}$$

and so

$$2\Gamma_{ijk} = \sigma_{ij}^{ab} \Gamma_{abk} + S_{ijk}$$

becomes

$$\Gamma_{ijk} = \frac{1}{2} [-\partial_k g_{ij} + \partial_j g_{ki} + \partial_i g_{kj} + S_{kij} + S_{kji} + S_{ijk} + \nabla_k g_{ij} - \nabla_j g_{ki} - \nabla_i g_{kj}]$$

If

$$\nabla_i g_{jk} = 0$$

the curvature tensor is labeled

$$C_{ijk}{}^l := \alpha_{ij}^{ab} [\partial_a \Gamma_{bk}^l + \Gamma_{bk}^p \Gamma_{ap}^l]$$

and satisfies

$$0 = \sigma_{kl}^{cd} C_{ijcd}$$

in addition. If

$$\begin{aligned} \nabla_i g_{jk} &= 0 \\ S_{ijk} &= 0 \end{aligned}$$

the curvature tensor is labeled

$$\begin{aligned} R_{ijk}{}^l &:= \alpha_{ij}^{ab} [\partial_a \Gamma_{bk}^l + \Gamma_{bk}^p \Gamma_{ap}^l] \\ R_{ijkl} &= \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} [\partial_a \partial_c g_{bd} + g_{pq} \Gamma_{ac}^p \Gamma_{bd}^q] \end{aligned}$$

and called *Riemannian Curvature tensor*, satisfying

$$\begin{aligned} 0 &= \sigma_{kl}^{cd} R_{ijcd} \\ 0 &= \alpha_{jkl}^{bcd} R_{bcd}{}^p \\ 0 &= \alpha_{ijk}^{abc} \nabla_a R_{bcl}{}^p \end{aligned}$$

3. C

3.1. C

3.1.1. π

The definition of **C** implies

$$0 = \sigma_{ij}^{ab} C_{abkl} \quad (3.1)$$

$$0 = \sigma_{kl}^{cd} C_{ijcd} \quad (3.2)$$

so

$$\begin{aligned} \pi_{ijkl}^{abcd} C_{abcd} &= \frac{1}{12} \alpha_{ij}^{pr} \alpha_{kl}^{qs} \sigma_{pq}^{ac} \sigma_{rs}^{bd} C_{abcd} + \frac{1}{8} \alpha_{kl}^{xy} \alpha_{ijx}^{pbc} \sigma_{py}^{ad} C_{abcd} + \frac{1}{24} \alpha_{ijkl}^{abcd} C_{abcd} \\ &= C_{ijkl}^{\boxplus} + C_{ijkl}^{\boxminus} + C_{ijkl}^{\boxdot} \end{aligned}$$

with

$$C_{ijkl}^{\boxplus} = \frac{1}{2} [C_{ijkl} + C_{klij}] - \frac{1}{6} [[C_{ijkl} + C_{klij}] + [C_{iljk} + C_{jkil}] + [C_{iklj} + C_{ljik}]] \quad (3.3)$$

$$C_{ijkl}^{\boxminus} = \frac{1}{2} [C_{ijkl} - C_{klij}] \quad (3.4)$$

$$C_{ijkl}^{\boxdot} = \frac{1}{6} [[C_{ijkl} + C_{klij}] + [C_{iljk} + C_{jkil}] + [C_{iklj} + C_{ljik}]] \quad (3.5)$$

The distribution of components reads $36 = 20 + 15 + 1$. (3.3) and (3.5) imply

$$C_{ijkl}^{\boxplus} + C_{ijkl}^{\boxdot} = \frac{1}{2} [C_{ijkl} + C_{klij}] \quad (3.6)$$

3.1.2. τ

$$\begin{aligned} \tau_{ijkl}^{abcd} C_{abcd} &= [\tau^{III}]_{ijkl}^{abcd} C_{abcd} + [\tau^{II}]_{ijkl}^{abcd} C_{abcd} + [\tau^I]_{ijkl}^{abcd} C_{abcd} \\ &=: C_{ijkl}^{III} + C_{ijkl}^{II} + C_{ijkl}^I \end{aligned}$$

with

$$\begin{aligned} C_{ijkl}^{III} &:= C_{ijkl} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} X_{bd} \\ C_{ijkl}^{II} &:= \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} [X_{bd} - \gamma_{bd}^{pq} X_{pq}] \\ C_{ijkl}^I &:= \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} X \end{aligned}$$

and

$$\begin{aligned} g^{ik} C_{ijkl} &=: C_{jl} = 2X_{bd} \\ g^{ik} g^{jl} C_{ijkl} &=: C = 24X \end{aligned}$$

so

$$\begin{aligned} X_{bd}^{II} &:= [X_{bd} - \gamma_{bd}^{pq} X_{pq}] = \frac{1}{2} [C_{jl} - \frac{1}{4} g_{jl} C] =: \frac{1}{2} C_{bd}^{II} \\ X &= \frac{1}{24} C \end{aligned}$$

The distribution of components reads $36 = 20 + 15 + 1$.

3.1.3. $\tau \circ \pi$

$$\begin{aligned}\tau \circ \pi \circ \mathbf{C} &= [\tau^{III} + \tau^{II} + \tau^I] \circ [\pi^{\boxplus} + \pi^{\boxminus} + \pi^{\boxdot}] \circ \mathbf{C} \\ &= [\pi^{\boxplus} + \pi^{\boxminus} + \pi^{\boxdot}] \circ [\tau^{III} + \tau^{II} + \tau^I] \circ \mathbf{C} = \pi \circ \tau \circ \mathbf{C}\end{aligned}$$

and in particular

$$\begin{aligned}C_{ijkl}^{\boxplus III} &= \frac{1}{2}[C_{ijkl}^{III} + C_{klij}^{III}] - C_{ijkl}^{\boxdot III} \\ C_{ijkl}^{\boxplus II} &= \frac{1}{2}[C_{ijkl}^{II} + C_{klij}^{II}] \\ &= \frac{1}{4}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}\sigma_{bd}^{pq}C_{pq}^{II} \\ C_{ijkl}^{\boxplus I} &= \frac{1}{2}[C_{ijkl}^I + C_{klij}^I] \\ &= \frac{1}{24}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}g_{bd}C \\ C_{ijkl}^{\boxminus III} &= \frac{1}{2}[C_{ijkl}^{III} - C_{klij}^{III}] \\ C_{ijkl}^{\boxminus II} &= \frac{1}{2}[C_{ijkl}^{II} - C_{klij}^{II}] \\ &= \frac{1}{4}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}\alpha_{bd}^{pq}C_{pq}^{II} \\ C_{ijkl}^{\boxminus I} &= 0 \\ C_{ijkl}^{\boxdot III} &= \frac{1}{6}[[C_{ijkl}^{III} + C_{klij}^{III}] + [C_{iljk}^{III} + C_{jkil}^{III}] + [C_{iklj}^{III} + C_{ljik}^{III}]] \\ C_{ijkl}^{\boxdot II} &= 0 \\ C_{ijkl}^{\boxdot I} &= 0\end{aligned}$$

The distribution of components reads $36 = 10 + 9 + 1 + 9 + 6 + 1$.

3.2. $*\mathbf{C}$

(3.1) and (3.2) imply

$$\begin{aligned}C_{ijkl} &= \frac{1}{2}\alpha_{ij}^{ab}C_{abkl} = \frac{1}{4}\alpha_{ijpq}^{abpq}C_{abkl} = -\frac{1}{4}\eta_{ijpq}\eta^{abpq}C_{abkl} =: -\frac{1}{2}\eta_{ijab}[*C]_{kl}^{ab} \quad (3.7) \\ C_{ijkl} &= \frac{1}{2}\alpha_{kl}^{cd}C_{ijcd} = \frac{1}{4}\alpha_{klrs}^{cdrs}C_{ijcd} = -\frac{1}{4}\eta_{ijrs}\eta^{cdrs}C_{ijcd} =: -\frac{1}{2}\eta_{klcd}[C*]_{ij}^{cd}\end{aligned}$$

3.2.1. π

(3.2) and (3.7) imply

$$\begin{aligned}0 &= \sigma_{ij}^{ab}[*C]_{abkl} \\ 0 &= \sigma_{kl}^{cd}[*C]_{ijcd}\end{aligned}$$

so

$$\pi \circ [*C] = [*C]^{\boxplus} + [*C]^{\boxminus} + [*C]^{\boxdot}$$

and in particular

$$\begin{aligned}
[*C]_{ijkl}^{\boxplus} &= \frac{1}{2}\eta_{ij}^{ab}C_{abkl}^{III} &= & * [C^{III}]_{ijkl} \\
[*C]_{ijkl}^{\boxminus} &= \frac{1}{2}\eta_{ijc}^p\alpha_{kl}^{cd}C_{pd}^{II} &= & * [C^{II}]_{ijkl} \\
[*C]_{ijkl}^{\boxdot} &= \frac{1}{12}\eta_{ijkl}C &= & * [C^I]_{ijkl}
\end{aligned}$$

summarized in

$$\pi \circ [*C] = *[\tau \circ C]$$

The substitution $C \rightarrow *C$ gives

$$\pi \circ [* * C] = -\pi \circ C = *[\tau \circ [*C]] \quad (3.8)$$

so

$$\begin{aligned}
C_{ijkl}^{\boxplus} &= -\frac{1}{2}\eta_{ij}^{ab}[*C]_{abkl}^{III} &= & - * [[*C]^{III}]_{ijkl} \\
C_{ijkl}^{\boxminus} &= -\frac{1}{2}\eta_{ijc}^p\alpha_{kl}^{cd}[*C]_{pd}^{II} &= & - * [[*C]^{II}]_{ijkl} \\
C_{ijkl}^{\boxdot} &= -\frac{1}{12}\eta_{ijkl} * C &= & - * [[*C]^I]_{ijkl}
\end{aligned}$$

The relations (3.4) and (3.6) become

$$\begin{aligned}
[*C]_{ijkl}^{\boxplus} &= \frac{1}{2}[[*C]_{ijkl} - [*C]_{klij}] &= & * [C^{II}]_{ijkl} \\
[*C]_{ijkl}^{\boxplus} + [*C]_{ijkl}^{\boxdot} &= \frac{1}{2}[[*C]_{ijkl} + [*C]_{klij}] &= & * [C^{III}]_{ijkl} + * [C^I]_{ijkl}
\end{aligned}$$

and

$$\begin{aligned}
- * [[*C]^{II}]_{ijkl} &= \frac{1}{2}[C_{ijkl} - C_{klij}] \\
- * [[*C]^{III}]_{ijkl} - * [[*C]^I]_{ijkl} &= \frac{1}{2}[C_{ijkl} + C_{klij}]
\end{aligned}$$

3.2.2. τ

Dualization of (3.8) gives

$$*[\pi \circ C] = \tau \circ [*C]$$

and in particular

$$\begin{aligned}
* [C^{\boxplus}]_{ijkl} &= [*C]_{ijkl} - [*C]_{ijkl}^I - [*C]_{ijkl}^{II} &= & [*C]_{ijkl}^{III} \\
* [C^{\boxminus}]_{ijkl} &= \frac{1}{2}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}[*C]_{bd}^{II} &= & [*C]_{ijkl}^{II} \\
* [C^{\boxdot}]_{ijkl} &= \frac{1}{24}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}g_{bd}[*C] &= & [*C]_{ijkl}^I
\end{aligned}$$

with

$$g^{jl}[*C]_{ijkl} =: [*C]_{ik}$$

$$g^{jl}g^{ik}[*C]_{ijkl} =: *C$$

The substitution $\mathbf{C} \rightarrow *C$ gives

$$\tau \circ [* * C] = -\tau \circ C = *[\pi \circ [*C]]$$

and in particular

$$\begin{aligned} C_{ijkl}^{III} &= \frac{1}{2}[C_{ijkl} - [*C^*]_{klij}] - C_{ijkl}^I \\ &= \frac{1}{6}[C_{ijkl} - [*C^*]_{klij}] - \frac{1}{3}\eta_{ij}^{ab}\alpha_{kl}^{cd}[*C]_{abcd} = -*[[*C]^{\boxplus}]_{ijkl} \\ C_{ijkl}^{II} &= \frac{1}{2}[C_{ijkl} + [*C^*]_{klij}] = -*[[*C]^{\boxminus}]_{ijkl} \\ C_{ijkl}^I &= \frac{1}{3}[C_{ijkl} - [*C^*]_{klij} + \eta_{ij}^{ab}\alpha_{kl}^{cd}[*C]_{abcd}] = -*[[*C]^{\boxdot}]_{ijkl} \end{aligned}$$

The relations (3.4) and (3.6) become

$$\begin{aligned} [*C]_{ijkl}^{II} &= \frac{1}{2}[[*C]_{ijkl} - [C^*]_{klij}] = *[[*C]^{\boxplus}]_{ijkl} \\ [*C]_{ijkl}^{III} + [*C]_{ijkl}^I &= \frac{1}{2}[[*C]_{ijkl} + [C^*]_{klij}] = *[[*C]^{\boxdot}]_{ijkl} \end{aligned}$$

and

$$\begin{aligned} C_{ijkl}^{II} &= \frac{1}{2}[C_{ijkl} + [*C^*]_{klij}] = -*[[*C]^{\boxminus}]_{ijkl} \\ C_{ijkl}^{III} + C_{ijkl}^I &= \frac{1}{2}[C_{ijkl} - [*C^*]_{klij}] = -*[[*C]^{\boxplus}]_{ijkl} - *[[*C]^{\boxdot}]_{ijkl} \end{aligned}$$

3.2.3. $\tau \circ \pi$

$$\begin{aligned} \tau \circ \pi \circ [*C] &= \tau \circ *[\tau \circ C] = *[\pi \circ \tau \circ C] \\ \pi \circ \tau \circ [*C] &= \pi \circ *[\pi \circ C] = *[\tau \circ \pi \circ C] \end{aligned}$$

so

$$\tau \circ *[\tau \circ C] = \pi \circ *[\pi \circ C]$$

and in particular

$$\begin{aligned} [*C]_{ijkl}^{\boxplus III} &= [*C^{III}]_{ijkl}^{III} = *[[*C]^{\boxplus III}]_{ijkl} = *[[*C]^{\boxplus}]_{ijkl}^{\boxplus} \\ [*C]_{ijkl}^{\boxplus II} &= [*C^{III}]_{ijkl}^{II} = *[[*C]^{\boxplus II}]_{ijkl} = *[[*C]^{\boxplus}]_{ijkl}^{\boxplus} \\ [*C]_{ijkl}^{\boxplus I} &= [*C^{III}]_{ijkl}^I = *[[*C]^{\boxplus I}]_{ijkl} = *[[*C]^{\boxplus}]_{ijkl}^{\boxplus} \\ [*C]_{ijkl}^{\boxminus III} &= [*C^{II}]_{ijkl}^{III} = *[[*C]^{\boxminus III}]_{ijkl} = *[[*C]^{\boxminus}]_{ijkl}^{\boxplus} \\ [*C]_{ijkl}^{\boxminus II} &= [*C^{II}]_{ijkl}^{II} = *[[*C]^{\boxminus II}]_{ijkl} = *[[*C]^{\boxminus}]_{ijkl}^{\boxplus} \\ [*C]_{ijkl}^{\boxdot III} &= [*C^I]_{ijkl}^{III} = *[[*C]^{\boxdot III}]_{ijkl} = *[[*C]^{\boxdot}]_{ijkl}^{\boxplus} \end{aligned}$$

3.3. $*\mathbf{C}*$

(3.1) and (3.2) imply

$$C_{ijkl} = \frac{1}{4}\eta_{ij}^{ab}\eta_{kl}^{cd}[*C*]_{abcd} \quad (3.9)$$

with

$$\begin{aligned} [*C*]_{ijkl} &:= \frac{1}{4}\eta_{ij}^{ab}\eta_{kl}^{cd}C_{abcd} \\ &= -C_{klij} + \alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}C_{db} - \frac{1}{4}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}g_{bd}C \end{aligned} \quad (3.10)$$

3.3.1. π

(3.9) implies

$$\begin{aligned} 0 &= \sigma_{ij}^{ab}[*C*]_{abkl} \\ 0 &= \sigma_{kl}^{cd}[*C*]_{ijcd} \end{aligned}$$

therefore

$$\pi \circ [*C*] = [*C*]^{\boxplus} + [*C*]^{\boxminus} + [*C*]^{\boxminus\boxminus} = *[\pi \circ \mathbf{C}]*$$

(3.10) implies

$$\begin{aligned} [*C*]^{\boxplus}_{ijkl} &= -C_{ijkl}^{\boxplus} + \frac{1}{2}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}\sigma_{bd}^{pq}C_{qp} - \frac{1}{4}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}g_{bd}C \\ [*C*]^{\boxminus}_{ijkl} &= C_{ijkl}^{\boxminus} + \frac{1}{2}\alpha_{ij}^{ab}\alpha_{kl}^{cd}g_{ac}\alpha_{bd}^{pq}C_{qp} \\ [*C*]^{\boxminus\boxminus}_{ijkl} &= -C_{ijkl}^{\boxminus\boxminus} \end{aligned}$$

3.3.2. τ

$$\tau \circ [*C*] = [*C*]^{\text{III}} + [*C*]^{\text{II}} + [*C*]^{\text{I}} = *[\tau \circ \mathbf{C}]*$$

and by means of (3.10)

$$[*C*]^{\text{III}}_{ijkl} = -C_{klij}^{\text{III}} \quad (3.11)$$

$$[*C*]^{\text{II}}_{ijkl} = C_{klij}^{\text{II}} \quad (3.12)$$

$$[*C*]^{\text{I}}_{ijkl} = -C_{klij}^{\text{I}} \quad (3.13)$$

with

$$\begin{aligned} g^{jl}[*C*]_{ijkl} &=: [*C*]_{ik} = C_{ki} - \frac{1}{2}Cg_{ik} \\ g^{jl}g^{ik}[*C*]_{ijkl} &=: *C* = -C \end{aligned}$$

3.3.3. $\tau \circ \pi$

(3.11), (3.12), (3.13), (3.4) and (3.6) imply

$$\begin{aligned}
[*C*]_{ijkl}^{\boxplus III} &= -C_{kl ij}^{\boxplus III} = -C_{ijkl}^{\boxplus III} \\
[*C*]_{ijkl}^{\boxplus II} &= C_{kl ij}^{\boxplus II} = C_{ijkl}^{\boxplus II} \\
[*C*]_{ijkl}^{\boxplus I} &= -C_{kl ij}^{\boxplus I} = -C_{ijkl}^{\boxplus I} \\
[*C*]_{ijkl}^{\boxminus III} &= -C_{kl ij}^{\boxminus III} = C_{ijkl}^{\boxminus III} \\
[*C*]_{ijkl}^{\boxminus II} &= C_{kl ij}^{\boxminus II} = -C_{ijkl}^{\boxminus II} \\
[*C*]_{ijkl}^{\boxminus I} &= -C_{kl ij}^{\boxminus I} = -C_{ijkl}^{\boxminus I}
\end{aligned}$$

and, equivalently

$$\begin{aligned}
\frac{1}{2}[C_{ijkl} - [*C*]_{ijkl}] &= C_{ijkl}^{\boxplus III} + C_{ijkl}^{\boxplus I} + C_{ijkl}^{\boxplus II} + C_{ijkl}^{\boxminus III} \\
\frac{1}{2}[C_{ijkl} + [*C*]_{ijkl}] &= C_{ijkl}^{\boxplus II} + C_{ijkl}^{\boxminus III}
\end{aligned}$$

3.4. ρ

3.4.1. \mathbf{C}

$$\begin{aligned}
\rho_{ijkl}^{abcd} C_{abcd} &= C^{abcd} n_{ia} n_{jb} n_{kc} n_{ld} &= [*C*]^{abcd} \eta_{ija} u_b \eta_{klc} u_d \\
&+ C^{abcd} n_{pa} u_b n_{rc} u_d [-\alpha_{ij}^{pq} u_q] [-\alpha_{kl}^{rs} u_s] &+ [*C*]^{abcd} n_{pa} u_b n_{rc} u_d [-\alpha_{ij}^{pq} u_q] [-\alpha_{kl}^{rs} u_s] \\
&+ C^{abcd} n_{pa} u_b n_{kc} n_{ld} [-\alpha_{ij}^{pq} u_q] &+ [C*]^{abcd} n_{pa} u_b \eta_{klc} u_d [-\alpha_{ij}^{pq} u_q] \\
&+ C^{abcd} n_{ia} n_{jb} n_{rc} u_d [-\alpha_{kl}^{rs} u_s] &+ [*C]^{abcd} \eta_{ija} u_b n_{rc} u_d [-\alpha_{kl}^{rs} u_s]
\end{aligned}$$

Set

$$\begin{aligned}
C_{ijkl}^{\mathbf{N}} &:= C_{abcd} [\rho^{\mathbf{N}}]_{ijkl}^{abcd} &:= C^{abcd} n_{ia} n_{jb} n_{kc} n_{ld} \\
&=: \mathbf{N}_{ijkl} &= [*\mathbf{N}*]_{ac} \eta_{ij}^a \eta_{kl}^c \\
C_{ijkl}^{\mathbf{W}} &:= C_{abcd} [\rho^{\mathbf{W}}]_{ijkl}^{abcd} &:= C^{pqrs} n_{ap} u_q n_{cr} u_s [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\
&=: \mathbf{W}_{ac} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\
C_{ijkl}^{\mathbf{LN}} &:= C_{abcd} [\rho^{\mathbf{N}}]_{ijkl}^{abcd} &:= C^{pqcd} n_{ap} u_q n_{kc} n_{ld} [-\alpha_{ij}^{ab} u_b] \\
&=: \mathbf{LN}_{akl} [-\alpha_{ij}^{ab} u_b] &= [\mathbf{LN}*]_{ac} \eta_{kl}^c [-\alpha_{ij}^{ab} u_b] \\
C_{ijkl}^{\mathbf{N}\mathbf{U}} &:= C_{abcd} [\rho^{\mathbf{N}}]_{ijkl}^{abcd} &:= C^{abrs} n_{ia} n_{jb} n_{cr} u_s [-\alpha_{kl}^{cd} u_d] \\
&=: \mathbf{N}\mathbf{U}_{ijc} [-\alpha_{kl}^{cd} u_d] &= [*\mathbf{N}\mathbf{U}]_{ac} \eta_{ij}^a [-\alpha_{kl}^{cd} u_d]
\end{aligned}$$

summarized in

$$\rho \circ \mathbf{C} = \mathbf{C}^{\mathbf{N}} + \mathbf{C}^{\mathbf{W}} + \mathbf{C}^{\mathbf{LN}} + \mathbf{C}^{\mathbf{N}\mathbf{U}}$$

The distribution of components reads $36 = 9 + 9 + 9 + 9$.

