Five Representations for Pi Constant

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Abstract

This note contains five representations for Pi constant

Representations

1. For $a > 0$, $1 < b < \sqrt{1+2a}$, we have

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) b^{-2n-3}}{2n+3} + \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(1+a)^{n+2}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k} b^{2k+1}}{2k+1}$$

2. For $a > 0$, $1 < b < \sqrt[4]{1+2a}$, we have

$$\frac{\pi}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{-4n-1}}{4n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k} b^{4k+3}}{4k+3}$$

3. For $a > 0$, $1 < b < \sqrt[6]{1+2a}$, we have

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{-6n-4}}{6n+4} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k} b^{6k+2}}{6k+2}$$

4. For $n \in \mathbb{N} \cup \{0\}$, we have

$$\frac{\pi}{2^{n+1} n!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(2k-1)!\,(n+2k-1)!\,2^{n+2k-1}} + \int_{-1}^{1} \frac{J_n(1+i\,x)}{(1+i\,x)^{n+1}}\,dx$$

Where $J_n(z)$ is the Bessel function and

$$\int_{-1}^{1} \frac{J_n(1+i\,x)}{(1+i\,x)^{n+1}}\,dx = \int_{-1}^{1} Re\left(\frac{J_n(1+i\,x)}{(1+i\,x)^{n+1}}\right) dx = 2\int_{0}^{1} Re\left(\frac{J_n(1+i\,x)}{(1+i\,x)^{n+1}}\right) dx$$

5. An integral involving the constant Pi

$$\pi = \int_0^{\pi} \cosh(\sin x)\cos(\cos x)\,dx$$

putting

$$u = F(u) = \int_0^{u} \cosh(\sin x)\cos(\cos x)\,dx$$

Must be $\pi = F(\pi)$, i.e. , π is a fixed point of $F(x)$. besides F is contractive :

$$F'(x) = \cosh(\sin x)\cos(\cos x)$$

$$0 < \cos 1 \ll F'(x) < 1 \quad \forall\, x \in I = (2.3147\ldots, 3.9684\ldots)$$

$$\pi \in I$$

The succession

$$u_{n+1} = \int_0^{u_n} \cosh(\sin x)\cos(\cos x)\,dx \quad , \qquad u_1 = 3$$

is convergent :

$$\lim_{n\to\infty} u_n = \pi$$

$$u_1 = 3\,, u_{10} = 3.1410304\ldots, u_{20} = 3.1415904\ldots, u_{30} = 3.1415926\ldots, \ldots$$

References

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