Some Formulas for e Constant

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March 17 , 2009

Abstract

In this paper we show some formulas for constant

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=2.718281\ldots$$

Introduction

In this paper infinite summations and products are shown for constant

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=2.718281\ldots$$

The number Pi , $\pi=3.1415\ldots$appears in some formulas ,also appear radicals of

$$\sqrt{2\pm\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}$$

The numbers of Bell appear in the formula (16) .

Formulas

(1) $e^{-1}=2\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(\pi/3)^{2n-2k}}{k!(2n-2k)!}$

(2) $e^{-1}=2\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(\pi/6)^{2n-2k+1}}{k!(2n-2k+1)}$

(3) $e=2\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(-1)^k(\pi/3)^{2n-2k}}{k!(2n-2k)!}$

(4) $e=2\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(-1)^k(\pi/6)^{2n-2k+1}}{k!(2n-2k+1)!}$

(5) $e\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicals}=2\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(-1)^k\left(\pi\,2^{-m-1}\right)^{2n-2k+1}}{k!(2n-2k+1)!}\quad ,m\in\mathbb{N}$

(6) $e^{\overbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}$ ...

$$(6)\quad e^{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicals}} = 2\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \frac{(-1)^k\left(\pi\, 2^{-m-1}\right)^{2n-2k}}{k!(2n-2k)!} \quad , m \in \mathbb{N}$$

$$(7)\quad e^{-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicals} = 2\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \frac{\left(\pi\, 2^{-m-1}\right)^{2n-2k+1}}{k!(2n-2k+1)!} \quad , m \in \mathbb{N}$$

$$(8)\quad e^{-1}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicals} = 2\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \frac{\left(\pi\, 2^{-m-1}\right)^{2n-2k}}{k!(2n-2k)!} \quad . m \in \mathbb{N}$$

$$(9)\quad e = \sum_{n=0}^{\infty} \frac{p_n}{n!(2\,b^2)^n}$$

In formula (9) is

$$p_{n+1} = 2(a^2+b^2)p_n - 8a^2b^2n\,p_{n-1}\,, n \in \mathbb{N}$$
$$p_0 = 1\,, p_1 = 2(a^2+b^2)\,;\; a, b \in \mathbb{Z}-\{0\}$$

$$(10)\quad e^{-1} = \sum_{n=0}^{\infty} \frac{q_n}{n!(2\,b^2)^n}$$

In formula (10) is

$$q_{n+1} = 2(a^2-b^2)q_n - 8a^2b^2n\,q_{n-1}\,, n \in \mathbb{N}$$
$$q_0 = 1\,, q_1 = 2(a^2-b^2)\;;\, a, b \in \mathbb{Z}-\{0\}$$

$$(11)\quad e = \sum_{n=0}^{\infty} \frac{c_n\, a^n}{n!}$$

In formula (11) is

$$c_{n+1} = c_n + 2nc_{n-1}\,, n \in \mathbb{N}\,, c_0 = c_1 = 1\,; a = \frac{2}{1+\sqrt{5}}$$

$$(12)\quad e = \sum_{n=0}^{\infty} \frac{c_n\, a^n}{n!}$$

In formula (12) is

$$c_{n+1} = 2nc_{n-1} + 3n(n-1)c_{n-2}\,, n \in \mathbb{N}-\{1\}, c_0 = 1, c_1 = 0, c_2 = 2$$
$$a = \frac{\left(\frac{1}{2}+\frac{\sqrt{69}}{18}\right)^{1/3}}{\frac{1}{3}+\left(\frac{25}{54}+\frac{\sqrt{69}}{18}\right)^{1/3}}$$

$$(13)\quad e = \frac{5}{3}\sqrt{\frac{5}{3}}\;\prod_{n=1}^{\infty}\left(\frac{2n+5}{2n+3}\right)\left(\frac{(2n+1)(2n+5)}{(2n+3)^2}\right)^n \frac{\sqrt{(2n+1)(2n+5)}}{2n+3}$$

$$(14)\quad e = \frac{7}{4}\sqrt[3]{\frac{7}{4}}\;\prod_{n=1}^{\infty}\left(\frac{3n+7}{3n+4}\right)\left(\frac{(3n+1)(3n+7)}{(3n+4)^2}\right)^n \sqrt[3]{\frac{(3n+1)(3n+7)}{(3n+4)^2}}$$