3.4.2. *C

$$\rho \circ [*C] = *[\rho \circ C]$$

but

$$\begin{aligned} [*C]_{ijkl}^{MN} &= -[*LN]_{ijkl} &= -[LN*]_{ac}\eta_{ij}^a\eta_{kl}^c &= *[C^{LN}]_{ijkl} \\ [*C]_{ijkl}^{UU} &= [*N]_{ac}[-\alpha_{ij}^{ab}u_b][-\alpha_{kl}^{cd}u_d] &= &= *[C^{NU}]_{ijkl} \\ [*C]_{ijkl}^{LN} &= [*MN]_{akl}[-\alpha_{ij}^{ab}u_b] &= [*MN*]_{ac}\eta_{kl}^c[-\alpha_{ij}^{ab}u_b] &= *[C^{MN}]_{ijkl} \\ [*C]_{ijkl}^{NU} &= -[*U]_{ijc}[-\alpha_{kl}^{cd}u_d] &= -U_{ac}\eta_{ij}^a[-\alpha_{kl}^{cd}u_d] &= *[C^U]_{ijkl} \end{aligned}$$

3.4.3. *C*

$$\rho \circ [*C*] = *[\rho \circ C]*$$

but

$$\begin{aligned} [*C*]_{ijkl}^{MN} &= [*U*]_{ijkl} &= U_{ac}\eta_{ij}^a\eta_{kl}^c &= *[C^U*]_{ijkl} \\ [*C*]_{ijkl}^{UU} &= [*N*]_{ac}[-\alpha_{ij}^{ab}u_b][-\alpha_{kl}^{cd}u_d] &= &= *[C^{MN}*]_{ijkl} \\ [*C*]_{ijkl}^{LN} &= -[*N*]_{akl}[-\alpha_{ij}^{ab}u_b] &= -[*N*]_{ac}\eta_{kl}^c[-\alpha_{ij}^{ab}u_b] &= *[C^{NU}*]_{ijkl} \\ [*C*]_{ijkl}^{NU} &= -[*U*]_{ijc}[-\alpha_{kl}^{cd}u_d] &= -[LN*]_{ac}\eta_{ij}^a[-\alpha_{kl}^{cd}u_d] &= *[C^U*]_{ijkl} \end{aligned}$$

3.5. $\rho \circ \pi$

$$\begin{aligned} \rho \circ \pi \circ C &= [\rho^{MN} + \rho^U + \rho^{LN} + \rho^{NU}] \circ [\pi^{\boxplus} + \pi^{\boxminus} + \pi^{\boxdot}] \circ C \\ &= [\pi^{\boxplus} + \pi^{\boxminus} + \pi^{\boxdot}] \circ [\rho^{MN} + \rho^U + \rho^{LN} + \rho^{NU}] \circ C = \pi \circ \rho \circ C \end{aligned}$$

and

$$\rho \circ \pi \circ C = -\rho \circ *[\tau \circ [*C]]$$

in particular

$$\begin{aligned} [*N*]_{ik}^{\boxplus} &= \frac{1}{2}\sigma_{ik}^{ac}[*N*]_{ac} & W_{ik}^{\boxplus} &= \frac{1}{2}\sigma_{ik}^{ac}W_{ac} \\ [*N*]_{ik}^{\boxminus} &= \frac{1}{2}\alpha_{ik}^{ac}[*N*]_{ac} & W_{ik}^{\boxminus} &= \frac{1}{2}\alpha_{ik}^{ac}W_{ac} \\ [*N*]_{ik}^{\boxdot} &= 0 & W_{ik}^{\boxdot} &= 0 \\ [UN*]_{ik}^{\boxplus} &= \frac{1}{2}[[UN*]_{ik} + [*N]_{ki}] - [UN*]_{ik}^{\boxdot} & [*N]_{ik}^{\boxplus} &= [UN*]_{ki}^{\boxplus} \\ [UN*]_{ik}^{\boxminus} &= \frac{1}{2}[[UN*]_{ik} - [*N]_{ki}] & [*N]_{ik}^{\boxminus} &= -[UN*]_{ki}^{\boxminus} \\ [UN*]_{ik}^{\boxdot} &= \frac{1}{2}\gamma_{ik}^{ac}[[UN*]_{ac} + [*N]_{ca}] & [*N]_{ik}^{\boxdot} &= [UN*]_{ki}^{\boxdot} \end{aligned}$$

and

$$[*N*]_{ik}^{\boxplus} = \frac{1}{2}\eta_i^{ab}u^p[*C]_{kpb}^{III} \quad W_{ik}^{\boxplus} = -\frac{1}{2}\eta_i^{ab}u^p[*C]_{abkp}^{III}$$

$$\begin{aligned}
[*\mathcal{N}^*]_{ik}^{\boxplus} &= -\frac{1}{2}\eta_{ik}{}^p u^q [*C]_{qp}^{II} & W_{ik}^{\boxplus} &= -\frac{1}{2}\eta_{ik}{}^p u^q [*C]_{pq}^{II} \\
[UN^*]_{ik}^{\boxplus} &= u^b u^d [*C]_{kbid}^{III} \\
[UN^*]_{ik}^{\boxplus\boxplus} &= \frac{1}{2}[n^a{}_i n^c{}_k - n_{ik} n^{ac}] [*C]_{ca}^{II} \\
[UN^*]_{ik}^{\boxplus\boxplus\boxplus} &= -\frac{1}{12}n_{ik} [*C]
\end{aligned}$$

The distribution of components is 6, 3, 6, 3, 8, 9, 1.

3.6. $\rho \circ \tau$

$$\begin{aligned}
\rho \circ \tau \circ \mathbf{C} &= [\rho^{\mathcal{N}^{\mathcal{W}}} + \rho^{\mathcal{W}} + \rho^{\mathcal{U}^{\mathcal{N}}} + \rho^{\mathcal{N}^{\mathcal{U}}}] \circ [\tau^{III} + \tau^{II} + \tau^I] \circ \mathbf{C} \\
&= [\tau^{III} + \tau^{II} + \tau^I] \circ [\rho^{\mathcal{N}^{\mathcal{W}}} + \rho^{\mathcal{W}} + \rho^{\mathcal{U}^{\mathcal{N}}} + \rho^{\mathcal{N}^{\mathcal{U}}}] \circ \mathbf{C} = \tau \circ \rho \circ \mathbf{C}
\end{aligned}$$

in particular

$$\begin{aligned}
[*\mathcal{N}^*]_{ik}^{III} &= \frac{1}{2}[[*\mathcal{N}^*]_{ik} - W_{ki}] - [*\mathcal{N}^*]_{ik}^I & W_{ik}^{III} &= -[*\mathcal{N}^*]_{ki}^{III} \\
[*\mathcal{N}^*]_{ik}^{II} &= \frac{1}{2}[[*\mathcal{N}^*]_{ik} + W_{ki}] & W_{ik}^{II} &= [*\mathcal{N}^*]_{ki}^{II} \\
[*\mathcal{N}^*]_{ik}^I &= \frac{1}{2}\gamma_{ik}^{ac} [[*\mathcal{N}^*]_{ac} - W_{ca}] & W_{ik}^I &= -[*\mathcal{N}^*]_{ki}^I \\
[UN^*]_{ik}^{III} &= \frac{1}{2}\sigma_{ik}^{ac} [UN^*]_{ac} & [*N^*]_{ik}^{III} &= \frac{1}{2}\sigma_{ik}^{ac} [*N^*]_{ac} \\
[UN^*]_{ik}^{II} &= \frac{1}{2}\alpha_{ik}^{ac} [UN^*]_{ac} & [*N^*]_{ik}^{II} &= \frac{1}{2}\alpha_{ik}^{ac} [*N^*]_{ac} \\
[UN^*]_{ik}^I &= 0 & [*N^*]_{ik}^I &= 0
\end{aligned}$$

and

$$\begin{aligned}
[*\mathcal{N}^*]_{ik}^{III} &= -u^p u^q C_{kpiq}^{III} \\
[*\mathcal{N}^*]_{ik}^{II} &= \frac{1}{2}[n_{ik} n^{ac} - n^c{}_i n^a{}_k] C_{ac}^{II} \\
[*\mathcal{N}^*]_{ik}^I &= \frac{1}{12}n_{ik} C \\
[UN^*]_{ik}^{III} &= \frac{1}{2}\eta_k{}^{cd} u^p C_{ipcd}^{III} & [*N^*]_{ik}^{III} &= \frac{1}{2}\eta_i{}^{ab} u^q C_{abkq}^{III} \\
[UN^*]_{ik}^{II} &= -\frac{1}{2}\eta_{ik}{}^c u^a C_{ac}^{II} & [*N^*]_{ik}^{II} &= \frac{1}{2}\eta_{ik}{}^a u^c C_{ac}^{II}
\end{aligned}$$

The distribution of components is 8, 9, 1, 6, 3, 6, 3.

3.7. $\rho \circ \tau \circ \pi$

$$\rho \circ \tau \circ \pi \circ \mathbf{C} = [\rho^{\mathcal{N}^{\mathcal{W}}} + \rho^{\mathcal{U}^{\mathcal{N}}} + \rho^{\mathcal{N}^{\mathcal{U}}} + \rho^{\mathcal{W}}] \circ [\tau^{III} + \tau^{II} + \tau^I] \circ [\rho^{\boxplus} + \rho^{\boxplus\boxplus} + \rho^{\boxplus\boxplus\boxplus}] \circ \mathbf{C}$$

and

$$[*\mathcal{N}^*]_{ik}^{\boxplus\boxplus\boxplus} = \frac{1}{4}\sigma_{ik}^{ac} [[*\mathcal{N}^*]_{ac} - W_{ac}] - [*\mathcal{N}^*]_{ik}^{\boxplus\boxplus I} \quad W_{ik}^{\boxplus\boxplus\boxplus} = -[*\mathcal{N}^*]_{ik}^{\boxplus\boxplus III}$$

$$\begin{aligned}
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} &= \frac{1}{4}\sigma_{ik}^{ac} [[*\mathcal{N}\mathcal{W}^*]_{ac} + U_{ac}] & W_{ik}^{\boxplus II} &= [*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus I} &= \frac{1}{4}\sigma_{ik}^{ac} \gamma^{pq} [[*\mathcal{N}\mathcal{W}^*]_{pq} - U_{pq}] & W_{ik}^{\boxplus I} &= -[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus I} \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus III} &= \frac{1}{4}\alpha_{ik}^{ac} [[*\mathcal{N}\mathcal{W}^*]_{ac} + U_{ac}] & W_{ik}^{\boxplus III} &= [*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus III} \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} &= \frac{1}{4}\alpha_{ik}^{ac} [[*\mathcal{N}\mathcal{W}^*]_{ac} - U_{ac}] & W_{ik}^{\boxplus II} &= -[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} \\
[UN^*]_{ik}^{\boxplus III \boxplus} &= \frac{1}{4}\sigma_{ik}^{ac} [[UN^*]_{ac} + [*\mathcal{N}U]_{ac}] - [UN^*]_{ik}^{\boxplus III \boxplus} & [*\mathcal{N}U]_{ik}^{\boxplus III \boxplus} &= [UN^*]_{ik}^{\boxplus III \boxplus} \\
[UN^*]_{ik}^{\boxplus III \boxplus} &= \frac{1}{4}\sigma_{ik}^{ac} [[UN^*]_{ac} - [*\mathcal{N}U]_{ac}] & [*\mathcal{N}U]_{ik}^{\boxplus III \boxplus} &= -[UN^*]_{ik}^{\boxplus III \boxplus} \\
[UN^*]_{ik}^{\boxplus III \boxplus} &= \frac{1}{4}\sigma_{ik}^{ac} \gamma^{pq} [[UN^*]_{pq} + [*\mathcal{N}U]_{pq}] & [*\mathcal{N}U]_{ik}^{\boxplus III \boxplus} &= [UN^*]_{ik}^{\boxplus III \boxplus} \\
[UN^*]_{ik}^{\boxplus II \boxplus} &= \frac{1}{4}\alpha_{ik}^{ac} [[UN^*]_{ac} - [*\mathcal{N}U]_{ac}] & [*\mathcal{N}U]_{ik}^{\boxplus II \boxplus} &= -[UN^*]_{ik}^{\boxplus II \boxplus} \\
[UN^*]_{ik}^{\boxplus II \boxplus} &= \frac{1}{4}\alpha_{ik}^{ac} [[UN^*]_{ac} + [*\mathcal{N}U]_{ac}] & [*\mathcal{N}U]_{ik}^{\boxplus II \boxplus} &= [UN^*]_{ik}^{\boxplus II \boxplus}
\end{aligned}$$

and

$$\rho \circ \tau \circ \pi \circ \mathbf{C} = -\rho \circ [* \pi \circ \tau \circ \mathbf{C}]$$

therefore

$$\begin{aligned}
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus III} &= -\frac{1}{2}\sigma_{ik}^{ac} u^b u^d C_{abcd}^{III} \\
&= \frac{1}{4}[\eta_i^{ab} n_k^c - \frac{1}{3}n_{ik}\eta^{abc}] u^d [[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} &= \frac{1}{4}\sigma_{ik}^{ac} [n_{ac} u^p u^q - n_a^p n_c^q] C_{pq}^{II} \\
&= -\frac{1}{4}\eta_i^{ab} u^d [[*C]_{abkd}^{III} - [*C]_{kdab}^{III}] \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus I} &= \frac{1}{12}n_{ik}C \\
&= \frac{1}{12}n_{ik}\eta^{abc} u^d [[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus III} &= \frac{1}{2}\alpha_{ik}^{ac} u^b u^d C_{abcd}^{III} \\
&= -\frac{1}{4}\eta_{ik}^{b} u^d \sigma_{bd}^{pq} [*C]_{pq}^{II} \\
[*\mathcal{N}\mathcal{W}^*]_{ik}^{\boxplus II} &= \frac{1}{4}\alpha_{ik}^{ac} n_a^p n_c^q C_{pq}^{II} \\
&= \frac{1}{4}\eta_{ik}^{b} u^d \alpha_{bd}^{pq} [*C]_{pq}^{II} \\
[UN^*]_{ik}^{\boxplus III \boxplus} &= \frac{1}{4}[\eta_i^{ab} n_k^c - \frac{1}{3}n_{ik}\eta^{abc}] u^d [C_{abcd}^{III} + C_{cdab}^{III}] \\
&= \frac{1}{2}\sigma_{ik}^{ac} u^b u^d [*C]_{abcd}^{III} \\
[UN^*]_{ik}^{\boxplus III \boxplus} &= -\frac{1}{4}\eta_i^{ab} u^d [C_{abkd}^{III} - C_{kdab}^{III}] \\
&= -\frac{1}{4}\sigma_{ik}^{ac} [n_{ac} u^p u^q - n_a^p n_c^q] [*C]_{pq}^{II}
\end{aligned}$$

$$\begin{aligned}
[UN*]_{ik}^{III\boxed{\square}} &= \frac{1}{12}n_{ik}\eta^{abc}u^d[C_{abcd}^{III} + C_{cdab}^{III}] \\
&= -\frac{1}{12}n_{ik}*C \\
[UN*]_{ik}^{II\boxplus} &= -\frac{1}{4}\eta_{ik}{}^b u^d \sigma_{bd}^{pq} C_{pq}^{II} \\
&= -\frac{1}{2}\alpha_{ik}^{ac} u^b u^d [*C]_{abcd}^{III} \\
[UN*]_{ik}^{II\boxminus} &= \frac{1}{4}\eta_{ik}{}^b u^d \alpha_{bd}^{pq} C_{pq}^{II} \\
&= -\frac{1}{4}\alpha_{ik}^{ac} n_a^p n_c^q [*C]_{pq}^{II}
\end{aligned}$$

and

$$\begin{aligned}
-\sigma_{ik}^{ac} u^b u^d C_{abcd}^{III} &= \frac{1}{2}[\eta_i{}^{ab} n_k^c - \frac{1}{3}n_{ik}\eta^{abc}]u^d[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
\sigma_{ik}^{ac}[n_{ac}u^p u^q - n_a^p n_c^q]C_{pq}^{II} &= -\eta_i{}^{ab} u^d[[*C]_{abkd}^{III} - [*C]_{kdab}^{III}] \\
C &= \eta^{abc} u^d[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
\alpha_{ik}^{ac} u^b u^d C_{abcd}^{III} &= -\frac{1}{2}\eta_{ik}{}^b u^d \sigma_{bd}^{pq} [*C]_{pq}^{II} \\
\alpha_{ik}^{ac} n_a^p n_c^q C_{pq}^{II} &= \eta_{ik}{}^b u^d \alpha_{bd}^{pq} [*C]_{pq}^{II} \\
\frac{1}{2}[\eta_i{}^{ab} n_k^c - \frac{1}{3}n_{ik}\eta^{abc}]u^d[C_{abcd}^{III} + C_{cdab}^{III}] &= \sigma_{ik}^{ac} u^b u^d [*C]_{abcd}^{III} \\
\eta_i{}^{ab} u^d[C_{abkd}^{III} - C_{kdab}^{III}] &= \sigma_{ik}^{ac}[n_{ac}u^p u^q - n_a^p n_c^q][*C]_{pq}^{II} \\
\eta^{abc} u^d[C_{abcd}^{III} + C_{cdab}^{III}] &= -*C \\
\frac{1}{2}\eta_{ik}{}^b u^d \sigma_{bd}^{pq} C_{pq}^{II} &= \alpha_{ik}^{ac} u^b u^d [*C]_{abcd}^{III} \\
\eta_{ik}{}^b u^d \alpha_{bd}^{pq} C_{pq}^{II} &= -\alpha_{ik}^{ac} n_a^p n_c^q [*C]_{pq}^{II}
\end{aligned}$$

or

$$\begin{aligned}
u^b u^d C_{ibkd}^{III\boxplus} &= -\frac{1}{2}\eta_i{}^{ab} u^d [*C]_{abkd}^{III\boxplus} \\
[n_{ik}u^a u^c - n_i^a n_k^c]g^{bd}C_{abcd}^{II\boxplus} &= -\eta_i{}^{ab} u^d [*C]_{abkd}^{III\boxplus} \\
g^{ac}g^{bd}C_{abcd}^{II\boxminus} &= 2\eta^{abc} u^d [*C]_{abcd}^{III\boxplus} \\
2u^b u^d C_{ibkd}^{III\boxminus} &= -\eta_{ik}{}^a u^c g^{bd} [*C]_{abcd}^{II\boxplus} \\
n_i^a n_k^c g^{bd} C_{abcd}^{II\boxminus} &= \eta_{ik}{}^a u^c g^{bd} [*C]_{abcd}^{II\boxplus} \\
\eta_i{}^{ab} u^d C_{abkd}^{III\boxplus} &= 2u^b u^d [*C]_{ibkd}^{III\boxplus} \\
\eta_i{}^{ab} u^d C_{abkd}^{III\boxminus} &= [n_{ik}u^a u^c - n_i^a n_k^c]g^{bd} [*C]_{abcd}^{II\boxplus} \\
2\eta^{abc} u^d C_{abcd}^{III\boxplus} &= -g^{ac}g^{bd} [*C]_{abcd}^{II\boxplus} \\
\eta_{ik}{}^a u^c g^{bd} C_{abcd}^{II\boxplus} &= 2u^b u^d [*C]_{ibkd}^{III\boxplus} \\
\eta_{ik}{}^a u^c g^{bd} C_{abcd}^{II\boxminus} &= -n_i^a n_k^c g^{bd} [*C]_{abcd}^{II\boxplus}
\end{aligned}$$

The distribution of components is 5, 6, 1, 3, 3, 5, 6, 1, 3, 3.

3.8. $\omega \circ \rho$

Let $u = (0, 0, 0, 1)^T$.

$$\begin{aligned}\omega_{ac}^{pq}[*N*]_{pq} &= \sum_{\alpha=1}^3 [*N]_{\alpha a}^{\hat{N}} [*N]_{\alpha c}^{\hat{N}} \\ \omega_{ac}^{pq}U_{pq} &= \sum_{\alpha=1}^3 U_{\alpha a}^{\hat{U}} U_{\alpha c}^{\hat{U}} \\ \omega_{ac}^{pq}[UN*]_{pq} &= \sum_{\alpha=1}^3 U_{\alpha a}^{UN} [*N]_{\alpha c}^{UN} \\ \omega_{ac}^{pq}[*N\mathcal{U}]_{pq} &= \sum_{\alpha=1}^3 [*N]_{\alpha a}^{N\mathcal{U}} U_{\alpha c}^{N\mathcal{U}}\end{aligned}$$

gives

$$\begin{aligned}\omega \circ C_{ijkl}^{N\mathcal{U}} &= \sum_{\alpha=1}^3 \eta_{ij}^a [*N]_{\alpha a}^{\hat{N}} \eta_{kl}^c [*N]_{\alpha c}^{\hat{N}} = \sum_{\alpha=1}^3 N_{\alpha ij}^{\hat{N}} N_{\alpha kl}^{\hat{N}} \\ \omega \circ C_{ijkl}^{U\mathcal{U}} &= \sum_{\alpha=1}^3 [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{\hat{U}}] [-\alpha_{kl}^{cd} u_d U_{\alpha c}^{\hat{U}}] = \sum_{\alpha=1}^3 U_{\alpha ij}^{\hat{U}} U_{\alpha kl}^{\hat{U}} \\ \omega \circ C_{ijkl}^{UN} &= \sum_{\alpha=1}^3 [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{UN}] \eta_{kl}^c [*N]_{\alpha c}^{UN} = \sum_{\alpha=1}^3 U_{\alpha ij}^{UN} N_{\alpha kl}^{UN} \\ \omega \circ C_{ijkl}^{N\mathcal{U}} &= \sum_{\alpha=1}^3 [-\alpha_{kl}^{cd} u_d U_{\alpha c}^{N\mathcal{U}}] \eta_{ij}^a [*N]_{\alpha a}^{N\mathcal{U}} = \sum_{\alpha=1}^3 N_{\alpha ij}^{N\mathcal{U}} U_{\alpha kl}^{N\mathcal{U}}\end{aligned}$$

so

$$\omega \circ \rho \circ C_{ijkl} = \sum_{\alpha=1}^3 [N_{\alpha ij}^{\hat{N}} N_{\alpha kl}^{\hat{N}} + U_{\alpha ij}^{\hat{U}} U_{\alpha kl}^{\hat{U}} + U_{\alpha ij}^{UN} N_{\alpha kl}^{UN} + N_{\alpha ij}^{N\mathcal{U}} U_{\alpha kl}^{N\mathcal{U}}]$$

N_α and U_α have rank 2. N_α are spacelike and U_α are timelike. Setting

$$\begin{aligned}N_{\alpha ij}^{+\hat{N}\mathcal{U}} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}} + N_{\alpha ij}^{N\mathcal{U}}] & U_{\alpha ij}^{+\hat{U}} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}} + U_{\alpha ij}^{UN}] \\ N_{\alpha ij}^{-\hat{N}\mathcal{U}} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}} - N_{\alpha ij}^{N\mathcal{U}}] & U_{\alpha ij}^{-\hat{U}} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}} - U_{\alpha ij}^{UN}] \\ N_{\alpha ij}^{+\hat{U}} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}} + N_{\alpha ij}^{UN}] & U_{\alpha ij}^{+\hat{N}\mathcal{U}} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}} + U_{\alpha ij}^{N\mathcal{U}}] \\ N_{\alpha ij}^{-\hat{U}} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}} - N_{\alpha ij}^{UN}] & U_{\alpha ij}^{-\hat{N}\mathcal{U}} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}} - U_{\alpha ij}^{N\mathcal{U}}]\end{aligned}$$

the factorization becomes

$$\begin{aligned}[\omega \circ \rho \circ C]_{ijkl} &= \sum_{\alpha=1}^3 [[N_{\alpha ij}^{+\hat{N}\mathcal{U}} + U_{\alpha ij}^{+\hat{U}}][N_{\alpha kl}^{+\hat{U}} + U_{\alpha kl}^{+\hat{N}\mathcal{U}}] + [N_{\alpha ij}^{-\hat{N}\mathcal{U}} + U_{\alpha ij}^{-\hat{U}}][N_{\alpha kl}^{-\hat{U}} + U_{\alpha kl}^{-\hat{N}\mathcal{U}}] \\ &\quad + [N_{\alpha ij}^{+\hat{U}} - U_{\alpha ij}^{+\hat{N}\mathcal{U}}][N_{\alpha kl}^{-\hat{U}} - U_{\alpha kl}^{-\hat{N}\mathcal{U}}] + [N_{\alpha ij}^{-\hat{U}} - U_{\alpha ij}^{-\hat{N}\mathcal{U}}][N_{\alpha kl}^{+\hat{U}} - U_{\alpha kl}^{+\hat{N}\mathcal{U}}]]\end{aligned}$$

$[N + U]_{\alpha ij}$ and $[N - U]_{\alpha ij}$ have rank 4.

4. \mathbf{R}

4.1. \mathbf{R}

4.1.1. π

$$\pi \circ \mathbf{R} = \mathbf{R}^{\boxplus}$$

4.1.2. τ

$$\tau \circ \mathbf{R} = \mathbf{R}^{III} + \mathbf{R}^{II} + \mathbf{R}^I$$

with

$$\begin{aligned} R_{ijkl}^{III} &:= R_{ijkl} - R_{ijkl}^{II} - R_{ijkl}^I \\ R_{ijkl}^{II} &:= \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} [R_{bd} - \frac{1}{4} g_{bd} R] =: \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} R_{bd}^{II} \\ R_{ijkl}^I &:= \frac{1}{24} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} R \end{aligned}$$

and

$$\begin{aligned} g^{jl} R_{ijkl} &=: R_{ik} \\ g^{jl} g^{ik} R_{ijkl} &=: R \end{aligned}$$

4.2. $*\mathbf{R}$

4.2.1. τ

$$\tau \circ [* \mathbf{R}] = [* \mathbf{R}]^{III}$$

4.2.2. π

$$\pi \circ [* \mathbf{R}] = [*R]_{ijkl}^{\boxplus} + [*R]_{ijkl}^{\boxminus} + [*R]_{ijkl}^{\boxtimes}$$

with

$$\begin{aligned} [*R]_{ijkl}^{\boxplus} &= \frac{1}{2} \eta_{ij}^{ab} R_{abkl}^{III} = [*R^{III}]_{ijkl} \\ [*R]_{ijkl}^{\boxminus} &= \frac{1}{2} \alpha_{kl}^{cd} \eta_{ijc}^p R_{pd}^{II} = [*R^{II}]_{ijkl} \\ [*R]_{ijkl}^{\boxtimes} &= \frac{1}{12} \eta_{ijkl} R = [*R^I]_{ijkl} \end{aligned}$$

and

$$\begin{aligned} [*R]_{ijkl}^{\boxminus} &= \frac{1}{2} [[*R]_{ijkl} - [R^*]_{ijkl}] = [*R^{II}]_{ijkl} \\ [*R]_{ijkl}^{\boxplus} + [*R]_{ijkl}^{\boxtimes} &= \frac{1}{2} [[*R]_{ijkl} + [R^*]_{ijkl}] = [*R^{III}]_{ijkl} + [*R^I]_{ijkl} \end{aligned}$$

4.3. *R*

4.3.1. π

$$\pi_{ijkl}^{abcd} [*R*]_{abcd} = [*R*]_{ijkl}^{\boxplus} = -R_{ijkl} + \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} R_{bd} - \frac{1}{4} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} R$$

4.3.2. τ

$$\tau_{ijkl}^{abcd} [*R*]_{abcd} = [*R*]_{ijkl}^{III} + [*R*]_{ijkl}^{II} + [*R*]_{ijkl}^I = -R_{ijkl}^{III} + R_{ijkl}^{II} - R_{ijkl}^I$$

and

$$\begin{aligned} R_{ijkl}^{III} + R_{ijkl}^I &= \frac{1}{2} [R_{ijkl} - [*R*]_{ijkl}] \\ R_{ijkl}^{II} &= \frac{1}{2} [R_{ijkl} + [*R*]_{ijkl}] \end{aligned}$$

with

$$\begin{aligned} g^{jl} [*R*]_{ijkl} &=: [*R*]_{ik} = R_{ik} - \frac{1}{2} R g_{ik} \\ g^{jl} g^{ik} [*R*]_{ijkl} &=: *R* = -R \end{aligned}$$

4.4. ρ

4.4.1. \mathbf{R}

$$\rho \circ \mathbf{R} = \mathbf{R}^{MN} + \mathbf{R}^{UN} + \mathbf{R}^{NU} + \mathbf{R}^W$$

with

$$\begin{aligned} R_{ijkl}^{MN} &:= R_{abcd} [\rho^{MN}]_{ijkl}^{abcd} &:= R^{abcd} n_{ia} n_{jb} n_{kc} n_{ld} \\ &=: MN_{ijkl}^{\boxplus} &= [*MN*]_{ac}^{\boxplus} \eta_{ij}^a \eta_{kl}^c \\ R_{ijkl}^{UN} &:= R_{abcd} [\rho^{UN}]_{ijkl}^{abcd} &:= R^{pqrs} n_{ap} u_q n_{cr} u_s [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\ &=: UN_{ac}^{\boxplus} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\ R_{ijkl}^{NU} &:= R_{abcd} [\rho^{NU}]_{ijkl}^{abcd} &:= R^{pqcd} n_{ap} u_q n_{kc} n_{ld} [-\alpha_{ij}^{ab} u_b] \\ &=: UN_{akl}^{\boxplus} [-\alpha_{ij}^{ab} u_b] &= [UN*]_{ac}^{\boxplus} [-\alpha_{ij}^{ab} u_b] \eta_{kl}^c \\ R_{ijkl}^{NU} &:= R_{abcd} [\rho^{NU}]_{ijkl}^{abcd} &:= R^{abrs} n_{ia} n_{jb} n_{cr} u_s [-\alpha_{kl}^{cd} u_d] \\ &=: NU_{ijc}^{\boxplus} [-\alpha_{kl}^{cd} u_d] &= [*NU]_{ac}^{\boxplus} \eta_{ij}^a [-\alpha_{kl}^{cd} u_d] \end{aligned}$$

4.4.2. *R

$$\begin{aligned} [*R]_{ijkl}^{MN} &= -[*UN]_{ijkl}^{\boxplus} &= -[UN*]_{ac}^{\boxplus} \eta_{ij}^a \eta_{kl}^c &= * [R^{UN}]_{ijkl} \\ [*R]_{ijkl}^{UN} &= [*NU]_{ac}^{\boxplus} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] &= &= * [R^{NU}]_{ijkl} \\ [*R]_{ijkl}^{UN} &= [*MN]_{akl}^{\boxplus} [-\alpha_{ij}^{ab} u_b] &= [*MN*]_{ac}^{\boxplus} \eta_{kl}^c [-\alpha_{ij}^{ab} u_b] &= * [R^{MN}]_{ijkl} \\ [*R]_{ijkl}^{NU} &= -[*UW]_{ijc}^{\boxplus} [-\alpha_{kl}^{cd} u_d] &= -U_{ac}^{\boxplus} \eta_{ij}^a [-\alpha_{kl}^{cd} u_d] &= * [R^W]_{ijkl} \end{aligned}$$

4.4.3. $*\mathbf{R}*$

$$\begin{aligned}
[*R*]_{ijkl}^{\mathbf{N}} &= [*U*]_{ijkl}^{\boxplus} &= W_{ac}^{\boxplus} \eta_{ij}{}^a \eta_{kl}{}^c &= [*R^U*]_{ijkl} \\
[*R*]_{ijkl}^{\mathbf{U}} &= [*N*]_{ac}^{\boxplus} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] &= [*R^{\mathbf{N}}*]_{ijkl} \\
[*R*]_{ijkl}^{\mathbf{N}} &= -[*N*]_{akl}^{\boxplus} [-\alpha_{ij}^{ab} u_b] &= -[*N]_{ac}^{\boxplus} \eta_{kl}{}^c [-\alpha_{ij}^{ab} u_b] &= [*R^{\mathbf{N}}*]_{ijkl} \\
[*R*]_{ijkl}^{\mathbf{U}} &= -[*UN*]_{ijc}^{\boxplus} [-\alpha_{kl}^{cd} u_d] &= -[UN*]_{ac}^{\boxplus} \eta_{ij}{}^a [-\alpha_{kl}^{cd} u_d] &= [*R^{\mathbf{U}}*]_{ijkl}
\end{aligned}$$

4.5. $\rho \circ \tau$

$$\rho \circ \tau \circ \mathbf{R} = [\rho^{\mathbf{N}} + \rho^{\mathbf{U}} + \rho^{\mathbf{N}} + \rho^{\mathbf{U}}] \circ [\tau^{\mathbf{III}} + \tau^{\mathbf{II}} + I] \circ \mathbf{R}$$

and

$$\begin{aligned}
[*N*]_{ik}^{\boxplus \mathbf{III}} &= \frac{1}{4} \sigma_{ik}^{ac} [[*N*]_{ac}^{\boxplus} - W_{ac}^{\boxplus}] - [*N*]_{ik}^{\boxplus I} & W_{ik}^{\boxplus \mathbf{III}} &= -[*N*]_{ik}^{\boxplus \mathbf{III}} \\
[*N*]_{ik}^{\boxplus \mathbf{II}} &= \frac{1}{4} \sigma_{ik}^{ac} [[*N*]_{ac}^{\boxplus} + W_{ac}^{\boxplus}] & W_{ik}^{\boxplus \mathbf{II}} &= [*N*]_{ik}^{\boxplus \mathbf{II}} \\
[*N*]_{ik}^{\boxplus I} &= \frac{1}{4} \sigma_{ik}^{ac} \gamma^{pq} [[*N*]_{pq}^{\boxplus} - W_{pq}^{\boxplus}] & W_{ik}^{\boxplus I} &= -[*N*]_{ik}^{\boxplus I} \\
[UN*]_{ik}^{\boxplus \mathbf{III}} &= \frac{1}{4} \sigma_{ik}^{ac} [[UN*]_{ac}^{\boxplus} + [*N]_{ac}^{\boxplus}] & [*N]_{ik}^{\boxplus \mathbf{III}} &= [UN*]_{ik}^{\boxplus \mathbf{III}} \\
[UN*]_{ik}^{\boxplus \mathbf{II}} &= \frac{1}{4} \alpha_{ik}^{ac} [[UN*]_{ac}^{\boxplus} - [*N]_{ac}^{\boxplus}] & [*N]_{ik}^{\boxplus \mathbf{II}} &= -[UN*]_{ik}^{\boxplus \mathbf{II}}
\end{aligned}$$

and

$$\begin{aligned}
[*N*]_{ik}^{\boxplus \mathbf{III}} &= -u^b u^d R_{ibkd}^{\mathbf{III}} \\
&= \frac{1}{4} [\eta_i{}^{ab} n^c{}_k - \frac{1}{3} n_{ik} \eta^{abc}] u^d [[*R]_{abcd} + [*R]_{cdab}] \\
[*N*]_{ik}^{\boxplus \mathbf{II}} &= \frac{1}{2} [n_{ik} u^p u^q - n_i^p n_k^q] R_{pq}^{\mathbf{II}} \\
&= -\frac{1}{4} \eta_i{}^{ab} u^d [[*R]_{abkd} - [*R]_{kdab}] \\
[*N*]_{ik}^{\boxplus I} &= \frac{1}{12} n_{ik} R \\
&= \frac{1}{12} n_{ik} \eta^{abc} u^d [[*R]_{abcd} + [*R]_{cdab}] \\
[UN*]_{ik}^{\boxplus \mathbf{III}} &= \frac{1}{2} \eta_i{}^{ab} u^d R_{kdab}^{\mathbf{III}} \\
&= \frac{1}{2} \sigma_{ik}^{ac} u^b u^d [*R]_{abcd} \\
[UN*]_{ik}^{\boxplus \mathbf{II}} &= -\frac{1}{2} \eta_{ik}{}^b u^d R_{bd}^{\mathbf{II}} \\
&= -\frac{1}{2} \alpha_{ik}^{ac} u^b u^d [*R]_{abcd}
\end{aligned}$$

or

$$\begin{aligned}
-u^b u^d R_{ibkd}^{\mathbf{III}} &= \frac{1}{2} \eta_i{}^{ab} u^d [*R]_{abkd}^{\boxplus} \\
-[n_{ik} u^p u^q - n_i^p n_k^q] R_{pq}^{\mathbf{II}} &= \eta_i{}^{ab} u^d [*R]_{abkd}^{\boxplus}
\end{aligned}$$

$$\begin{aligned}
R &= 2\eta^{abc}u^d[*R]_{abcd}^{\boxplus} \\
\eta_i^{ab}u^d R_{abkd}^{III} &= 2u^b u^d [*R]_{ibkd}^{\boxplus} \\
\eta_{ik}^a u^c R_{ac}^{II} &= 2u^b u^d [*R]_{ibkd}^{\boxplus}
\end{aligned}$$

4.6. $\omega \circ \rho$

$$\begin{aligned}
[\omega \circ \sigma]_{ac}^{pq} [*MN]_{pq}^{\boxplus} &= \sum_{\alpha=1}^3 *N_{\alpha a}^{\boxplus MN} *N_{\alpha c}^{\boxplus MN} \\
[\omega \circ \sigma]_{ac}^{pq} U_{pq}^{\boxplus} &= \sum_{\alpha=1}^3 U_{\alpha a}^{\boxplus W} U_{\alpha c}^{\boxplus W} \\
\omega_{ac}^{pq} [UN]_{pq}^{\boxplus} &= \sum_{\alpha=1}^3 U_{\alpha a}^{\boxplus UN} *N_{\alpha c}^{\boxplus UN} \\
\omega_{ac}^{pq} [*NU]_{pq}^{\boxplus} &= \sum_{\alpha=1}^3 *N_{\alpha a}^{\boxplus UN} U_{\alpha c}^{\boxplus UN}
\end{aligned}$$

implies

$$\begin{aligned}
\omega \circ R_{ijkl}^{NW} &= \sum_{\alpha=1}^3 \eta_{ij}^a *N_{\alpha a}^{\boxplus NW} \eta_{kl}^c *N_{\alpha c}^{\boxplus NW} = \sum_{\alpha=1}^3 N_{\alpha ij}^{\boxplus NW} N_{\alpha kl}^{\boxplus NW} \\
\omega \circ R_{ijkl}^{WU} &= \sum_{\alpha=1}^3 [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{\boxplus W}] [-\alpha_{kl}^{cd} u_d U_{\alpha c}^{\boxplus W}] = \sum_{\alpha=1}^3 U_{\alpha ij}^{\boxplus W} U_{\alpha kl}^{\boxplus W} \\
\omega \circ R_{ijkl}^{UN} &= \sum_{\alpha=1}^3 [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{\boxplus UN}] \eta_{kl}^c *N_{\alpha c}^{\boxplus UN} = \sum_{\alpha=1}^3 U_{\alpha ij}^{\boxplus UN} N_{\alpha kl}^{\boxplus UN} \\
\omega \circ R_{ijkl}^{NU} &= \sum_{\alpha=1}^3 \eta_{ij}^a *N_{\alpha a}^{\boxplus UN} [-\alpha_{kl}^{cd} u_d U_{\alpha c}^{\boxplus UN}] = \sum_{\alpha=1}^3 N_{\alpha ij}^{\boxplus UN} U_{\alpha kl}^{\boxplus UN}
\end{aligned}$$

so

$$\begin{aligned}
[\omega \circ \rho \circ R]_{ijkl} &= \sum_{\alpha=1}^3 [N_{\alpha ij}^{\boxplus NW} N_{\alpha kl}^{\boxplus NW} + U_{\alpha ij}^{\boxplus W} U_{\alpha kl}^{\boxplus W} + N_{\alpha ij}^{\boxplus UN} U_{\alpha kl}^{\boxplus UN} + U_{\alpha ij}^{\boxplus UN} N_{\alpha kl}^{\boxplus UN}] \\
&= \sum_{\alpha=1}^3 [N_{\alpha ij}^{\boxplus NW} N_{\alpha kl}^{\boxplus NW} + U_{\alpha ij}^{\boxplus W} U_{\alpha kl}^{\boxplus W} + N_{\alpha ij}^{\boxplus UN} N_{\alpha kl}^{\boxplus UN} + U_{\alpha ij}^{\boxplus UN} U_{\alpha kl}^{\boxplus UN} - Z_{\alpha ij}^{-UN} Z_{\alpha kl}^{-UN}] \\
&= \sum_{\alpha=1}^3 [N_{\alpha ij}^{\boxplus NW} N_{\alpha kl}^{\boxplus NW} + U_{\alpha ij}^{\boxplus W} U_{\alpha kl}^{\boxplus W} - N_{\alpha ij}^{\boxplus UN} N_{\alpha kl}^{\boxplus UN} - U_{\alpha ij}^{\boxplus UN} U_{\alpha kl}^{\boxplus UN} + Z_{\alpha ij}^{+UN} Z_{\alpha kl}^{+UN}]
\end{aligned}$$

with

$$\begin{aligned}
Z_{\alpha ij}^{-UN} &:= N_{\alpha ij}^{\boxplus UN} - U_{\alpha ij}^{\boxplus UN} \\
Z_{\alpha ij}^{+UN} &:= N_{\alpha ij}^{\boxplus UN} + U_{\alpha ij}^{\boxplus UN}
\end{aligned}$$

N_{α} and U_{α} have rank 2, $Z_{\alpha}^{\pm UN}$ have rank 4. N_{α} are spacelike, while U_{α} are timelike. Setting

$$N_{\alpha ij}^+ := \frac{1}{2}[N_{\alpha ij}^{\boxplus NW} + N_{\alpha ij}^{\boxplus UN}] \quad U_{\alpha ij}^+ := \frac{1}{2}[U_{\alpha ij}^{\boxplus W} + U_{\alpha ij}^{\boxplus UN}]$$

$$N_{\alpha ij}^- := \frac{1}{2}[N_{\alpha ij}^{\boxplus MN} - N_{\alpha ij}^{\boxplus UN}] \quad U_{\alpha ij}^- := \frac{1}{2}[U_{\alpha ij}^{\boxplus W} - U_{\alpha ij}^{\boxplus UN}]$$

the factorization becomes

$$\begin{aligned} [\omega \circ \rho \circ R]_{ijkl} &= \sum_{\alpha=1}^3 [2N_{\alpha ij}^+ N_{\alpha kl}^+ + 2N_{\alpha ij}^- N_{\alpha kl}^- + 2U_{\alpha ij}^+ U_{\alpha kl}^+ + 2U_{\alpha ij}^- U_{\alpha kl}^- - Z_{\alpha ij}^{-UN} Z_{\alpha kl}^{-UN}] \\ &= \sum_{\alpha=1}^3 [2N_{\alpha ij}^+ N_{\alpha kl}^- + 2N_{\alpha ij}^- N_{\alpha kl}^+ + 2U_{\alpha ij}^+ U_{\alpha kl}^- + 2U_{\alpha ij}^- U_{\alpha kl}^+ + Z_{\alpha ij}^{+UN} Z_{\alpha kl}^{+UN}] \end{aligned}$$