(15) $e=\left(\frac{2m+1}{m+1}\right)^{1+\frac{1}{m}}\ \prod_{n=1}^{\infty}\left(\frac{m\,n+2m+1}{m\,n+m+1}\right)\left(\frac{(m\,n+1)(m\,n+2m+1)}{(m\,n+m+1)^2}\right)^{n+\frac{1}{m}}\ ,m\in\mathbb{N}$

(16) $e=\frac{1}{B_m}\prod_{n=1}^{\infty}\left(1+\frac{(n+1)^{m-1}}{A_n}\right)\ ,m\in\mathbb{N}$

In formula (16) is

$$A_n=\sum_{k=1}^{n}\binom{n}{k}(n-k)!\,k^m\ ,n,m\in\mathbb{N}$$
$$A_{n+1}=(n+1)A_n+(n+1)^m\ ,A_1=1$$
$$B_m=\{1,2,5,15,52,203,877,4140,\ldots\}$$
$$B_{m+1}=\sum_{k=0}^{m}\binom{m}{k}B_k\ ,B_0=B_1=1\ \text{, Bell numbers}$$

(17) $e=2\left(\sqrt{2}\right)^{\ln 2}\left(\sqrt[3]{\sqrt[2]{2}}\right)^{(\ln 2)^2}\left(\sqrt[4]{\sqrt[3]{\sqrt[2]{2}}}\right)^{(\ln 2)^3}\ \ldots$

(18) $e^{k-1}=\prod_{n=0}^{\infty}k^{(\ln k)^n/(n+1)!}=k\left(\sqrt[2]{k}\right)^{\ln k}\left(\sqrt[3]{\sqrt[2]{k}}\right)^{(\ln k)^2}\left(\sqrt[4]{\sqrt[3]{\sqrt[2]{k}}}\right)^{(\ln k)^3}\ \ldots,\ k\in\mathbb{N}-\{1\}$

(19) $e=2\cdot 2^{1-\ln 2}\cdot 2^{(1-\ln 2)^2}\cdot 2^{(1-\ln 2)^3}\cdots$

(20) $e=3\cdot 3^{1-\ln 3}\cdot 3^{(1-\ln 3)^2}\cdot 3^{(1-\ln 3)^3}\cdots$

(21) $e=2^{\ln 2}\cdot 4^{(1-\ln 2)\ln 2}\cdot 8^{(1-\ln 2)^2\ln 2}\cdot 16^{(1-\ln 2)^3\ln 2}\cdots$

(22) $e=1+\sum_{n=0}^{\infty}\frac{1}{(m(n+1))!}\sum_{k=1}^{m}\binom{m(n+1)}{mn+k}(m-k)!\ \ ,m\in\mathbb{N}$

(23) $e=2\cdot 4^{1/5}\cdot 8^{1/71}\cdot 16^{1/9060}\cdots=\prod_{n=1}^{\infty}2^{n\,a_n}$

In formula (23) is

$$a_{n+1}=1+\left[(n+1)\left(\frac{1}{\ln 2}-\sum_{k=1}^{n}\frac{k}{a_k}\right)^{-1}\right]\ ,a_1=1,n\in\mathbb{N}$$

The function $[x]=$ integer part of x .

(24) $e=\left(1+\frac{1}{m}\right)^m/\left(1-\sum_{n=0}^{\infty}\frac{a_n}{(n+1)m^{n+1}}\right)\ ,m\in\mathbb{N}-\{1\}$

In formula (24) is

$$a_n=\sum_{k=0}^{n}\frac{(-1)^{n-k}(n-k+1)b_k}{n-k+2}\quad ,n\in\mathbb{N}\cup\{0\}$$
$$b_{n+1}=-\frac{1}{n+1}\sum_{k=0}^{n}\frac{(-1)^{n-k}(n-k+1)b_k}{n-k+2}\quad ,b_0=1,n\in\mathbb{N}\cup\{0\}$$
$$a_n=\{\frac{1}{2},-\frac{11}{12},\frac{21}{16},-\frac{2447}{1440},\frac{4795}{2304},\ldots\}$$

(25) $\left(1+\frac{1}{m}\right)^m\left(2-m\ln\left(1+\frac{1}{m}\right)\right)\le e\le\left(1+\frac{1}{m}\right)^m+3\left(1-m\ln\left(1+\frac{1}{m}\right)\right)\ ,m\in\mathbb{N}$

References

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