Setting

$$\begin{aligned} Z_{\alpha ij}^{++} &= [N_{\alpha ij}^+ + U_{\alpha ij}^+] & Z_{\alpha ij}^{--} &= [N_{\alpha ij}^- + U_{\alpha ij}^-] \\ Z_{\alpha ij}^{+-} &= [N_{\alpha ij}^+ - U_{\alpha ij}^+] & Z_{\alpha ij}^{-+} &= [N_{\alpha ij}^- - U_{\alpha ij}^-] \\ Z_{\alpha ij}^{+0} &= [N_{\alpha ij}^+ - U_{\alpha ij}^-] & Z_{\alpha ij}^{-0} &= [N_{\alpha ij}^- - U_{\alpha ij}^+] \\ Z_{\alpha ij}^{0+} &= [N_{\alpha ij}^+ + U_{\alpha ij}^-] & Z_{\alpha ij}^{0-} &= [N_{\alpha ij}^- + U_{\alpha ij}^+] \end{aligned}$$

the factorization becomes

$$\begin{aligned} [\omega \circ \rho \circ R]_{ijkl} &= \sum_{\alpha=1}^3 [Z_{\alpha ij}^{++} Z_{\alpha kl}^{++} + Z_{\alpha ij}^{--} Z_{\alpha kl}^{--} + Z_{\alpha ij}^{+-} Z_{\alpha kl}^{--} + Z_{\alpha ij}^{-+} Z_{\alpha kl}^{++}] \\ &= \sum_{\alpha=1}^3 [Z_{\alpha ij}^{+-} Z_{\alpha kl}^{+-} + Z_{\alpha ij}^{-+} Z_{\alpha kl}^{-+} + Z_{\alpha ij}^{+0} Z_{\alpha kl}^{+0} + Z_{\alpha ij}^{0+} Z_{\alpha kl}^{0+}] \\ &= \sum_{\alpha=1}^3 [Z_{\alpha ij}^{++} Z_{\alpha kl}^{++} + Z_{\alpha ij}^{--} Z_{\alpha kl}^{--} + Z_{\alpha ij}^{+-} Z_{\alpha kl}^{+-} + Z_{\alpha ij}^{-+} Z_{\alpha kl}^{-+} - N_{\alpha ij}^{\boxplus UN} N_{\alpha kl}^{\boxplus UN} - U_{\alpha ij}^{\boxplus UN} U_{\alpha kl}^{\boxplus UN}] \\ &=: \sum_{\beta=1}^{18} Z_{\beta ij} Z_{\beta kl} \end{aligned}$$

The $Z_{\alpha ij}^{\pm\pm}$ have rank 4. Comparing

$$[\tau \circ R]_{ijkl} = [\tau \circ [\omega \circ \rho \circ R]]_{ijkl}$$

gives

$$\begin{aligned} R_{ijkl}^{III} &= \sum_{\beta} [Z_{\beta ij} Z_{\beta kl} - \gamma_{ijkl}^{abcd} Z_{\beta ab} Z_{\beta cd}] & =: \sum_{\beta} [Z_{\beta ij} Z_{\beta kl}]^{III} \\ R_{ijkl}^{II} &= \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g^{pr} [\delta_b^q \delta_d^s - \frac{1}{4} g_{bd} g^{qs}] \sum_{\beta} Z_{\beta pq} Z_{\beta rs} \\ &=: \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} \sum_{\beta} [Z_{\beta b}^q Z_{\beta dq}]^{II} & =: \sum_{\beta} [Z_{\beta ij} Z_{\beta kl}]^{II} \\ R_{ijkl}^I &= \frac{1}{24} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} g^{pr} g^{qs} [\sum_{\beta} Z_{\beta pq} Z_{\beta rs}] \\ &=: \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} \sum_{\beta} Z_{\beta}^{pq} Z_{\beta pq} & =: \sum_{\beta} [Z_{\beta ij} Z_{\beta kl}]^I \end{aligned}$$

5. $\nabla \mathbf{C}$

5.0.1. $\nabla[\pi \circ \mathbf{C}]$

$$\nabla[\pi \circ \mathbf{C}] = \nabla \mathbf{C}^{\boxplus} + \nabla \mathbf{C}^{\boxplus\boxplus} + \nabla \mathbf{C}^{\boxplus\boxplus\boxplus}$$

The distribution of components reads $144 = 80 + 60 + 4$.

5.0.2. $\nabla[\tau \circ \mathbf{C}]$

$$\nabla[\tau \circ \mathbf{C}] = \nabla \mathbf{C}^{III} + \nabla \mathbf{C}^{II} + \nabla \mathbf{C}^I$$

with

$$\begin{aligned} \nabla_m C_{ijkl}^{III} &= \nabla_m C_{ijkl} - \nabla_m C_{ijkl}^{II} - \nabla_m C_{ijkl}^I \\ \nabla_m C_{ijkl}^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} \frac{1}{2} \nabla_m C_{bd}^{II}] \\ \nabla_m C_{ijkl}^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} \frac{1}{24} \nabla_m C] \end{aligned}$$

The distribution of components reads $144 = 80 + 60 + 4$.

5.0.3. $\nabla[\tau \circ \pi \circ \mathbf{C}]$

$$\nabla[\tau \circ \pi \circ \mathbf{C}] = \nabla[\mathbf{C}^{\boxplus III} + \mathbf{C}^{\boxplus II} + \mathbf{C}^{\boxplus I}] + \nabla[\mathbf{C}^{\boxplus\boxplus III} + \mathbf{C}^{\boxplus\boxplus II}] + \nabla \mathbf{C}^{\boxplus\boxplus\boxplus III}$$

The distribution of components reads $144 = 40 + 36 + 4 + 36 + 24 + 4$.

5.0.4. $\nabla[\rho \circ \mathbf{C}]$

$$\nabla[\rho \circ \mathbf{C}] = \nabla \mathbf{C}^{MV} + \nabla \mathbf{C}^{WU} + \nabla \mathbf{C}^{UN} + \nabla \mathbf{C}^{NU}$$

The distribution of components reads $144 = 36 + 36 + 36 + 36$.

5.1. π

$$\pi \circ \nabla \mathbf{C} = [\nabla \mathbf{C}]^{\boxplus\boxplus\boxplus} + [\nabla \mathbf{C}]^{\boxplus\boxplus} + [\nabla \mathbf{C}]^{\boxplus} + [\nabla \mathbf{C}]^{\boxplus\boxplus\boxplus}$$

with

$$\begin{aligned} [\nabla_m C_{ijkl}]^{\boxplus\boxplus\boxplus} &= \frac{1}{24} \alpha_{ij}^{pr} \alpha_{kl}^{qs} \sigma_{rs}^{bd} \sigma_{pqm}^{ace} \nabla_e C_{abcd} \\ &= \frac{1}{24} \alpha_{ij}^{ab} \alpha_{kl}^{cd} \sigma_{bd}^{rs} [\nabla_m [C_{arcs} + C_{csar}] + \nabla_c [C_{mras} + C_{asmr}] + \nabla_a [C_{crms} + C_{mscr}]] \\ [\nabla_m C_{ijkl}]^{\boxplus\boxplus} &= \frac{1}{20} \alpha_{kl}^{xy} \alpha_{ijx}^{pbc} \sigma_{pym}^{ade} \nabla_e C_{abcd} \\ &= \frac{1}{20} \alpha_{kl}^{xy} \alpha_{ijx}^{pbc} [\nabla_m [C_{pbcy} - C_{cy pb}] + \nabla_y [C_{mbcp} - C_{cpmb}] + \nabla_p [C_{ybcm} - C_{cm yb}]] \\ [\nabla_m C_{ijkl}]^{\boxplus} &= \frac{1}{24} [\alpha_{ijm}^{pre} \alpha_{kl}^{qs} \sigma_{rs}^{bd} \sigma_{pq}^{ac} + \alpha_{kl}^{xy} \alpha_{ijx}^{prc} \alpha_{ym}^{qs} \sigma_{rs}^{be} \sigma_{pq}^{ad}] \nabla_e C_{abcd} \end{aligned}$$

$$\begin{aligned}
&= [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} + \alpha_{kl}^{xy} [\nabla_m C_{ijxy}]_{\overline{x}}^{\overline{y}} \\
&= \frac{1}{24} \alpha_{kl}^{qs} [\alpha_{ijm}^{pre} \sigma_{rs}^{bd} \nabla_e [C_{pbqd} + C_{qdpb}] + \alpha_{ijq}^{prc} \sigma_{mc}^{xy} \nabla_r [C_{pxys} + C_{yspx}]] \\
&\quad + \frac{1}{24} \alpha_{kl}^{qs} \alpha_{ijq}^{prc} [\alpha_{sm}^{xy} \nabla_y [C_{prcx} - C_{cxpr}] + \nabla_r [C_{smcp} - C_{cpsm}]] \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{30} [\alpha_{kl}^{xy} \alpha_{ijxm}^{pbce} \sigma_{py}^{ad} + \alpha_{ijkl}^{abcd} \sigma_{pm}^{ae}] \nabla_e C_{abcd} \\
&= \alpha_{kl}^{xy} [\nabla_m C_{ijxy}]_{\overline{lm}}^{\overline{ik}} + [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} \\
&= \frac{1}{30} [\alpha_{kl}^{qs} \alpha_{ijqm}^{pbce} \nabla_e [C_{pbcs} - C_{cspb}] + \frac{1}{2} \alpha_{ijkl}^{abcd} \nabla_p [C_{mbcd} - C_{cdmb}]] \\
&\quad + \frac{1}{30} \alpha_{ijkl}^{abcd} [\nabla_m C_{pbcd} + \frac{1}{2} \nabla_p [C_{mbcd} + C_{cdmb}]]
\end{aligned}$$

The distribution of components reads $144 = 60 + 36 + 2 \cdot 20 + 2 \cdot 4$.

5.1.1. $\pi \circ \nabla[\pi \circ \mathbf{C}]$

$$\pi \circ \nabla[\pi \circ \mathbf{C}] = [\pi^{\boxplus\boxplus} + \pi^{\boxplus}] \circ \nabla \mathbf{C}^{\boxplus} + [\pi^{\boxplus\boxplus} + \pi^{\boxplus} + \pi^{\boxplus}] \circ \nabla \mathbf{C}^{\boxplus} + \pi^{\boxplus} \circ \nabla \mathbf{C}^{\boxplus}$$

with

$$\begin{aligned}
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{2} \nabla_m C_{ijkl}^{\boxplus} - \frac{1}{8} [\alpha_{ij}^{ab} \alpha_{kl}^{cd} + \alpha_{kl}^{ab} \alpha_{ij}^{cd}] \nabla_a C_{bmc d}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{2} \nabla_m C_{ijkl}^{\boxplus} + \frac{1}{8} [\alpha_{ij}^{ab} \alpha_{kl}^{cd} + \alpha_{kl}^{ab} \alpha_{ij}^{cd}] \nabla_a C_{bmc d}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{4}{10} \nabla_m C_{ijkl}^{\boxplus} - \frac{1}{10} [\alpha_{ij}^{ab} \alpha_{kl}^{cd} - \alpha_{kl}^{ab} \alpha_{ij}^{cd}] \nabla_a [C_{bmc d}^{\boxplus} + C_{dmcb}^{\boxplus}] \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{3} \nabla_m C_{ijkl}^{\boxplus} - \frac{1}{6} [\alpha_{ij}^{ab} \alpha_{kl}^{cd} - \alpha_{kl}^{ab} \alpha_{ij}^{cd}] \nabla_a C_{dmcb}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{4}{15} \nabla_m C_{ijkl}^{\boxplus} + \frac{1}{30} [\alpha_{ij}^{ab} \alpha_{kl}^{cd} - \alpha_{kl}^{ab} \alpha_{ij}^{cd}] \nabla_a [3C_{bmc d}^{\boxplus} + 2C_{dmcb}^{\boxplus}] \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \nabla_m C_{ijkl}^{\boxplus}
\end{aligned}$$

which can be written as

$$\begin{aligned}
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \nabla_m C_{ijkl}^{\boxplus} - \frac{1}{16} [\alpha_{mij}^{eab} \alpha_{kl}^{cd} + \alpha_{mkl}^{eab} \alpha_{ij}^{cd}] \nabla_e C_{abcd}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{16} [\alpha_{mij}^{eab} \alpha_{kl}^{cd} + \alpha_{mkl}^{eab} \alpha_{ij}^{cd}] \nabla_e C_{abcd}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \nabla_m C_{ijkl}^{\boxplus} - [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} - [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= -\frac{2}{5} [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} + \frac{1}{20} [\alpha_{mij}^{eab} \alpha_{kl}^{cd} - \alpha_{mkl}^{eab} \alpha_{ij}^{cd}] \nabla_e C_{abcd}^{\boxplus} \\
[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} &= \frac{1}{12} \alpha_{kl}^{qs} \alpha_{ijq}^{prc} [\alpha_{sm}^{xy} \nabla_y C_{prcx} + \nabla_r C_{smcp}^{\boxplus}]
\end{aligned}$$

or as

$$[\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} = \nabla_m C_{ijkl}^{\boxplus} - [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}} - [\nabla_m C_{ijkl}]_{\overline{lm}}^{\overline{ik}}$$

$$\begin{aligned}
[\nabla_m C_{ijkl}^{\square}]^{\square\square} &= \frac{3}{2}[\nabla_m C_{ijkl}^{\square}]^{\square} - \frac{1}{4}[\alpha_{ij}^{ab}\alpha_{kl}^{cd} - \alpha_{kl}^{ab}\alpha_{ij}^{cd}]\nabla_a C_{bmc d}^{\square} \\
[\nabla_m C_{ijkl}^{\square}]^{\square} &= \frac{1}{30}[2\alpha_{kl}^{qs}\alpha_{ijqm}^{abce}\nabla_e C_{abcs}^{\square} + \alpha_{ijkl}^{abcd}\nabla_a C_{mbcd}^{\square}]
\end{aligned}$$

and

$$\begin{aligned}
[\nabla_m C_{ijkl}^{\square}]^{\square\square} &= \frac{1}{30}[-2\alpha_{kl}^{qs}\alpha_{ijqm}^{abcd}g_{as} + \alpha_{ijkl}^{abcd}g_{am}]\nabla^p[*[C^{\square}]^*]_{pbcd} \\
[\nabla_m C_{ijkl}^{\square\square}]^{\square} &= -\frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{eab}\alpha_{ij}^{cd}]g_{ac}\nabla^p[[*C^{\square\square}]^*]_{pdbe} + g_{bd}[*[C^{\square\square}]^*]_{pe} \\
\frac{1}{20}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{ecd}\alpha_{ij}^{ab}]\nabla_e C_{abcd}^{\square} &= -\frac{1}{10}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} - \alpha_{mkl}^{eab}\alpha_{ij}^{cd}]g_{ac}\nabla^p[[*C^{\square}]^*]_{pdbe} + g_{bd}[*[C^{\square}]^*]_{pe}
\end{aligned}$$

The distribution of components reads $144 = 60 + 20 + 36 + 20 + 4 + 4$.

5.2. τ

$$\tau \circ \nabla \mathbf{C} = [\nabla \mathbf{C}]^{III} + [\nabla \mathbf{C}]^{II} + [\nabla \mathbf{C}]^I$$

with

$$\begin{aligned}
[\nabla_m C_{ijkl}]^{III} &= \nabla_m C_{ijkl} - \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}X_{bdm} + g_{am}Y_{bcd} + g_{cm}Z_{dab}] \\
[\nabla_m C_{ijkl}]^{II} &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}[X_{bdm}]^{II} + g_{am}[Y_{bcd}]^{II} + g_{cm}[Z_{dab}]^{II}] \\
[\nabla_m C_{ijkl}]^I &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}[X_{bdm}]^I + g_{am}[Y_{bcd}]^I + g_{cm}[Z_{dab}]^I] \\
&=: \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}g_{bd}X_m + g_{bm}g_{ac}Y_d + g_{dm}g_{ac}Z_b]
\end{aligned}$$

and

$$\begin{aligned}
0 &= \sigma_{kl}^{cd}Y_{bcd} \\
0 &= \sigma_{ij}^{ab}Z_{abd}
\end{aligned}$$

The equations for the coefficients read

$$\begin{aligned}
g^{jl}[\nabla_m C_{ijkl}]^{II} &= [\nabla_m C_{ik}]^{II} &= 2[X_{ikm} - Y_{ikm} - Z_{kim}]^{II} \\
g^{mi}[\nabla_m C_{ijkl}]^{II} &= [\nabla^p C_{pjkl}]^{II} &= [\alpha_{kl}^{cd}[X_{jdc} + 2Z_{dcj}] + 6Y_{jkl}]^{II} \\
g^{mk}[\nabla_m C_{ijkl}]^{II} &= [\nabla^p C_{ijpl}]^{II} &= [\alpha_{ij}^{ab}[X_{bla} + 2Y_{bal}] + 6Z_{lij}]^{II} \\
g^{ik}g^{jl}[\nabla_m C_{ijkl}]^I &= \nabla_m C &= 24X_m + 6Y_m + 6Z_m \\
g^{mi}g^{jl}[\nabla_m C_{ijkl}]^I &= \nabla^p C_{pk} &= 6X_k + 9Y_k + 3Z_k \\
g^{mk}g^{jl}[\nabla_m C_{ijkl}]^I &= \nabla^p C_{ip} &= 6X_i + 3Y_i + 9Z_i
\end{aligned}$$

so with

$$\begin{aligned}
[X_{ikm}]^{II} &= [X_{ikm}]^{II\square\square} + [X_{ikm}]^{\square II} + [X_{ikm}]^{\square} \\
[Y_{ikm}]^{II} &= [Y_{ikm}]^{\square II} + [Y_{ikm}]^{\square} \\
[Z_{ikm}]^{II} &= [Z_{ikm}]^{\square II} + [Z_{ikm}]^{\square}
\end{aligned}$$

the solution reads

$$[X_{ikm}]^{II\square\square} = \frac{1}{2}[\nabla_m C_{ik}]^{II\square\square}$$

$$\begin{aligned}
[X_{ikm}]^{\boxplus} &= \frac{1}{6}[5\nabla_m C_{ik} + \nabla^p C_{pikm} - \nabla^p C_{ikpm}]^{\boxplus} \\
[Y_{ikm}]^{\boxplus} &= \frac{1}{6}[\nabla_m C_{ik} + 2\nabla^p C_{pikm} + \nabla^p C_{kmp i}]^{\boxplus} \\
[Z_{ikm}]^{\boxplus} &= \frac{1}{6}[-\nabla_m C_{ik} + \nabla^p C_{pikm} + 2\nabla^p C_{kmp i}]^{\boxplus} \\
[X_{ikm}]^{II\boxplus} &= \frac{1}{15}[10\nabla_m C_{ik} + \sigma_{ik}^{ac}\nabla_m C_{ac} + \frac{1}{4}\nabla^p[11C_{pikm} + C_{kmp i} + 11C_{impk} + C_{pkim}]]^{II\boxplus} \\
&\quad + \frac{1}{12}\nabla^p[C_{ikpm} - C_{pmik}]^{II\boxplus} \\
[Y_{ikm}]^{II\boxplus} &= \frac{1}{15}[4\nabla^p C_{pikm} - \nabla^p C_{kmp i} + \alpha_{km}^{ce}[20\nabla_e C_{ic} + 2\sigma_{ic}^{rs}\nabla_e C_{rs} - 10\nabla_i C_{ce}]]^{II\boxplus} \\
[Z_{ikm}]^{II\boxplus} &= \frac{1}{15}[4\nabla^p C_{kmp i} - \nabla^p C_{pikm} + \alpha_{km}^{ce}[20\nabla_e C_{ci} + 2\sigma_{ic}^{rs}\nabla_e C_{rs} + 10\nabla_i C_{ce}]]^{II\boxplus}
\end{aligned}$$

and

$$\begin{aligned}
X_k &= \frac{1}{18}[5X_{ak}^a - X_{ka}^a - X_{ka}^a] &= \frac{1}{36}[2\nabla_k C - \nabla^a C_{ak} - \nabla^k C_{ka}] \\
Y_k &= \frac{1}{18}[-X_{ak}^a + 5X_{ka}^a - X_{ka}^a] + \frac{1}{3}Y_{ka}^a &= \frac{1}{36}[-\nabla_k C + 5\nabla^a C_{ak} - \nabla^k C_{ka}] \\
Z_k &= \frac{1}{18}[-X_{ak}^a - X_{ka}^a + 5X_{ka}^a] + \frac{1}{3}Z_{ka}^a &= \frac{1}{36}[-\nabla_k C - \nabla^a C_{ak} + 5\nabla^k C_{ka}]
\end{aligned}$$

The distribution of components reads $144 = 32 + 100 + 12$.

5.2.1. $\tau \circ \nabla[\pi \circ \mathbf{C}]$

$$\tau \circ \nabla[\pi \circ \mathbf{C}] = [\tau^{III} + \tau^{II} + \tau^I] \circ [\nabla \mathbf{C}^{\boxplus} + \nabla \mathbf{C}^{\boxminus}] + [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{C}^{\boxplus}$$

with

$$\begin{aligned}
[\nabla_m C_{ijkl}^{\boxplus}]^{III} &= \nabla_m C_{ijkl}^{\boxplus} - \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}X_{bdm}^{\boxplus} + g_{am}Y_{bcd}^{\boxplus} + g_{cm}Y_{dab}^{\boxplus}] \\
[\nabla_m C_{ijkl}^{\boxplus}]^{II} &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}[X_{bdm}^{\boxplus}]^{II} + g_{am}[Y_{bcd}^{\boxplus}]^{II} + g_{cm}[Y_{dab}^{\boxplus}]^{II}] \\
[\nabla_m C_{ijkl}^{\boxplus}]^I &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}g_{bd}X_m^{\boxplus} + g_{bm}g_{ac}Y_d^{\boxplus} + g_{dm}g_{ac}Y_b^{\boxplus}] \\
[\nabla_m C_{ijkl}^{\boxminus}]^{III} &= \nabla_m C_{ijkl}^{\boxminus} - \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}X_{bdm}^{\boxminus} + g_{am}Y_{bcd}^{\boxminus} - g_{cm}Y_{dab}^{\boxminus}] \\
[\nabla_m C_{ijkl}^{\boxminus}]^{II} &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}[X_{bdm}^{\boxminus}]^{II} + g_{am}[Y_{bcd}^{\boxminus}]^{II} - g_{cm}[Y_{dab}^{\boxminus}]^{II}] \\
[\nabla_m C_{ijkl}^{\boxminus}]^I &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{bm}g_{ac}Y_d^{\boxminus} - g_{dm}g_{ac}Y_b^{\boxminus}] \\
[\nabla_m C_{ijkl}^{\boxplus}]^{III} &= \nabla_m C_{abcd}^{\boxplus} - \alpha_{ij}^{ab}\alpha_{kl}^{cd}[2g_{am}Y_{bcd}^{\boxplus}] \\
[\nabla_m C_{ijkl}^{\boxplus}]^{II} &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[2g_{am}Y_{bcd}^{\boxplus}]
\end{aligned}$$

and

$$\begin{aligned}
0 &= \sigma_{kl}^{cd}Y_{bcd}^{\boxplus} \\
0 &= \sigma_{kl}^{cd}Y_{bcd}^{\boxminus} \\
0 &= \alpha_{jl}^{bd}X_{bdm}^{\boxplus}
\end{aligned}$$

$$0 = \alpha_{jkl}^{bcd} Y_{bcd}^{\boxplus}$$

$$0 = \sigma_{jl}^{bd} X_{bdm}^{\boxplus}$$

The equations for the coefficients read

$$\begin{aligned}
g^{jl} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= [\nabla_m C_{ik}^{\boxplus}]^{II} = 2[X_{ikm}^{\boxplus} - \sigma_{ik}^{ac} Y_{acm}^{\boxplus}]^{II} \\
g^{mi} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= [\nabla^p C_{pjkl}^{\boxplus}]^{II} = [\alpha_{kl}^{cd} X_{jdc}^{\boxplus} + 8Y_{jkl}^{\boxplus}]^{II} \\
g^{jl} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= [\nabla_m C_{ik}^{\boxplus}]^{II} = 2[X_{ikm}^{\boxplus} - \alpha_{ik}^{ac} Y_{acm}^{\boxplus}]^{II} \\
g^{mi} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= [\nabla^p C_{pjkl}^{\boxplus}]^{II} = [\alpha_{kl}^{cd} [X_{jdc}^{\boxplus} - Y_{dcj}^{\boxplus}] + 6Y_{jkl}^{\boxplus}]^{II} \\
g^{mi} \nabla_m C_{ijkl}^{\boxplus} &= \nabla^p C_{pjkl}^{\boxplus} = 2Y_{jkl}^{\boxplus} \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\boxplus}]^I &= \nabla_m C = 24X_m^{\boxplus} + 12Y_m^{\boxplus} \\
g^{mk} g^{jl} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= \nabla^p C_{ip}^{\boxplus} = 6X_i^{\boxplus} + 12Y_i^{\boxplus} \\
g^{mi} g^{jl} [\nabla_m C_{ijkl}^{\boxplus}]^{II} &= \nabla^p C_{pk}^{\boxplus} = 6Y_k^{\boxplus}
\end{aligned}$$

so with

$$\begin{aligned}
[X_{ikm}^{\boxplus}]^{II} &= [X_{ikm}^{\boxplus}]^{\boxplus II} + [X_{ikm}^{\boxplus}]^{\boxplus II} \\
[Y_{ikm}^{\boxplus}]^{II} &= [Y_{ikm}^{\boxplus}]^{\boxplus II} \\
[X_{ikm}^{\boxplus}]^{II} &= [X_{ikm}^{\boxplus}]^{\boxplus II} + [X_{ikm}^{\boxplus}]^{\boxplus} \\
[Y_{ikm}^{\boxplus}]^{II} &= [Y_{ikm}^{\boxplus}]^{\boxplus II} + [Y_{ikm}^{\boxplus}]^{\boxplus} \\
Y_{ikm}^{\boxplus} &= [Y_{ikm}^{\boxplus}]^{\boxplus}
\end{aligned}$$

the solution reads

$$\begin{aligned}
[X_{ikm}^{\boxplus}]^{II \boxplus} &= \frac{1}{2} [\nabla_m C_{ik}^{\boxplus}]^{II \boxplus} \\
[X_{ikm}^{\boxplus}]^{\boxplus} &= \frac{1}{6} [5\nabla_m C_{ik}^{\boxplus} + 2\nabla^p C_{pikm}^{\boxplus}]^{II} \\
[Y_{ikm}^{\boxplus}]^{\boxplus} &= \frac{1}{6} [\nabla_m C_{ik}^{\boxplus} + \nabla^p C_{pikm}^{\boxplus}]^{II} \\
Y_{ikm}^{\boxplus} &= \frac{1}{2} \nabla^p C_{pikm}^{\boxplus} \\
[X_{ikm}^{\boxplus}]^{II \boxplus} &= \frac{4}{5} [\nabla_m C_{ik}^{\boxplus} + \frac{1}{4} \sigma_{ik}^{ac} \nabla^p C_{pacm}^{\boxplus}]^{II \boxplus} \\
[Y_{ikm}^{\boxplus}]^{II \boxplus} &= \frac{1}{5} [\nabla^p C_{pikm}^{\boxplus} + \frac{1}{2} \alpha_{km}^{ce} \nabla_e C_{ic}^{\boxplus}]^{II \boxplus} \\
[X_{ikm}^{\boxplus}]^{II \boxplus} &= \frac{2}{3} [\nabla_m C_{ik}^{\boxplus} - \frac{1}{2} \nabla^p C_{pmik}^{\boxplus}]^{II \boxplus} \\
[Y_{ikm}^{\boxplus}]^{II \boxplus} &= \frac{1}{3} [\nabla^p C_{pikm}^{\boxplus} - \frac{1}{2} \nabla_i C_{km}^{\boxplus}]^{II \boxplus} \\
X_k^{\boxplus} &= \frac{1}{18} [\nabla_k C - \nabla^a C_{ak}^{\boxplus}] \\
Y_k^{\boxplus} &= \frac{1}{36} [-\nabla_k C + 4\nabla^a C_{ak}^{\boxplus}] \\
Y_k^{\boxplus} &= \frac{1}{6} \nabla^a C_{ak}^{\boxplus}
\end{aligned}$$

5.2.2. $\tau \circ \nabla[\tau \circ \mathbf{C}]$

$$\tau \circ \nabla[\tau \circ \mathbf{C}] = [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{C}^{III} + [\tau^{II} + \tau^I] \circ \nabla \mathbf{C}^{II} + \tau^I \circ \nabla \mathbf{C}^I$$

with

$$\begin{aligned} [\nabla_m C_{ijkl}^{III}]^{III} &= \nabla_m C_{ijkl}^{III} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{III} + g_{am} Y_{bcd}^{III} + g_{cm} Z_{dab}^{III}] \\ [\nabla_m C_{ijkl}^{III}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} [X_{bdm}^{III}]^{II} + g_{am} [Y_{bcd}^{III}]^{II} + g_{cm} [Z_{dab}^{III}]^{II}] \\ [\nabla_m C_{ijkl}^{II}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} [X_{bdm}^{III}]^{II}] \\ [\nabla_m C_{ijkl}^{II}]^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} X_m^{II} + g_{bm} g_{ac} Y_d^{II} + g_{dm} g_{ac} Z_b^{II}] \\ [\nabla_m C_{ijkl}^I]^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} X_m^I] \end{aligned}$$

and

$$\begin{aligned} 0 &= \sigma_{kl}^{cd} Y_{bcd}^{III} \\ 0 &= \sigma_{ij}^{ab} Z_{abd}^{III} \\ 0 &= [X_{bdm}^{III}]^I \\ 0 &= [Y_{bcd}^{III}]^I \\ 0 &= [Z_{dab}^{III}]^I \end{aligned}$$

The equations for the coefficients read

$$\begin{aligned} g^{jl} [\nabla_m C_{ijkl}^{III}]^{II} &= 0 &= 2X_{ikm}^{III} - 2Y_{ikm}^{III} - 2Z_{kim}^{III} \\ g^{mi} [\nabla_m C_{ijkl}^{III}]^{II} &= \nabla^p C_{pjkl}^{III} &= \alpha_{kl}^{cd} [X_{jdc}^{III} + 2Z_{dcj}^{III}] + 6Y_{jkl}^{III} \\ g^{mk} [\nabla_m C_{ijkl}^{III}]^{II} &= \nabla^p C_{ijpl}^{III} &= \alpha_{ij}^{ab} [X_{bla}^{III} + 2Y_{bal}^{III}] + 6Z_{lij}^{III} \\ g^{jl} [\nabla_m C_{ijkl}^{II}]^{II} &= [\nabla_m C_{ik}^{II}]^{II} &= 2[X_{ikm}^{II}]^{II} \\ g^{ik} g^{jl} [\nabla_m C_{ijkl}^{II}]^I &= 0 &= 24X_i^{II} + 6Y_i^{II} + 6Z_i^{II} \\ g^{mk} g^{jl} [\nabla_m C_{ijkl}^{II}]^I &= \nabla^p C_{ip}^{II} &= 6X_i^{II} + 3Y_i^{II} + 9Z_i^{II} \\ g^{mi} g^{jl} [\nabla_m C_{ijkl}^{II}]^I &= \nabla^p C_{pi}^{II} &= 6X_i^{II} + 9Y_i^{II} + 3Z_i^{II} \\ g^{ik} g^{jl} [\nabla_m C_{ijkl}^I]^I &= \nabla_m C &= 24X_m^I \end{aligned}$$

so with

$$\begin{aligned} Y_{ikm}^{III} &= [Y_{ikm}^{III}]^{\boxplus} + [Y_{ikm}^{III}]^{\boxminus} \\ Z_{ikm}^{III} &= [Z_{ikm}^{III}]^{\boxplus} + [Z_{ikm}^{III}]^{\boxminus} \\ [X_{ikm}^{II}]^{II} &= [X_{ikm}^{II}]^{\boxplus II} + [X_{ikm}^{II}]^{\boxplus II} + [X_{ikm}^{II}]^{\boxminus} \end{aligned}$$

the solution reads

$$\begin{aligned} X_{ikm}^{III} &= Y_{ikm}^{III} + Z_{kim}^{III} \\ [Y_{ikm}^{III}]^{\boxminus} &= \frac{1}{6} [2\nabla^p C_{pikm}^{III} + \nabla^p C_{kmpi}^{III}]^{\boxminus} \\ [Z_{ikm}^{III}]^{\boxminus} &= \frac{1}{6} [\nabla^p C_{pikm}^{III} + 2\nabla^p C_{kmpi}^{III}]^{\boxminus} \\ [Y_{ikm}^{III}]^{II \boxplus} &= \frac{1}{15} [4\nabla^p C_{pikm}^{III} - \nabla^p C_{kmpi}^{III}]^{II \boxplus} \end{aligned}$$

$$\begin{aligned}
[Z_{ikm}^{III}]^{II\boxplus} &= \frac{1}{15}[4\nabla^p C_{kmp i}^{III} - \nabla^p C_{pikm}^{III}]^{II\boxplus} \\
[X_{ikm}^{II}]^{II} &= \frac{1}{2}[\nabla_m C_{ik}^{II}]^{II} \\
X_m^{II} &= -\frac{1}{4}[Y_m^{II} + Z_m^{II}] \\
Y_d^{II} &= \frac{1}{36}[5\nabla^a C_{ak}^{II} - \nabla^k C_{ka}^{II}] \\
Z_b^{II} &= \frac{1}{36}[5\nabla^k C_{ka}^{II} - \nabla^a C_{ak}^{II}] \\
\nabla_m C &= \frac{1}{24}X_m^I
\end{aligned}$$

The distribution of components reads $144 = 32 + 48 + 52 + 8 + 4$.

5.2.3. $\tau \circ \nabla[\tau \circ \pi \circ \mathbf{C}]$

$$\pi \circ \nabla[\tau \circ \pi \circ \mathbf{C}] = [\tau^{III} + \tau^{II}] \circ [\mathbf{C}^{\boxplus III} \mathbf{C}^{\boxplus III} + \mathbf{C}^{\boxplus III}] + [\tau^{II} + \tau^I] \circ [\mathbf{C}^{\boxplus II} + \mathbf{C}^{\boxplus II}] + \tau^I \circ \nabla \mathbf{C}^{\boxplus I}$$

with

$$\begin{aligned}
[\nabla_m C_{ijkl}^{\boxplus III}]^{III} &= \nabla_m C_{ijkl}^{\boxplus III} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} - g_{am} Y_{bcd}^{\boxplus III} - g_{cm} Y_{dab}^{\boxplus III}] \\
[\nabla_m C_{ijkl}^{\boxplus III}]^{III} &= \nabla_m C_{ijkl}^{\boxplus III} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} - g_{am} Y_{bcd}^{\boxplus III} + g_{cm} Y_{dab}^{\boxplus III}] \\
[\nabla_m C_{ijkl}^{\boxplus III}]^{III} &= [\nabla_m C_{ijkl}^{\boxplus III}]^{III} \\
[\nabla_m C_{ijkl}^{\boxplus III}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} + g_{am} Y_{bcd}^{\boxplus III} + g_{cm} Y_{dab}^{\boxplus III}] \\
[\nabla_m C_{ijkl}^{\boxplus III}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} + g_{am} Y_{bcd}^{\boxplus III} - g_{cm} Y_{dab}^{\boxplus III}] \\
[\nabla_m C_{ijkl}^{\boxplus III}]^{II} &= [\nabla_m C_{ijkl}^{\boxplus III}]^{II} \\
[\nabla_m C_{ijkl}^{\boxplus II}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} [X_{bdm}^{\boxplus II}]^{II}] \\
[\nabla_m C_{ijkl}^{\boxplus II}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} [X_{bdm}^{\boxplus II}]^{II}] \\
[\nabla_m C_{ijkl}^{\boxplus II}]^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} X_m^{\boxplus II} + g_{bm} g_{ac} Y_d^{\boxplus II} + g_{dm} g_{ac} Y_b^{\boxplus II}] \\
[\nabla_m C_{ijkl}^{\boxplus II}]^I &= [\nabla_m C_{ijkl}^{\boxplus II}]^I \\
[\nabla_m C_{ijkl}^{\boxplus I}]^I &= [\nabla_m C_{ijkl}^{\boxplus I}]^I
\end{aligned}$$

and

$$\begin{aligned}
0 &= \sigma_{kl}^{cd} Y_{bcd}^{\boxplus III} \\
0 &= \sigma_{kl}^{cd} Y_{bcd}^{\boxplus III} \\
0 &= \alpha_{jl}^{bd} X_{bdm}^{\boxplus III} \\
0 &= \alpha_{jl}^{bd} X_{bdm}^{\boxplus II} \\
0 &= \alpha_{jkl}^{bcd} Y_{bcd}^{\boxplus III} \\
0 &= \sigma_{jl}^{bd} X_{bdm}^{\boxplus III}
\end{aligned}$$

$$\begin{aligned}
0 &= \sigma_{jl}^{bd} X_{bdm}^{\square II} \\
0 &= [X_{bdm}^{\square III}]^{II} \\
0 &= [X_{bdm}^{\square III}]^{II} \\
0 &= [Y_{bkl}^{\square III}]^{II} \\
0 &= [Y_{bkl}^{\square III}]^{II}
\end{aligned}$$

The equations for the coefficients read

$$\begin{aligned}
g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{II} &= 0 &= 2[X_{ikm}^{\square III} - \sigma_{ik}^{ac} Y_{acm}^{\square III}] \\
g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{II} &= \nabla^p C_{pjkl}^{\square III} &= \alpha_{kl}^{cd} X_{jdc}^{\square III} + 8Y_{jkl}^{\square III} \\
g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{II} &= 0 &= 2[X_{ikm}^{\square III} - \alpha_{ik}^{ac} Y_{acm}^{\square III}] \\
g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{II} &= \nabla^p C_{pjkl}^{\square III} &= \alpha_{kl}^{cd} [X_{jdc}^{\square III} - 2Y_{dcj}^{\square III}] + 6Y_{jkl}^{\square III} \\
g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{II} &= [\nabla^p C_{pjkl}^{\square III}]^{II} &= 2Y_{jkl}^{\square III} \\
g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{II} &= [\nabla_m C_{ik}^{\square II}]^{II} &= 2[X_{ikm}^{\square II}]^{II} \\
g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{II} &= [\nabla_m C_{ik}^{\square II}]^{II} &= 2[X_{ikm}^{\square II}]^{II} \\
g^{ik} g^{jl}[\nabla_m C_{ijkl}^{\square II}]^I &= 0 &= 24X_m^{\square II} + 12Y_m^{\square II} \\
g^{mk} g^{jl}[\nabla_m C_{ijkl}^{\square II}]^I &= \nabla^p C_{ip}^{\square II} &= 6X_i^{\square II} + 12Y_i^{\square II} \\
g^{mi} g^{jl}[\nabla_m C_{ijkl}^{\square II}]^I &= \nabla^p C_{pk}^{\square II} &= 6Y_k^{\square II} \\
g^{ik} g^{jl}[\nabla_m C_{ijkl}^{\square I}] &= \nabla_m C &= 24[X^{\square I}]_m
\end{aligned}$$

so with

$$\begin{aligned}
Y_{ikm}^{\square III} &= [Y_{ikm}^{\square III}]^{\square} \\
Y_{ikm}^{\square III} &= [Y_{ikm}^{\square III}]^{\square} + [Y_{ikm}^{\square III}]^{\square} \\
Y_{ikm}^{\square III} &= [Y_{ikm}^{\square III}]^{\square} \\
[X_{ikm}^{\square II}]^{II} &= [X_{ikm}^{\square II}]^{\square II} + [X_{ikm}^{\square II}]^{\square II} \\
[X_{ikm}^{\square II}]^{II} &= [X_{ikm}^{\square II}]^{\square II} + [X_{ikm}^{\square II}]^{\square}
\end{aligned}$$

the solution becomes

$$\begin{aligned}
X_{ikm}^{\square III} &= \sigma_{ik}^{ac} Y_{acm}^{\square III} \\
Y_{ikm}^{\square III} &= \frac{1}{5} \nabla^p C_{pikm}^{\square III} \\
X_{ikm}^{\square III} &= \alpha_{ik}^{ac} Y_{acm}^{\square III} \\
[Y_{ikm}^{\square III}]^{\square} &= \frac{1}{6} [\nabla^p C_{pikm}^{\square III}]^{\square} \\
[Y_{ikm}^{\square III}]^{\square} &= \frac{1}{3} [\nabla^p C_{pikm}^{\square III}]^{\square} \\
Y_{ikm}^{\square III} &= \frac{1}{2} \nabla^p C_{pikm}^{\square III}
\end{aligned}$$

$$\begin{aligned}
[X_{ikm}^{\boxplus II}] &= \frac{1}{2}[\nabla_m C_{ik}^{\boxplus II}] \\
[X_{ikm}^{\boxminus II}] &= \frac{1}{2}[\nabla_m C_{ik}^{\boxminus II}] \\
X_m^{\boxplus II} &= -\frac{1}{2}Y_m^{\boxplus II} \\
Y_m^{\boxplus II} &= \frac{1}{9}\nabla^a C_{am}^{\boxplus II} \\
Y_m^{\boxminus II} &= \frac{1}{6}\nabla^a C_{am}^{\boxminus II} \\
X_m^{\boxplus I} &= \frac{1}{24}\nabla_m C
\end{aligned}$$

5.3. $\tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}]$

$$\begin{aligned}
\tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}] &= [\tau^{III} + \tau^{II} + \tau^I] \circ [[\nabla \mathbf{C}^{\boxplus}]^{\boxplus} + [\nabla \mathbf{C}^{\boxplus}]^{\boxminus} + [\nabla \mathbf{C}^{\boxminus}]^{\boxplus}] \\
&+ [\tau^{III} + \tau^{II}] \circ [[\nabla \mathbf{C}^{\boxminus}]^{\boxplus} + [\nabla \mathbf{C}^{\boxminus}]^{\boxminus} + [\nabla \mathbf{C}^{\boxplus}]^{\boxplus}]
\end{aligned}$$

The relations between the coefficients $X^{\boxplus\boxplus}$ and $Y^{\boxplus\boxplus}$ and the traces remain unchanged, so $X^{\boxplus\boxplus}$ and $Y^{\boxplus\boxplus}$ are determined by

$$\begin{aligned}
[g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus}]^{\boxplus II} &= [\nabla_m C_{ik}^{\boxplus}]^{\boxplus II} - \frac{1}{4}\sigma_{ik}^{ac}[\nabla^p C_{pacm}^{\boxplus} - \alpha_{cm}^{xy}\nabla_x C_{ay}^{\boxplus}]^{\boxplus II} \\
[g^{mi}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus}]^{\boxplus II} &= [\nabla^p C_{pjkl}^{\boxplus}]^{\boxplus II} - \frac{1}{4}[\nabla^p C_{pjkl}^{\boxplus} - \alpha_{kl}^{xy}\nabla_x C_{jy}^{\boxplus}]^{\boxplus II} \\
[g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxminus}]^{\boxplus II} &= \frac{1}{4}\sigma_{ik}^{ac}[\nabla^p C_{pacm}^{\boxplus} - \alpha_{cm}^{xy}\nabla_x C_{ay}^{\boxplus}]^{\boxplus II} \\
[g^{mi}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxminus}]^{\boxplus II} &= \frac{1}{4}[\nabla^p C_{pjkl}^{\boxplus} - \alpha_{kl}^{xy}\nabla_x C_{jy}^{\boxplus}]^{\boxplus II} \\
g^{ik}g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus} &= \nabla_m C + [\nabla^p C_{pm}^{\boxplus} - \frac{1}{2}\nabla_m C] \\
g^{mi}g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus} &= \nabla^p C_{pk}^{\boxplus} - \frac{1}{2}[\nabla^p C_{pk}^{\boxplus} - \frac{1}{2}\nabla_k C] \\
g^{ik}g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxminus} &= -[\nabla^p C_{pm}^{\boxplus} - \frac{1}{2}\nabla_m C] \\
g^{mi}g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxminus} &= \frac{1}{2}[\nabla^p C_{pk}^{\boxplus} - \frac{1}{2}\nabla_k C] \\
[g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus}]^{\boxplus II} &= [\nabla_m C_{ik}^{\boxplus}]^{\boxplus II} - \frac{1}{2}[\nabla^p C_{pmik}^{\boxplus} + \nabla_m C_{ik}^{\boxplus}] + \frac{1}{12}\alpha_{ikm}^{ace}[\nabla^p C_{pace}^{\boxplus} - \nabla_a C_{ce}^{\boxplus}] \\
[g^{mi}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus}]^{\boxplus II} &= [\nabla^p C_{pjkl}^{\boxplus}]^{\boxplus II} - \frac{1}{2}[\nabla^p C_{pjkl}^{\boxplus} + \nabla_j C_{kl}^{\boxplus}] - \frac{1}{12}\alpha_{jkl}^{bcd}[\nabla^p C_{pbcd}^{\boxplus} - \nabla_b C_{cd}^{\boxplus}] \\
g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus\boxplus} &= \frac{1}{2}[\nabla^p C_{pmik}^{\boxplus} + \nabla_m C_{ik}^{\boxplus}] - \frac{1}{20}\alpha_{ikm}^{ace}[3\nabla^p C_{pace}^{\boxplus} + \nabla_a C_{ce}^{\boxplus}] \\
g^{mi}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus\boxplus} &= \frac{1}{2}[\nabla^p C_{pjkl}^{\boxplus} + \nabla_j C_{kl}^{\boxplus}] + \frac{1}{20}\alpha_{jkl}^{bcd}[\nabla^p C_{pbcd}^{\boxplus} - 3\nabla_b C_{cd}^{\boxplus}] \\
g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus\boxminus} &= \frac{1}{15}\alpha_{ikm}^{ace}[\nabla^p C_{pace}^{\boxplus} + 2\nabla_a C_{ce}^{\boxplus}] \\
g^{mi}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus\boxminus} &= \frac{1}{30}\alpha_{jkl}^{bcd}[\nabla^p C_{pbcd}^{\boxplus} + 2\nabla_b C_{cd}^{\boxplus}] \\
g^{ik}g^{jl}[\nabla_m C_{ijkl}^{\boxplus}]^{\boxplus\boxplus} &= 0
\end{aligned}$$

$$g^{mi}g^{jl}[\nabla_m C_{ijkl}^{\square}]^{\square} = \nabla^p C_{pk}^{\square}$$

5.3.1. $\tau \circ \pi \circ \nabla[\tau \circ \pi \circ \mathbf{C}]$

$$\begin{aligned} \tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}] &= [\tau^{III} + \tau^{II} + \tau^I] \circ [[\nabla[\tau \circ \mathbf{C}^{\square}]]^{\square} + [\nabla[\tau \circ \mathbf{C}^{\square}]]^{\square}] \\ &+ [\tau^{III} + \tau^{II} + \tau^I] \circ [[\nabla[\tau \circ \mathbf{C}^{\square}]]^{\square}] \\ &+ [\tau^{III} + \tau^{II}] \circ [[\nabla[\tau \circ \mathbf{C}^{\square}]]^{\square}]^{\square} + [\nabla[\tau \circ \mathbf{C}^{\square}]]^{\square} \\ &+ [\tau^{III} + \tau^{II}] \circ [[\nabla \mathbf{C}^{\square}]^{\square}] \end{aligned}$$

with

$$\begin{aligned} g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= -\frac{1}{4}\sigma_{ik}^{ac}\nabla^p C_{pacm}^{\square III} \\ g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \nabla^p C_{pjkl}^{\square III} - \frac{1}{4}\nabla^p C_{pjkl}^{\square III} \\ g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{4}\sigma_{ik}^{ac}\nabla^p C_{pacm}^{\square III} \\ g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{4}\nabla^p C_{pjkl}^{\square III} \\ g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= -\frac{1}{2}\nabla^p C_{pmik}^{\square III} + \frac{1}{12}\alpha_{ikm}^{ace}\nabla^p C_{pace}^{\square III} \\ g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \nabla^p C_{pjkl}^{\square III} - \frac{1}{2}\nabla^p C_{pjkl}^{\square III} - \frac{1}{12}\alpha_{jkl}^{bcd}\nabla^p C_{pbcd}^{\square III} \\ g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{2}\nabla^p C_{pmik}^{\square III} - \frac{3}{20}\alpha_{ikm}^{ace}\nabla^p C_{pace}^{\square III} \\ g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{2}\nabla^p C_{pjkl}^{\square III} + \frac{1}{20}\alpha_{jkl}^{bcd}\nabla^p C_{pbcd}^{\square III} \\ g^{jl}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{15}\alpha_{ikm}^{ace}\nabla^p C_{pace}^{\square III} \\ g^{mi}[\nabla_m C_{ijkl}^{\square III}]^{\square} &= \frac{1}{30}\alpha_{jkl}^{bcd}\nabla^p C_{pbcd}^{\square III} \\ [g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= [\nabla_m C_{ik}^{\square II}]^{\square} + \frac{1}{8}\sigma_{ik}^{ac}\alpha_{cm}^{xy}[\nabla_x C_{ay}^{\square II}]^{\square} \\ [g^{mi}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= \frac{1}{2}\alpha_{kl}^{cd}[\nabla_c C_{jd}^{\square II}]^{\square} + \frac{1}{8}\alpha_{kl}^{xy}[\nabla_x C_{jy}^{\square II}]^{\square} \\ [g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= -\frac{1}{8}\sigma_{ik}^{ac}\alpha_{cm}^{xy}[\nabla_x C_{ay}^{\square II}]^{\square} \\ [g^{mi}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= -\frac{1}{8}\alpha_{kl}^{xy}[\nabla_x C_{jy}^{\square II}]^{\square} \\ [g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= [\nabla_m C_{ik}^{\square II}]^{\square} - \frac{1}{4}\alpha_{ik}^{ac}\sigma_{cm}^{ep}[\nabla_e C_{ap}^{\square II}]^{\square} - \frac{1}{6}\alpha_{ikm}^{ace}\nabla_a C_{ce}^{\square II} \\ [g^{mi}[\nabla_m C_{ijkl}^{\square II}]^{\square}]^{\square} &= \frac{1}{2}\alpha_{kl}^{cd}[\nabla_c C_{jd}^{\square II}]^{\square} - \frac{1}{4}\alpha_{kl}^{ac}\sigma_{jc}^{ep}[\nabla_e C_{ap}^{\square II}]^{\square} + \frac{1}{6}\alpha_{jkl}^{bcd}\nabla_b C_{cd}^{\square II} \\ g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{\square} &= \frac{1}{4}\alpha_{ik}^{ac}\sigma_{cm}^{ep}[\nabla_e C_{ap}^{\square II}]^{\square} + \frac{1}{10}\alpha_{ikm}^{ace}\nabla_a C_{ce}^{\square II} \\ g^{mi}[\nabla_m C_{ijkl}^{\square II}]^{\square} &= \frac{1}{4}\alpha_{kl}^{ac}\sigma_{jc}^{ep}[\nabla_e C_{ap}^{\square II}]^{\square} - \frac{1}{5}\alpha_{jkl}^{bcd}\nabla_b C_{cd}^{\square II} \\ g^{jl}[\nabla_m C_{ijkl}^{\square II}]^{\square} &= \frac{1}{15}\alpha_{ikm}^{ace}\nabla_a C_{ce}^{\square II} \end{aligned}$$

$$\begin{aligned}
g^{mi}[\nabla_m C_{ijkl}^{\square II}]^{\square} &= && \frac{1}{30} \alpha_{jkl}^{bcd} \nabla_b C_{cd}^{\square} \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && \nabla^p C_{pm}^{\square II} \\
g^{mi} g^{jl} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && \nabla^p C_{pk}^{\square II} - \frac{1}{2} \nabla^p C_{pk}^{\square II} \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && - \nabla^p C_{pm}^{\square II} \\
g^{mi} g^{jl} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && \frac{1}{2} \nabla^p C_{pk}^{\square II} \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && 0 \\
g^{jl} g^{mi} [\nabla_m C_{ijkl}^{\square II}]^{\square} &= && \nabla^p C_{pk}^{\square} \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\square I}]^{\square} &= && \nabla_m C - \frac{1}{4} \nabla_m C \\
g^{mi} g^{jl} [\nabla_m C_{ijkl}^{\square I}]^{\square} &= && \frac{1}{4} \nabla_k C + \frac{1}{8} \nabla_k C \\
g^{ik} g^{jl} [\nabla_m C_{ijkl}^{\square I}]^{\square} &= && \frac{1}{4} \nabla_m C \\
g^{mi} g^{jl} [\nabla_m C_{ijkl}^{\square I}]^{\square} &= && - \frac{1}{8} \nabla_k C
\end{aligned}$$

5.4. ρ

$$\begin{aligned}
\rho_{ijklm}^{abcde} \nabla_e C_{abcd} &= \nabla^e C^{abcd} [n_{me} - u_m u_e] n_{ia} n_{jb} n_{kc} n_{ld} \\
&+ \nabla^e C^{pqrs} [n_{me} - u_m u_e] n_{ap} u_q n_{cr} u_s [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\
&+ \nabla^e C^{pqcd} [n_{me} - u_m u_e] n_{ap} u_q n_{kc} n_{ld} [-\alpha_{ij}^{ab} u_b] \\
&+ \nabla^e C^{abrs} [n_{me} - u_m u_e] n_{ia} n_{jb} n_{cr} u_s [-\alpha_{kl}^{cd} u_d] \\
&=: [\nabla_m C_{ijkl}]^{NW} + [\nabla_m C_{ijkl}]^{WU} + [\nabla_m C_{ijkl}]^{LN} + [\nabla_m C_{ijkl}]^{NU} \\
&= \rho_{ijkl}^{abcd} \delta_m^e \nabla_e C_{abcd} \\
&= \nabla_m [\rho_{ijkl}^{abcd} C_{abcd}] = \nabla_m C_{ijkl}^{NW} + \nabla_m C_{ijkl}^{WU} + \nabla_m C_{ijkl}^{LN} + \nabla_m C_{ijkl}^{NU}
\end{aligned}$$

in more detail

$$\rho_{ijklm}^{abcde} \nabla_e [\rho_{abcd}^{pqrs} C_{pqrs}] = \nabla_m [\rho_{ijkl}^{abcd} C_{abcd}]$$

but

$$\begin{aligned}
[\nabla C^{MN}]^{MN} &= \nabla C^{MN} - \nabla \rho^{MN} \circ C^{MN} & [\nabla C^{LN}]^{LN} &= \nabla C^{LN} - \nabla \rho^{LN} \circ C^{LN} \\
[\nabla C^{WU}]^{MN} &= 0 & [\nabla C^{NU}]^{LN} &= 0 \\
[\nabla C^{LN}]^{MN} &= -\nabla \rho^{MN} \circ C^{LN} & [\nabla C^{MN}]^{LN} &= -\nabla \rho^{LN} \circ C^{MN} \\
[\nabla C^{NU}]^{MN} &= -\nabla \rho^{MN} \circ C^{NU} & [\nabla C^{WU}]^{LN} &= -\nabla \rho^{LN} \circ C^{WU} \\
[\nabla C^{WU}]^{WU} &= \nabla C^{WU} - \nabla \rho^{WU} \circ C^{WU} & [\nabla C^{NU}]^{NU} &= \nabla C^{NU} - \nabla \rho^{NU} \circ C^{NU} \\
[\nabla C^{MN}]^{WU} &= 0 & [\nabla C^{LN}]^{NU} &= 0 \\
[\nabla C^{LN}]^{WU} &= -\nabla \rho^{WU} \circ C^{LN} & [\nabla C^{MN}]^{NU} &= -\nabla \rho^{NU} \circ C^{MN} \\
[\nabla C^{NU}]^{WU} &= -\nabla \rho^{WU} \circ C^{NU} & [\nabla C^{WU}]^{NU} &= -\nabla \rho^{NU} \circ C^{WU}
\end{aligned}$$

which implies

$$\nabla[\rho^{NW} + \rho^W + \rho^{LN} + \rho^{NU}] \circ [[\rho^{NW} + \rho^W + \rho^{LN} + \rho^{NU}] \circ \mathbf{C}] = \nabla\rho \circ \mathbf{C} = 0$$

with

$$\begin{aligned} -\nabla_m[\rho^{NW}]_{ijkl}^{abcd} C_{abcd}^{NW} &= -\nabla_m u^n [-\alpha_{ij}^{ab} u_b] \mathbb{N}_{nakl} + [-\alpha_{kl}^{cd} u_d] \mathbb{N}_{ijn} \\ -\nabla_m[\rho^{NW}]_{ijkl}^{abcd} C_{abcd}^{LN} &= [-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{akl} \\ -\nabla_m[\rho^{NW}]_{ijkl}^{abcd} C_{abcd}^{NU} &= [-\alpha_{kl}^{cd} \nabla_m u_d] \mathbb{N}_{ijc} \\ -\nabla_m[\rho^W]_{ijkl}^{abcd} C_{abcd}^{UU} &= - [[-\alpha_{ij}^{ab} \nabla_m u_b] [-\alpha_{kl}^{cd} u_d] + [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} \nabla_m u_d]] \mathbb{U}_{ac} \\ -\nabla_m[\rho^W]_{ijkl}^{abcd} C_{abcd}^{LN} &= [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \mathbb{U}_{anc} \\ -\nabla_m[\rho^W]_{ijkl}^{abcd} C_{abcd}^{NU} &= [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \mathbb{N}_{nac} \\ -\nabla_m[\rho^{LN}]_{ijkl}^{abcd} C_{abcd}^{LN} &= - [[-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{akl} + [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \mathbb{U}_{anc}] \\ -\nabla_m[\rho^{LN}]_{ijkl}^{abcd} C_{abcd}^{NW} &= [-\alpha_{ij}^{ab} u_b] \nabla_m u^n \mathbb{N}_{nakl} \\ -\nabla_m[\rho^{LN}]_{ijkl}^{abcd} C_{abcd}^{UU} &= [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} \nabla_m u_d] \mathbb{U}_{ac} \\ -\nabla_m[\rho^{NU}]_{ijkl}^{abcd} C_{abcd}^{NU} &= - [[-\alpha_{kl}^{cd} \nabla_m u_d] \mathbb{N}_{ijc} + [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \mathbb{N}_{nac}] \\ -\nabla_m[\rho^{NU}]_{ijkl}^{abcd} C_{abcd}^{NW} &= [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \mathbb{N}_{ijn} \\ -\nabla_m[\rho^{NU}]_{ijkl}^{abcd} C_{abcd}^{UU} &= [-\alpha_{kl}^{cd} u_d] [-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{ac} \end{aligned}$$

and

$$\begin{aligned} \nabla_m C_{ijkl}^{NW} &= \nabla_m [\rho^{NW}]_{ijkl}^{abcd} C_{abcd}^{NW} + \nabla_m \mathbb{N}_{ijkl} - u_m \bullet \mathbb{N}_{ijkl} \\ \nabla_m C_{ijkl}^{UU} &= \nabla_m [\rho^W]_{ijkl}^{abcd} C_{abcd}^{UU} + [\nabla_m \mathbb{U}_{ac} - u_m \bullet \mathbb{U}_{ac}] [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\ \nabla_m C_{ijkl}^{LN} &= \nabla_m [\rho^{LN}]_{ijkl}^{abcd} C_{abcd}^{LN} + [\nabla_m \mathbb{U}_{akl} - u_m \bullet \mathbb{U}_{akl}] [-\alpha_{ij}^{ab} u_b] \\ \nabla_m C_{ijkl}^{NU} &= \nabla_m [\rho^{NU}]_{ijkl}^{abcd} C_{abcd}^{NU} + [\nabla_m \mathbb{N}_{ijc} - u_m \bullet \mathbb{N}_{ijc}] [-\alpha_{kl}^{cd} u_d] \end{aligned}$$

with the definitions

$$\begin{aligned} \nabla_i X_{i_1 \dots i_n} &:= n_i^a n_{i_1}^{a_1} \dots n_{i_n}^{a_n} \nabla_a X_{a_1 \dots a_n} \\ \bullet X_{i_1 \dots i_n} &:= u^a n_{i_1}^{a_1} \dots n_{i_n}^{a_n} \nabla_a X_{a_1 \dots a_n} \end{aligned}$$

and so

$$\begin{aligned} [\nabla_m C_{ijkl}^{NW}]^{NW} &= \nabla_m \mathbb{N}_{ijkl} - u_m \bullet \mathbb{N}_{ijkl} \\ [\nabla_m C_{ijkl}^{LN}]^{NW} &= \mathbb{U}_{akl} [-\alpha_{ij}^{ab} \nabla_m u_b] \\ [\nabla_m C_{ijkl}^{NU}]^{NW} &= \mathbb{N}_{ijc} [-\alpha_{kl}^{cd} \nabla_m u_d] \\ [\nabla_m C_{ijkl}^{UU}]^{UU} &= [\nabla_m \mathbb{U}_{ac} - u_m \bullet \mathbb{U}_{ac}] [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \\ [\nabla_m C_{ijkl}^{LN}]^{UU} &= \mathbb{U}_{anc} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \\ [\nabla_m C_{ijkl}^{NU}]^{UU} &= \mathbb{N}_{nac} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \\ [\nabla_m C_{ijkl}^{LN}]^{LN} &= [\nabla_m \mathbb{U}_{akl} - u_m \bullet \mathbb{U}_{akl}] [-\alpha_{ij}^{ab} u_b] \\ [\nabla_m C_{ijkl}^{NW}]^{LN} &= \mathbb{N}_{nakl} [-\alpha_{ij}^{ab} u_b] \nabla_m u^n \\ [\nabla_m C_{ijkl}^{UU}]^{LN} &= \mathbb{U}_{ac} [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} \nabla_m u_d] \\ [\nabla_m C_{ijkl}^{NU}]^{NU} &= [\nabla_m \mathbb{N}_{ijc} - u_m \bullet \mathbb{N}_{ijc}] [-\alpha_{kl}^{cd} u_d] \\ [\nabla_m C_{ijkl}^{NW}]^{NU} &= \mathbb{N}_{ijn} [-\alpha_{kl}^{cd} u_d] \nabla_m u^n \\ [\nabla_m C_{ijkl}^{UU}]^{NU} &= \mathbb{U}_{ac} [-\alpha_{kl}^{cd} u_d] [-\alpha_{ij}^{ab} \nabla_m u_b] \end{aligned}$$

6. $\nabla \mathbf{R}$

See [12] for more details.

6.0.1. $\nabla[\tau \circ \mathbf{R}]$

$$\nabla[\tau \circ \mathbf{R}] = \nabla \mathbf{R}^{III} + \nabla \mathbf{R}^{II} + \nabla \mathbf{R}^I$$

The distribution of components reads $80 = 40 + 36 + 4$.

6.0.2. $\nabla[\rho \circ \mathbf{R}]$

$$\nabla[\rho \circ \mathbf{R}] = \nabla \mathbf{R}^{NV} + \nabla \mathbf{R}^{LN} + \nabla \mathbf{R}^{NJ} + \nabla \mathbf{R}^W$$

The distribution of components reads $80 = 24 + 32 + 24$.

6.1. π

$$\pi \circ [\nabla \mathbf{R}] = [\nabla \mathbf{R}]^{\boxplus\boxplus} + [\nabla \mathbf{R}]^{\boxplus}$$

with

$$\begin{aligned} [\nabla_m R_{ijkl}]^{\boxplus\boxplus} &= \nabla_m R_{ijkl} - \frac{1}{8} [\alpha_{mij}^{eab} \nabla_e R_{abkl} + \alpha_{mkl}^{ecd} \nabla_e R_{ijcd}] \\ [\nabla_m R_{ijkl}]^{\boxplus} &= \frac{1}{8} [\alpha_{mij}^{eab} \nabla_e R_{abkl} + \alpha_{mkl}^{ecd} \nabla_e R_{ijcd}] \\ &= \frac{1}{4} [\eta_{ijm}^a \nabla^b [*R]_{abkl} + \eta_{klm}^a \nabla^b [*R]_{abij}] \\ &= \frac{1}{8} [\eta_{ijm}^b \eta_{kl}^{cd} + \eta_{klm}^b \eta_{ij}^{cd}] \nabla^a [*R^*]_{abcd} \\ &= - \frac{1}{8} [\alpha_{mij}^{eab} \alpha_{kl}^{cd} + \alpha_{mkl}^{eab} \alpha_{ij}^{cd}] g_{ac} \nabla^p [[*R^*]_{pdbe} + g_{bd} [*R^*]_{pe}] \\ &= 0 \end{aligned}$$

because of

$$\begin{aligned} 0 &= \alpha_{mij}^{eab} \nabla_e R_{abkl} \\ \Leftrightarrow 0 &= \nabla^p [*R]_{pjkl} \\ \Leftrightarrow 0 &= \nabla^p [*R^*]_{pjkl} \end{aligned}$$

6.1.1. $\pi \circ \nabla[\tau \circ \mathbf{R}]$

$$\pi \circ \nabla[\tau \circ \mathbf{R}] = [\pi^{\boxplus\boxplus} + \pi^{\boxplus}] \circ [\nabla \mathbf{R}^{III} + \nabla \mathbf{R}^{II} + \nabla \mathbf{R}^I]$$

with

$$\begin{aligned} [\nabla_m R_{ijkl}^{III}]^{\boxplus\boxplus} &= \nabla_m R_{ijkl}^{III} - \frac{1}{8} [\alpha_{mij}^{eab} \nabla_e R_{abkl}^{III} + \alpha_{mkl}^{ecd} \nabla_e R_{ijcd}^{III}] \\ [\nabla_m R_{ijkl}^{II}]^{\boxplus\boxplus} &= \nabla_m R_{ijkl}^{II} - \frac{1}{8} [\alpha_{mij}^{eab} \alpha_{kl}^{cd} + \alpha_{mkl}^{ecd} \alpha_{ij}^{ab}] g_{ac} \nabla_e R_{bd}^{II} \end{aligned}$$

$$\begin{aligned}
[\nabla_m R_{ijkl}^I]^\boxplus &= \nabla_m R_{ijkl}^I - \frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{ecd}\alpha_{ij}^{ab}]\frac{1}{12}g_{ac}g_{bd}\nabla_e R \\
[\nabla_m R_{ijkl}^{III}]^\boxplus &= \frac{1}{8}[\alpha_{mij}^{eab}\nabla_e R_{abkl}^{III} + \alpha_{mkl}^{ecd}\nabla_e R_{ijcd}^{III}] \\
&= \frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{eab}\alpha_{ij}^{cd}]g_{ac}\nabla^p R_{pdbe}^{III} \\
[\nabla_m R_{ijkl}^{II}]^\boxplus &= \frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{ecd}\alpha_{ij}^{ab}]g_{ac}\nabla_e R_{bd}^{II} \\
&= -\frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{eab}\alpha_{ij}^{cd}]g_{ac}[\frac{1}{2}\alpha_{be}^{rs}[\nabla_r R_{ds}^{II} + g_{ds}\nabla^p R_{pr}^{II}] + g_{bd}\nabla^p R_{pe}^{II}] \\
[\nabla_m R_{ijkl}^I]^\boxplus &= \frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{ecd}\alpha_{ij}^{ab}]\frac{1}{12}g_{ac}g_{bd}\nabla_e R \\
&= \frac{1}{8}[\alpha_{mij}^{eab}\alpha_{kl}^{cd} + \alpha_{mkl}^{ecd}\alpha_{ij}^{ab}]g_{ac}[\frac{1}{12}\alpha_{be}^{rs}g_{ds}\nabla_r R + \frac{1}{4}g_{bd}\nabla_e R]
\end{aligned}$$

and

$$\begin{aligned}
0 &= [\nabla_m R_{ijkl}^{III} + \nabla_m R_{ijkl}^{II} + \nabla_m R_{ijkl}^I]^\boxplus \\
0 &= \nabla^p [[*R*]_{pjkl}^{III} + [*R*]_{pjkl}^{II} + [*R*]_{pjkl}^I]
\end{aligned}$$

in particular

$$\begin{aligned}
0 &= \alpha_{mij}^{eab}[\nabla_e R_{abkl}^{III} + \alpha_{kl}^{cd}g_{ac}\nabla_e R_{bd}^{II} + \alpha_{kl}^{cd}\frac{1}{12}g_{ac}g_{bd}\nabla_e R] \\
0 &= -\nabla^p R_{pjkl}^{III} + \alpha_{kl}^{cd}\frac{1}{2}[\nabla_c R_{jd}^{II} + g_{jd}\nabla^p R_{pc}^{II}] - \alpha_{kl}^{cd}\frac{1}{12}g_{jd}\nabla_c R
\end{aligned}$$

6.2. τ

$$\tau \circ \nabla \mathbf{R} = [\nabla \mathbf{R}]^{III} + [\nabla \mathbf{R}]^{II} + [\nabla \mathbf{R}]^I$$

with

$$\begin{aligned}
[\nabla_m R_{ijkl}]^{III} &= \nabla_m R_{ijkl} - \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}X_{bdm}^\boxplus + g_{am}Y_{bcd}^\boxplus + g_{cm}Y_{dab}^\boxplus] \\
[\nabla_m R_{ijkl}]^{II} &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}[X_{bdm}^\boxplus]^{II} + g_{am}[Y_{bcd}^\boxplus]^{II} + g_{cm}[Y_{dab}^\boxplus]^{II}] \\
[\nabla_m R_{ijkl}]^I &= \alpha_{ij}^{ab}\alpha_{kl}^{cd}[g_{ac}g_{bd}X_m^\boxplus + g_{bm}g_{ac}Y_d^\boxplus + g_{dm}g_{ac}Y_b^\boxplus]
\end{aligned}$$

and

$$\begin{aligned}
[X_{ikm}^\boxplus]^{II\ \square\square} &= \frac{1}{2}[\nabla_m R_{ik}]^{II\ \square\square} \\
[X_{ikm}^\boxplus]^{II\ \boxplus} &= \frac{4}{5}[\nabla_m R_{ik} + \frac{1}{4}\sigma_{ik}^{ac}\nabla^p R_{pacm}]^{II\ \boxplus} \\
[Y_{ikm}^\boxplus]^{II\ \boxplus} &= \frac{1}{5}[\nabla^p R_{pikm} + \frac{1}{2}\alpha_{km}^{ce}\nabla_e R_{ic}]^{II\ \boxplus} \\
X_k^\boxplus &= \frac{1}{18}[\nabla_k R - \nabla^a R_{ak}] \\
Y_k^\boxplus &= \frac{1}{36}[-\nabla_k R + 4\nabla^a R_{ak}]
\end{aligned}$$

6.2.1. $\tau \circ \nabla[\tau \circ \mathbf{R}]$

$$\tau \circ \nabla[\tau \circ \mathbf{R}] = [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{R}^{III} + [\tau^{II} + \tau^I] \circ \nabla \mathbf{R}^{II} + \tau^I \circ \nabla \mathbf{R}^I$$

with

$$\begin{aligned} [\nabla_m R_{ijkl}^{III}]^{III} &= \nabla_m R_{ijkl}^{III} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} - g_{am} Y_{bcd}^{\boxplus III} - g_{cm} Y_{dab}^{\boxplus III}] \\ [\nabla_m R_{ijkl}^{III}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} X_{bdm}^{\boxplus III} + g_{am} Y_{bcd}^{\boxplus III} + g_{cm} Y_{dab}^{\boxplus III}] \\ [\nabla_m R_{ijkl}^{II}]^{II} &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} [X_{bdm}^{\boxplus II}]^{II}] \\ [\nabla_m R_{ijkl}^{II}]^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} X_m^{\boxplus II} + g_{bm} g_{ac} Y_d^{\boxplus II} + g_{dm} g_{ac} Y_b^{\boxplus II}] \\ [\nabla_m R_{ijkl}^I]^I &= \alpha_{ij}^{ab} \alpha_{kl}^{cd} [g_{ac} g_{bd} X_m^{\boxplus I}] \end{aligned}$$

and

$$\begin{aligned} X_{ikm}^{\boxplus III} &= \sigma_{ik}^{ac} Y_{acm}^{\boxplus III} \\ Y_{ikm}^{\boxplus III} &= \frac{1}{5} \nabla^p R_{pikm}^{III} \\ [X_{ikm}^{\boxplus II}]^{II} &= \frac{1}{2} [\nabla_m R_{ik}]^{II} \\ X_m^{\boxplus II} &= -\frac{1}{2} Y_m^{\boxplus II} \\ Y_m^{\boxplus II} &= \frac{1}{9} \nabla^a R_{am}^{II} \\ X_m^{\boxplus I} &= \frac{1}{24} \nabla_m R \end{aligned}$$

6.3. $\tau \circ \pi$

$$\tau \circ \pi \circ \nabla \mathbf{R} = [\tau^{III} + \tau^{II} + \tau^I] \circ [[\nabla \mathbf{R}]^{\boxplus III} + [\nabla \mathbf{R}]^{\boxplus II}]$$

with

$$\begin{aligned} [g^{jl} [\nabla_m R_{ijkl}]^{\boxplus III}]^{II} &= [\nabla_m R_{ik}]^{II} - \frac{1}{4} \sigma_{ik}^{ac} [\nabla^p R_{pacm} - \alpha_{cm}^{xy} \nabla_x R_{ay}]^{II} \\ [g^{mi} [\nabla_m R_{ijkl}]^{\boxplus III}]^{II} &= [\nabla^p R_{pjkl}]^{II} - \frac{1}{4} [\nabla^p R_{pjkl} - \alpha_{kl}^{xy} \nabla_x R_{jy}]^{II} \\ [g^{jl} [\nabla_m R_{ijkl}]^{\boxplus II}]^{II} &= \frac{1}{4} \sigma_{ik}^{ac} [\nabla^p R_{pacm} - \alpha_{cm}^{xy} \nabla_x R_{ay}]^{II} \\ [g^{mi} [\nabla_m R_{ijkl}]^{\boxplus II}]^{II} &= \frac{1}{4} [\nabla^p R_{pjkl} - \alpha_{kl}^{xy} \nabla_x R_{jy}]^{II} \\ g^{ik} g^{jl} [\nabla_m R_{ijkl}]^{\boxplus III} &= \nabla_m R + [\nabla^p R_{pm} - \frac{1}{2} \nabla_m R] \\ g^{mi} g^{jl} [\nabla_m R_{ijkl}]^{\boxplus III} &= \nabla^p R_{pk} - \frac{1}{2} [\nabla^p R_{pk} - \nabla_k R] \\ g^{ik} g^{jl} [\nabla_m R_{ijkl}]^{\boxplus II} &= -[\nabla^p R_{pm} - \frac{1}{2} \nabla_m R] \\ g^{mi} g^{jl} [\nabla_m R_{ijkl}]^{\boxplus II} &= \frac{1}{2} [\nabla^p R_{pk} - \frac{1}{2} \nabla_k R] \end{aligned}$$

in particular

$$\begin{aligned} 0 &= [\nabla^p R_{pjkl} - \alpha_{kl}^{xy} \nabla_x R_{jy}]^{II} \\ 0 &= \nabla^p R_{pm} - \frac{1}{2} \nabla_m R \end{aligned}$$

6.3.1. $\tau \circ \pi \circ \nabla[\tau \circ \mathbf{R}]$

$$\begin{aligned} \tau \circ \pi \circ \nabla[\tau \circ \mathbf{R}] &= [\tau^{III} + \tau^{II}] \circ [\nabla \mathbf{R}^{III}]^{\boxplus} + [\tau^{II} + \tau^I] \circ [\nabla \mathbf{R}^{II}]^{\boxplus} + \tau^I \circ [\nabla \mathbf{R}^I]^{\boxplus} \\ &+ [\tau^{III} + \tau^{II}] \circ [\nabla \mathbf{R}^{III}]^{\boxminus} + [\tau^{II} + \tau^I] \circ [\nabla \mathbf{R}^{II}]^{\boxminus} + \tau^I \circ [\nabla \mathbf{R}^I]^{\boxminus} \end{aligned}$$

with the corresponding traces

$$\begin{aligned} g^{jl}[\nabla_m R_{ijkl}]^{\boxplus} &= \frac{1}{4} \sigma_{ik}^{ac} \nabla^p R_{pacm}^{III} \\ g^{mi}[\nabla_m R_{ijkl}]^{\boxplus} &= \frac{1}{4} \nabla^p R_{pjkl}^{III} \\ [g^{jl}[\nabla_m R_{ijkl}]^{\boxplus}]^{II} &= -\frac{1}{8} \sigma_{ik}^{ac} \alpha_{cm}^{xy} [\nabla_x R_{ay}^{II}]^{II} \\ [g^{mi}[\nabla_m R_{ijkl}]^{\boxplus}]^{II} &= -\frac{1}{8} \alpha_{kl}^{xy} [\nabla_x R_{jy}^{II}]^{II} \\ g^{ik} g^{jl}[\nabla_m R_{ijkl}]^{\boxplus} &= -\nabla^p R_{pm}^{II} \\ g^{mi} g^{jl}[\nabla_m R_{ijkl}]^{\boxplus} &= \frac{1}{2} \nabla^p R_{pk}^{II} \\ g^{ik} g^{jl}[\nabla_m R_{ijkl}]^{\boxplus} &= \frac{1}{4} \nabla_m R \\ g^{mi} g^{jl}[\nabla_m R_{ijkl}]^{\boxplus} &= -\frac{1}{8} \nabla_k R \end{aligned}$$

in particular

$$\begin{aligned} 0 &= -\nabla^p R_{pjkl}^{III} + \frac{1}{2} \alpha_{kl}^{cd} [\nabla_c R_{jd}^{II} + \frac{1}{3} g_{jd} \nabla^p R_{pc}^{II}] \\ 0 &= -\nabla^p R_{pm}^{II} + \frac{1}{4} \nabla_m R \end{aligned}$$

6.4. ρ

$$\begin{aligned} \rho_{ijklm}^{abcde} \nabla_e R_{abcd} &= [\nabla_m R_{ijkl}]^{NV} + [\nabla_m R_{ijkl}]^{UW} + [\nabla_m R_{ijkl}]^{LN} + [\nabla_m R_{ijkl}]^{NU} \\ = \nabla_m [\rho_{ijkl}^{abcd} R_{abcd}] &= \nabla_m R_{ijkl}^{NV} + \nabla_m R_{ijkl}^{UW} + \nabla_m R_{ijkl}^{LN} + \nabla_m R_{ijkl}^{NU} \end{aligned}$$

but

$$\begin{aligned} [\nabla \mathbf{R}]^{NV} &= \nabla \mathbf{R}^{NV} - \nabla \rho^{NV} \circ \mathbf{R} \\ [\nabla \mathbf{R}]^{UW} &= \nabla \mathbf{R}^{UW} - \nabla \rho^{UW} \circ \mathbf{R} \\ [\nabla \mathbf{R}]^{LN} &= \nabla \mathbf{R}^{LN} - \nabla \rho^{LN} \circ \mathbf{R} \\ [\nabla \mathbf{R}]^{NU} &= \nabla \mathbf{R}^{NU} - \nabla \rho^{NU} \circ \mathbf{R} \end{aligned}$$

in particular

$$[\nabla_m R_{ijkl}]^{NV} = \nabla_m \mathcal{N}_{ijkl}^{\boxplus} - u_m \bullet \mathcal{N}_{ijkl}^{\boxplus} + [-\alpha_{ij}^{ab} \nabla_m u_b] \mathcal{U}_{akl}^{\boxplus} + [-\alpha_{kl}^{cd} \nabla_m u_d] \mathcal{N}_{ijc}^{\boxplus} \quad (6.1)$$

$$[\nabla_m R_{ijkl}]^{UW} = [\nabla_m \mathcal{U}_{ac}^{\boxplus} - u_m \bullet \mathcal{U}_{ac}^{\boxplus} + \nabla_m u^n [\mathcal{U}_{anc}^{\boxplus} + \mathcal{N}_{nac}^{\boxplus}]] [-\alpha_{ij}^{ab} u_b] [-\alpha_{kl}^{cd} u_d] \quad (6.2)$$

$$[\nabla_m R_{ijkl}]^{LN} = [\nabla_m \mathcal{U}_{akl}^{\boxplus} - u_m \bullet \mathcal{U}_{akl}^{\boxplus} + \nabla_m u^n \mathcal{N}_{nakl}^{\boxplus} + [-\alpha_{kl}^{cd} \nabla_m u_d] \mathcal{U}_{ac}^{\boxplus}] [-\alpha_{ij}^{ab} u_b] \quad (6.3)$$

$$[\nabla_m R_{ijkl}]^{NU} = [\nabla_m \mathcal{N}_{ijc}^{\boxplus} - u_m \bullet \mathcal{N}_{ijc}^{\boxplus} + \nabla_m u^n \mathcal{N}_{ijn}^{\boxplus} + [-\alpha_{ij}^{ab} \nabla_m u_b] \mathcal{U}_{ac}^{\boxplus}] [-\alpha_{kl}^{cd} u_d] \quad (6.4)$$

and

$$\begin{aligned}
0 &= \nabla^l [*R*]_{ijkl} \\
&= \nabla_m [*R*]_{abc} \rho_{ijkl}^{labc} \\
&= \nabla_m [*R*]_{abcl} [n^{ml} - u^m u^l] n_i^a n_j^b n_k^c \\
&\quad - \nabla_m [*R*]_{abcl} n^{ml} n_{ia} n_{jb} u_k u_c \\
&\quad - \nabla_m [*R*]_{pqcl} [n^{ml} - u^m u^l] \alpha_{ij}^{ab} n_a^p u_b u^q n_k^c \\
&\quad + \nabla_m [*R*]_{pqcl} n^{ml} \alpha_{ij}^{ab} n_a^p u_b u^q u_k u^c
\end{aligned}$$

or

$$0 = n_i^a n_j^b n_k^c n^{ml} \nabla_m [*R*]_{abcl} \quad (6.5)$$

$$-u^l [n_i^a n_j^b n_k^c u^m \nabla_m [*R*]_{abcl}]$$

$$0 = u_k [n_i^a n_j^b n_k^c n^{ml} \nabla_m [*R*]_{abcl}] \quad (6.6)$$

$$0 = u_j [n_i^a n_j^b n_k^c n^{ml} \nabla_m [*R*]_{abcl}] \quad (6.7)$$

$$-u_j u^l [n_i^a n_j^b n_k^c u^m \nabla_m [*R*]_{abcl}]$$

$$0 = u_j u_k [n_i^a n_j^b n_k^c n^{ml} \nabla_m [*R*]_{abcl}] \quad (6.8)$$

Equating the coefficients of (6.5) - (6.8) with those of (6.1) - (6.4) leads to

$$\begin{aligned}
0 &\neq \nabla_m \mathbb{N}_{ijkl}^{\boxplus} - u_m \bullet \mathbb{N}_{ijkl}^{\boxplus} + [-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{akl}^{\boxplus} + [-\alpha_{kl}^{cd} \nabla_m u_d] \mathbb{U}_{ijc}^{\boxplus} \\
&\quad - u_l [\nabla_m \mathbb{N}_{ijk}^{\boxplus} - u_m \bullet \mathbb{N}_{ijk}^{\boxplus} + \nabla_m u^n \mathbb{N}_{ijnk}^{\boxplus} + [-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{ak}^{\boxplus}] \\
0 &\neq u_k [\nabla_m \mathbb{N}_{ijl}^{\boxplus} - u_m \bullet \mathbb{N}_{ijl}^{\boxplus} + \nabla_m u^n \mathbb{N}_{ijnl}^{\boxplus} + [-\alpha_{ij}^{ab} \nabla_m u_b] \mathbb{U}_{al}^{\boxplus}] \\
0 &\neq -u_j [\nabla_m \mathbb{U}_{ikl}^{\boxplus} - u_m \bullet \mathbb{U}_{ikl}^{\boxplus} + \nabla_m u^n \mathbb{U}_{nikl}^{\boxplus} + [-\alpha_{kl}^{cd} \nabla_m u_d] \mathbb{U}_{ic}^{\boxplus}] \\
&\quad + u_j u_l [\nabla_m \mathbb{U}_{ik}^{\boxplus} - u_m \bullet \mathbb{U}_{ik}^{\boxplus} + \nabla_m u^n [\mathbb{U}_{ink}^{\boxplus} + \mathbb{N}_{nik}^{\boxplus}]] \\
0 &\neq -u_j u_k [\nabla_m \mathbb{U}_{il}^{\boxplus} - u_m \bullet \mathbb{U}_{il}^{\boxplus} + \nabla_m u^n [\mathbb{U}_{inl}^{\boxplus} + \mathbb{N}_{nil}^{\boxplus}]]
\end{aligned}$$

Substitution $R \rightarrow [*R*]$ and taking the trace over m, l gives

$$\begin{aligned}
0 &= \nabla^l [*U*]_{ijkl}^{\boxplus} - [-\alpha_{ij}^{ab} \nabla^l u_b] [*N*]_{akl}^{\boxplus} - [-\alpha_{kl}^{cd} \nabla^l u_d] [*UN*]_{ijc}^{\boxplus} \\
&\quad + \bullet [*UN*]_{ijk}^{\boxplus} - \bullet u^n [*U*]_{ijnk}^{\boxplus} - [-\alpha_{ij}^{ab} \bullet u_b] [*N*]_{ak}^{\boxplus} \\
0 &= \nabla^l [*UN*]_{ijl}^{\boxplus} - \nabla^l u^n [*U*]_{ijnl}^{\boxplus} - [-\alpha_{ij}^{ab} \nabla^l u_b] [*N*]_{al}^{\boxplus} \\
0 &= \nabla^l [*N*]_{ikl}^{\boxplus} - \nabla^l u^n [*U*]_{nikl}^{\boxplus} - [-\alpha_{kl}^{cd} \nabla^l u_d] [*N*]_{ic}^{\boxplus} \\
&\quad + \bullet [*N*]_{ik}^{\boxplus} - \bullet u^n [*N*]_{ink}^{\boxplus} - \bullet u^n [*UN*]_{nik}^{\boxplus} \\
0 &= \nabla^l [*N*]_{il}^{\boxplus} - \nabla^m u^n [*N*]_{inl}^{\boxplus} - \nabla^m u^n [*UN*]_{nil}^{\boxplus}
\end{aligned}$$

or

$$\begin{aligned}
0 &= \eta_{kl}^c [\nabla^l \mathbb{U}_{ic}^{\boxplus}] + \nabla^l u^n [\eta_{mi}^b [*N*]_{bc}^{\boxplus} + \eta_{nc}^d [\mathbb{U}_{id}^{\boxplus}]] \\
&\quad + \bullet [\mathbb{U}_{ik}^{\boxplus}] + \bullet u^n [\eta_{kn}^c \mathbb{U}_{ic}^{\boxplus} + \eta_{mi}^b [*N*]_{bk}^{\boxplus}] \\
0 &= \nabla^l [\mathbb{U}_{il}^{\boxplus}] + \nabla^l u^n [\eta_{ln}^c \mathbb{U}_{ic}^{\boxplus} + \eta_{mi}^b [*N*]_{bl}^{\boxplus}] \\
0 &= \eta_{kl}^c [\nabla^l [*N*]_{ic}^{\boxplus}] + \nabla^l u^n [\eta_{in}^a \mathbb{U}_{ac}^{\boxplus} + \eta_{nc}^d [*N*]_{id}^{\boxplus}] \\
&\quad + \bullet [*N*]_{ik}^{\boxplus} - \bullet u^n [\eta_{mk}^c [*N*]_{ic}^{\boxplus} + \eta_{ni}^a [\mathbb{U}_{ak}^{\boxplus}]] \\
0 &= \nabla^l [*N*]_{il}^{\boxplus} - \nabla^l u^n [\eta_{nl}^c [*N*]_{ic}^{\boxplus} + \eta_{ni}^a [\mathbb{U}_{al}^{\boxplus}]]
\end{aligned}$$

6.5. $\nabla[\omega \circ \rho \circ \mathbf{R}]$

$$\begin{aligned}
\nabla_m[\omega \circ \rho \circ R]_{ijkl} &= \nabla_m \sum_{\beta} Z_{\beta ij} Z_{\beta kl} \\
&= \sum_{\beta} [\nabla_m Z_{\beta ij} Z_{\beta kl} + Z_{\beta ij} \nabla_m Z_{\beta kl}] \\
&=: \sum_{\beta} [h_{\beta mij} Z_{\beta kl} + Z_{\beta ij} h_{\beta mkl}]
\end{aligned}$$

The identity $\nabla^p[*R*]_{pjkl} = 0$ becomes

$$\begin{aligned}
0 &= \nabla^p \sum_{\beta} *Z_{\beta pj} *Z_{\beta kl} \\
&=: \sum_{\beta} [*h_{\beta j} *Z_{\beta kl} + *Z_{\beta pj} [*h_{\beta}]^p_{kl}] \\
&= \sum_{\beta} [*h_{\beta j} *Z_{\beta kl} - h_{\beta j} Z_{\beta kl} + \alpha_{kl}^{cd} [\frac{1}{2} g_{cj} [h_{\beta}]_d^{pq} Z_{\beta pq} + [h_{\beta}]_{dj}^p Z_{\beta pc} + g_{cj} h_{\beta}^b Z_{\beta bd}]]
\end{aligned}$$

and

$$\begin{aligned}
0 &= g^{mi} [\nabla_m R_{ijkl}^{III}]^{\boxplus} + [g^{mi} [\nabla_m R_{ijkl}^{II}]]^{\boxplus II} \\
0 &= g^{mi} g^{jl} [\nabla_m R_{ijkl}^{II}]^{\boxplus} + g^{mi} g^{jl} [\nabla_m R_{ijkl}^I]^{\boxplus}
\end{aligned}$$

can be written as

$$\begin{aligned}
\nabla^p R_{pjkl}^{III} &= \frac{1}{2} [-\alpha_{kl}^{cd} \nabla_c R_{jd}^{II}]^{\boxplus II} = \frac{1}{2} \sum_{\beta} [*h_{\beta j} *Z_{\beta kl} - h_{\beta j} Z_{\beta kl} + \alpha_{kl}^{cd} Z_{\beta j}^b h_{\beta dbc}]^{\boxplus II} \\
\nabla^p R_{pk}^{II} &= \frac{1}{4} \nabla_k R = \sum_{\beta} [-*h_{\beta}^b *Z_{\beta bk} - h_{\beta}^b Z_{\beta bk}]
\end{aligned}$$

The condition

$$Z_{\beta ij} := \alpha_{ij}^{ab} \nabla_a P_{\beta b}$$

implies

$$\begin{aligned}
h_{\beta abc} &= \nabla_a \nabla_b P_{\beta c} - \nabla_a \nabla_c P_{\beta b} \\
h_{\beta k} &= \nabla^p \nabla_p P_{\beta k} + R_k^p P_{\beta p} \\
*h_{\beta k} &= \eta_{ak}^{pq} \nabla^a \nabla_p P_{\beta q} \\
&= -\frac{1}{2} \eta_k^{apq} R_{apq}^x P_{\beta x} \\
&= -[*R]_k^q{}^x P_{\beta x} \\
&= 0
\end{aligned}$$

so

$$\begin{aligned}
0 &= \sum_{\beta} *Z_{\beta pj} [*h_{\beta}]^p_{kl} \\
\nabla^p R_{pjkl}^{III} &= \frac{1}{2} \sum_{\beta} [-h_{\beta j} Z_{\beta kl} + \alpha_{kl}^{cd} Z_{\beta j}^b h_{\beta dbc}]^{\boxplus II} \\
\nabla^p R_{pk}^{II} &= \sum_{\beta} [-h_{\beta}^b Z_{\beta bk}]
\end{aligned}$$

Part I.

See [5], [6], [7] and [8] for comparison.

7. \mathbf{T}

Setting

$$\frac{[*\mathbf{R}^*]}{\kappa} =: T$$

implies

$$\begin{aligned} \pi \circ [*\mathbf{R}^*] &= \pi \circ \kappa \mathbf{T} \\ \tau \circ [*\mathbf{R}^*] &= \tau \circ \kappa \mathbf{T} \\ \rho \circ [*\mathbf{R}^*] &= \rho \circ \kappa \mathbf{T} \\ \rho \circ \tau \circ [*\mathbf{R}^*] &= \rho \circ \tau \circ \kappa \mathbf{T} \end{aligned}$$

in particular

$$\begin{aligned} -\mathbf{R}^{III} &= \frac{[*\mathbf{R}^*]^{\boxplus}}{[*\mathbf{R}^*]^{III}} = \kappa \mathbf{T}^{\boxplus} \\ -\mathbf{R}^{II} &= \frac{[*\mathbf{R}^*]^{II}}{[*\mathbf{R}^*]^{II}} = \kappa \mathbf{T}^{II} \\ -\mathbf{R}^I &= \frac{[*\mathbf{R}^*]^I}{[*\mathbf{R}^*]^I} = \kappa \mathbf{T}^I \\ [*[\mathbf{R}^{NW}]^*] &= \frac{[*\mathbf{R}^*]^{UN}}{[*\mathbf{R}^*]^{UN}} = \kappa \mathbf{T}^{EE} \\ [*[\mathbf{R}^{U}]^*] &= \frac{[*\mathbf{R}^*]^{NW}}{[*\mathbf{R}^*]^{NW}} = \kappa \mathbf{T}^{HH} \\ [*[\mathbf{R}^{LN}]^*] &= \frac{[*\mathbf{R}^*]^{NU}}{[*\mathbf{R}^*]^{NU}} = \kappa \mathbf{T}^{HE} \\ [*[\mathbf{R}^{N}]^*] &= \frac{[*\mathbf{R}^*]^{UN}}{[*\mathbf{R}^*]^{UN}} = \kappa \mathbf{T}^{EH} \end{aligned}$$

with the definitions

$$\begin{aligned} u^j u^l T_{ijkl} &=: EE_{ik}^{\boxplus} \\ n_i^a n_j^b u^l T_{abkl} &=: HE_{ijc}^{\boxplus} \\ u^j n_k^c n_l^d T_{ijcd} &=: EH_{akl}^{\boxplus} \\ n_i^a n_j^b n_k^c n_l^d T_{abcd} &=: HH_{ijkl}^{\boxplus} \end{aligned}$$

The same holds for the covariant derivatives

$$\nabla \frac{[*\mathbf{R}^*]}{\kappa} = \nabla \mathbf{T}$$

so

$$\begin{aligned} \pi \circ \nabla [* \mathbf{R}^*] &= \pi \circ \kappa \nabla \mathbf{T} \\ \tau \circ \nabla [* \mathbf{R}^*] &= \tau \circ \kappa \nabla \mathbf{T} \\ \rho \circ \nabla [* \mathbf{R}^*] &= \rho \circ \kappa \nabla \mathbf{T} \end{aligned}$$

in particular

$$\begin{aligned}
[\nabla[*\mathbf{R}*}]^{\boxplus\boxplus} &= \kappa[\nabla\mathbf{T}]^{\boxplus\boxplus} \\
[\nabla[*\mathbf{R}*}]^{\boxplus} &= \kappa[\nabla\mathbf{T}]^{\boxplus} \\
[\nabla[*\mathbf{R}*}]^{III} &= \kappa[\nabla\mathbf{T}]^{III} \\
[\nabla[*\mathbf{R}*}]^{II} &= \kappa[\nabla\mathbf{T}]^{II} \\
[\nabla[*\mathbf{R}*}]^I &= \kappa[\nabla\mathbf{T}]^I \\
[\nabla[*\mathbf{R}*}]^{NW} &= \kappa[\nabla\mathbf{T}]^{HH} \\
[\nabla[*\mathbf{R}*}]^{UN} &= \kappa[\nabla\mathbf{T}]^{EH} \\
[\nabla[*\mathbf{R}*}]^{NU} &= \kappa[\nabla\mathbf{T}]^{HE} \\
[\nabla[*\mathbf{R}*}]^{UW} &= \kappa[\nabla\mathbf{T}]^{EE}
\end{aligned}$$

and for the corresponding identity

$$0 = \nabla^p \frac{[*R*]_{pjkl}}{\kappa} = \nabla^p T_{pjkl}$$

in particular

$$\begin{aligned}
0 &= \nabla^p T_{pjkl}^{III} + \alpha_{kl}^{cd} \frac{1}{2} [\nabla_c T_{jd}^{II} + g_{jd} \nabla^p T_{pc}^{II}] + \alpha_{kl}^{cd} \frac{1}{12} g_{jd} \nabla_c T \\
0 &= \nabla^p T_{pjkl}^{III} + \frac{1}{2} \alpha_{kl}^{cd} [\nabla_c T_{jd}^{II}]^{II} \\
0 &= \nabla^l T_{jl}^{II} + \frac{1}{4} \nabla_j T \\
0 &= \eta_{kl}^c [\nabla^l [*HH*]_{ic}^{\boxplus\boxplus} - \nabla^l u^n [\eta_{ni}^b [EH*]_{bc}^{\boxplus\boxplus} + \eta_{nc}^d [*HE]_{id}^{\boxplus\boxplus}] \\
&\quad - \bullet [*HE]_{ik}^{\boxplus\boxplus} + \bullet u^n [\eta_{kn}^c [*HH*]_{ic}^{\boxplus\boxplus} + \eta_{ni}^b EE_{bk}^{\boxplus\boxplus}] \\
0 &= -\nabla^l [*HE]_{il}^{\boxplus\boxplus} + \nabla^l u^n [\eta_{ln}^c [*HH*]_{ic}^{\boxplus\boxplus} + \eta_{ni}^b EE_{bl}^{\boxplus\boxplus}] \\
0 &= \eta_{kl}^c [-\nabla^l [EH*]_{ic}^{\boxplus\boxplus} + \nabla^l u^n [\eta_{in}^a [*HH*]_{ac}^{\boxplus\boxplus} + \eta_{nc}^d EE_{id}^{\boxplus\boxplus}] \\
&\quad + \bullet EE_{ik}^{\boxplus\boxplus} + \bullet u^n [\eta_{nk}^c [EH*]_{ic}^{\boxplus\boxplus} + \eta_{ni}^a [*HE]_{ak}^{\boxplus\boxplus}] \\
0 &= \nabla^l EE_{il}^{\boxplus\boxplus} + \nabla^l u^n [\eta_{nl}^c [EH*]_{ic}^{\boxplus\boxplus} + \eta_{ni}^a [*HE]_{al}^{\boxplus\boxplus}]
\end{aligned}$$

The difference between

$$[*R*]_{ijkl} = \kappa T_{ijkl}$$

and the Einstein equations

$$[*R*]_{ik} = \kappa T_{ik}$$

can be emphasized by considering $\tau \circ [*R*]$ in both cases:

$$\tau \circ [*R*] \left\{ \begin{array}{l} [*R*]^{III} = \kappa \mathbf{T}^{III} \\ [*R*]^{II} = \kappa \mathbf{T}^{II} \\ [*R*]^I = \kappa \mathbf{T}^I \end{array} \right\} \kappa \mathbf{T} \circ \mathbf{T}$$

and

$$\tau \circ [*R*] \left\{ \begin{array}{l} [*R*]^{III} = [*R*]^{III} \\ [*R*]^{II} = \kappa \mathbf{T}^{II} \\ [*R*]^I = \kappa \mathbf{T}^I \end{array} \right\} \kappa \mathbf{T}^I + \kappa \mathbf{T}^{II} + [*R*]^{III}$$

Part II.

Repeating the reasoning of 1.2 in the developed notation.

$$g^{jl}[*R*]_{ijkl}^I = g^{jl}\kappa T_{ijkl}^I = g^{jl}[\sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}]^I$$

Removing the projector I gives

$$g^{jl}[*R*]_{ijkl} = g^{jl}\kappa T_{ijkl} = g^{jl}[\sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}]$$

Removing the trace gives

$$[*R*]_{ijkl} = \kappa T_{ijkl} = \sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}$$

8. $\omega \circ \rho \circ \mathbf{T}$

Setting

$$\frac{[*R*]}{\kappa} =: T$$

gives

$$\omega \circ \rho \circ [*R*] = \kappa \omega \circ \rho \circ \mathbf{T}$$

in particular

$$\begin{aligned} *N_{\alpha ij}^{\boxplus MN} *N_{\alpha kl}^{\boxplus MN} &= [-\alpha_{ij}^{ab}u_b] *N_{\alpha a}^{\boxplus MN} [-\alpha_{kl}^{cd}u_d] *N_{\alpha c}^{\boxplus MN} \\ &=: [-\alpha_{ij}^{ab}u_b] \sqrt{\kappa} E_{\alpha a}^{\boxplus EE} [-\alpha_{kl}^{cd}u_d] \sqrt{\kappa} E_{\alpha c}^{\boxplus EE} =: \sqrt{\kappa} E_{\alpha ij}^{\boxplus EE} \sqrt{\kappa} E_{\alpha kl}^{\boxplus EE} \\ *U_{\alpha ij}^{\boxplus UW} *U_{\alpha kl}^{\boxplus UW} &= [-\eta_{ij}^a U_{\alpha a}^{\boxplus UW}] [-\eta_{kl}^c U_{\alpha c}^{\boxplus UW}] \\ &=: \eta_{ij}^a \sqrt{\kappa} *H_{\alpha a}^{\boxplus HH} \eta_{kl}^c \sqrt{\kappa} *H_{\alpha c}^{\boxplus HH} =: \sqrt{\kappa} H_{\alpha ij}^{\boxplus HH} \sqrt{\kappa} H_{\alpha kl}^{\boxplus HH} \\ *U_{\alpha ij}^{\boxplus LN} *N_{\alpha kl}^{\boxplus LN} &= [-\eta_{ij}^a U_{\alpha a}^{\boxplus LN}] [-\alpha_{kl}^{cd}u_d] *N_{\alpha c}^{\boxplus LN} \\ &=: \eta_{ij}^a \sqrt{\kappa} *H_{\alpha a}^{\boxplus HE} [-\alpha_{kl}^{cd}u_d] \sqrt{\kappa} E_{\alpha c}^{\boxplus HE} =: \sqrt{\kappa} H_{\alpha ij}^{\boxplus HE} \sqrt{\kappa} E_{\alpha kl}^{\boxplus HE} \\ *N_{\alpha ij}^{\boxplus LN} *U_{\alpha kl}^{\boxplus LN} &= [-\alpha_{ij}^{ab}u_b] *N_{\alpha a}^{\boxplus LN} [-\eta_{kl}^c U_{\alpha c}^{\boxplus LN}] \\ &=: [-\alpha_{ij}^{ab}u_b] \sqrt{\kappa} E_{\alpha a}^{\boxplus HE} \eta_{kl}^c \sqrt{\kappa} *H_{\alpha c}^{\boxplus HE} =: \sqrt{\kappa} E_{\alpha ij}^{\boxplus HE} \sqrt{\kappa} H_{\alpha kl}^{\boxplus HE} \end{aligned}$$

and

$$*Z_{\beta ij} *Z_{\beta kl} =: \sqrt{\kappa} F_{\beta ij} \sqrt{\kappa} F_{\beta kl}$$

The identity $\nabla^p T_{pjkl} = 0$ becomes

$$\begin{aligned} 0 &= \nabla^p \sum_{\beta} F_{\beta pj} F_{\beta kl} \\ &=: \sum_{\beta} [j_{\beta j} F_{\beta kl} + F_{\beta pj} [j_{\beta}^p]_{kl}] \end{aligned}$$

or

$$\begin{aligned}
-\nabla^p T_{pjkl}^{III} &= \frac{1}{2}[-\alpha_{kl}^{cd} \nabla_c T_{jd}^{II}]^{II} = \frac{1}{2} \sum_{\beta} [j_{\beta j} F_{\beta kl} - *j_{\beta j} *F_{\beta kl} + \alpha_{kl}^{cd} *F_{\beta j}{}^b *j_{\beta dbc}]^{II} \\
\nabla^p T_{pk}^{II} &= -\frac{1}{4} \nabla_k T = \sum_{\beta} [-j_{\beta}^b F_{\beta bk} - *j_{\beta}^b *F_{\beta bk}]
\end{aligned}$$

The condition

$$F_{\beta ij} := \alpha_{ij}^{ab} \nabla_a A_{\beta b}$$

implies

$$\begin{aligned}
j_{\beta abc} &= \nabla_a \nabla_b A_{\beta c} - \nabla_a \nabla_c A_{\beta b} \\
j_{\beta k} &= \nabla^p \nabla_p A_{\beta k} + R_k{}^p A_{\beta p} \\
*j_{\beta k} &= \eta_{ak}{}^{pq} \nabla^a \nabla_p A_{\beta q} \\
&= -\frac{1}{2} \eta_k{}^{apq} R_{apq}{}^x A_{\beta x} \\
&= -[*R]_k{}^q{}^x A_{\beta x} \\
&= 0
\end{aligned}$$

so

$$\begin{aligned}
-\nabla^p T_{pjkl}^{III} &= \frac{1}{2}[-\alpha_{kl}^{cd} \nabla_c T_{jd}^{II}]^{II} = \frac{1}{2} \sum_{\beta} [j_{\beta j} F_{\beta kl} + \alpha_{kl}^{cd} *F_{\beta j}{}^b *j_{\beta dbc}]^{II} \\
\nabla^p T_{pk}^{II} &= -\frac{1}{4} \nabla_k T = \sum_{\beta} [-j_{\beta}^b F_{\beta bk}]
\end{aligned}$$

Part III.

9. κ

Let the units *meter* 'm', *second* 's', *kilogram* 'kg' and the *electrostatic unit of charge* 'esu' be defined. Let π denote in this (and only in this) section the number π and set 'S' for the unit of the *action* S (usually 'Joule second'). Set

$$\begin{aligned}
 c &:= 299792458 \cdot \frac{\text{m}}{\text{s}} & (9.1) \\
 G &:= 6,67259 \cdot 10^{-20} \cdot \left[\frac{\text{m}}{\text{esu}} \right]^2 \cdot \left[\frac{\text{m}}{\text{s}} \right]^4 \\
 \text{alternatively } G &:= 6,67259 \cdot 10^{-11} \cdot \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \\
 K_{[1]} &:= \sqrt{\frac{c^4}{8\pi G}} \cdot \frac{\text{m}}{\text{esu}} \\
 &\approx 6,940 \cdot 10^{25} \\
 K_{[2]} &:= \frac{c^2}{8\pi G} \cdot \frac{\text{m}}{\text{kg}} \\
 &\approx 5,359 \cdot 10^{25} \\
 K_{[3]} &:= \frac{c^4}{8\pi G} \cdot \frac{\text{m}}{\text{kg}} \cdot \frac{\text{s}^2}{\text{m}^2} \\
 &\approx 4,817 \cdot 10^{42}
 \end{aligned}$$

The equations

$$\begin{aligned}
 R_{ijkl} &= \frac{1}{2} \alpha_{ij}^{ab} \alpha_{kl}^{cd} [\partial_a \partial_c g_{bd} + g_{pq} \Gamma_{ac}^p \Gamma_{bd}^q] \\
 R_{ijkl} &= \kappa T_{ijkl} \\
 T_{ijkl} &= \sum_{\beta} F_{\beta ij} F_{\beta kl} \\
 T_{ijkl} &= T_{ijkl}^{HH} + T_{ijkl}^{EH} + T_{ijkl}^{HE} + T_{ijkl}^{EE}
 \end{aligned}$$

imply

$$\begin{aligned}
 [R_{ijkl}] &= \frac{1}{\text{m}^2} \\
 [R_{ijkl}] &= [\kappa] [T_{ijkl}] \\
 [F_{\beta ij}] &= \sqrt{[T_{ijkl}]} \\
 [T_{ijkl}] &= [EE_{ik}] + [EH_{ijk}] + [HE_{ijk}] + [HH_{ijkl}]
 \end{aligned}$$

9.1. X, S

$$[T_{ijkl}] = : \frac{X}{\text{m}^4}$$

$$\begin{aligned}
[\kappa] &= \frac{\text{m}^2}{\text{X}} \\
[F_{\beta ij}] &= \frac{\sqrt{\text{X}}}{\text{m}^2} \\
\left[\int EE^a_a \frac{1}{6} \eta_{ijk} dx^i dx^j dx^k \right] &= \frac{\text{X}}{\text{m}}
\end{aligned}$$

Alternatively

$$\begin{aligned}
[T_{ijkl}] &=: K_{[3]} \cdot \frac{\text{S}}{\text{s} \cdot \text{m}^3} = K_{[3]} \cdot \text{kg} \cdot \frac{\text{m}^2}{\text{s}^2} \cdot \frac{1}{\text{m}^3} \\
[\kappa] &= \frac{1}{K_{[3]}} \cdot \frac{\text{m} \cdot \text{s}}{\text{S}} \\
[F_{\beta ij}] &= \sqrt{K_{[3]} \cdot \frac{\text{S}}{\text{s} \cdot \text{m}^3}} \\
\left[\int EE^a_a \frac{1}{6} \eta_{ijk} dx^i dx^j dx^k \right] &= K_{[3]} \cdot \frac{\text{S}}{\text{s}} = K_{[3]} \cdot \text{kg} \cdot \frac{\text{m}^2}{\text{s}^2}
\end{aligned}$$

9.2. esu

$$\begin{aligned}
[F_{\beta ij}] &=: K_{[1]} \cdot \frac{\text{esu}}{\text{m}^2} \\
[T_{ijkl}] &= K_{[1]}^2 \cdot \frac{\text{esu}^2}{\text{m}^4} \\
[\kappa] &= \frac{\text{m}^2}{K_{[1]}^2 \cdot \text{esu}^2} \\
\left[\int EE^a_a \frac{1}{6} \eta_{ijk} dx^i dx^j dx^k \right] &= K_{[1]}^2 \cdot \frac{\text{esu}^2}{\text{m}}
\end{aligned}$$

9.3. kg

$$\begin{aligned}
\left[\int EE^a_a \frac{1}{6} \eta_{ijk} dx^i dx^j dx^k \right] &=: K_{[2]} \cdot \text{kg} \\
[T_{ijkl}] &= K_{[2]} \cdot \frac{\text{kg}}{\text{m}^3} \\
[\kappa] &= \frac{1}{K_{[2]}} \cdot \frac{\text{m}}{\text{kg}} \\
[F_{\beta ij}] &= \sqrt{K_{[2]} \cdot \frac{\text{kg}}{\text{m}^3}}
\end{aligned}$$

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