# Interpreting the summation notation when the lower limit is greater than the upper limit *

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**Abstract**

In interpreting the sigma notation for finite summation, it is generally assumed that the lower limit of summation is less than or equal to the upper limit. This presumption has led to certain misconceptions, especially concerning what constitutes an empty sum. This paper addresses how to construe the sigma notation when the lower limit is greater than the upper limit.

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# 1 Introduction

Consider a sequence $(f_i)$, with domain $[a, b]$. When $b \geq a$, with $a, b \in \mathbb{Z}$, the notation $\sum_{i=a}^{b} f_i$ is understood to mean $f_a + f_{a+1} + \ldots + f_b$. In particular

$$\sum_{i=a}^{a} f_i \equiv f_a \,. \tag{1.1}$$

As long as the conditions $c \geq a$ and $b \geq c + 1$ hold, the following splitting of summation is intuitive and straightforward:

$$\sum_{i=a}^{c} f_i + \sum_{i=c+1}^{b} f_i = \sum_{i=a}^{b} f_i \,. \tag{1.2}$$

The aim of this paper is to extend the interpretation of the sigma notation so that $\sum_{i=a}^{b} f_i$ becomes meaningful for $f_i$ defined in the domain $[a, b]$ if $a \leq b$ (or domain $(b, a)$ if $a > b$).

# 2 Empty sum

To achieve the purpose stated in the last paragraph of the Introduction section we first define an *empty sum* by setting $c = b$ in (1.2) to obtain

$$\sum_{i=a}^{b} f_i + \sum_{i=b+1}^{b} f_i = \sum_{i=a}^{b} f_i \,,$$

which makes sense only if we adopt the following familiar interpretation:

$$\boxed{\sum_{i=b+1}^{b} f_i = 0, \quad b \in \mathbb{Z}} \,. \tag{2.1}$$

It appears that it is in view of (2.1) that many authors adopt the misconceived interpretation that whenever the upper limit of a summation is less than its lower limit, the sum evaluates to zero. It must be emphasized that the empty sum, as defined in (2.1), requires the lower limit to be exactly 1 greater than the upper limit. The misconstrued interpretation of declaring a sum empty once the upper limit is lower than the lower limit of summation, which is inconsistent with the theory of summation, is found in scientific literature (see for example [1]) and software (as implemented in PARI-GP and GNU Emacs Calc, for example), as well as in various informal writings and posts on the internet (see [2, 3, 4]). In the next section we give an interpretation of the sigma notation, with lower limit greater than upper limit, that is consistent with summation theory and which subsumes the *empty sum* (2.1).

## 3 Summation with lower limit greater than upper limit

On setting $b = a - 1$ in (1.2) we obtain

$$\sum_{i=a}^{c} f_i + \sum_{i=c+1}^{a-1} f_i = \sum_{i=a}^{a-1} f_i\,,$$

which on account of (2.1) gives

$$\sum_{i=a}^{c} f_i + \sum_{i=c+1}^{a-1} f_i = 0\,. \tag{3.1}$$

Since $a > c$ whenever $a - 1 > c + 1$, (3.1) allows the interpretation of the summation notation whenever the lower limit is greater than the upper limit of summation, and we have

$$\boxed{\sum_{i=a}^{c} f_i \equiv -\sum_{i=c+1}^{a-1} f_i, \quad a, c \in \mathbb{Z} \text{ and } a > c}\,, \tag{3.2}$$

provided the sequence $(f_i)$ is defined in the interval $(c, a)$.

Setting $c = a$ in (3.1) and using (1.1) we obtain

$$\boxed{\sum_{i=a+1}^{a-1} f_i \equiv -f_a}\,.$$

# 4 Examples

The essence of a proper interpretation of the sigma notation becomes evident when a summation can be expressed in closed form by a formula. We give two examples here.

## 4.1 The sum of the natural numbers in an interval

Consider the sum

$$S = \sum_{i=m}^{n} i \,, \quad m, n \in \mathbb{Z} \,. \tag{4.1}$$

If $n \geq m$ in (4.1) it is easy to show that

$$S = (n + m)(n - m + 1)/2 \,. \tag{4.2}$$

On the other hand, if $m > n$ in (4.1) then we must invoke (3.2) and thus

$$\begin{aligned} S = -\sum_{i=n+1}^{m-1} i &= -\left((m-1) + (n+1)\right)\left((m-1) - (n+1) + 1\right)/2 \\ &= (n + m)(n - m + 1)/2 \,, \end{aligned}$$

which provides the same evaluation formula as (4.2), underscoring the correctness of our interpretation of the sum with lower limit greater than upper limit.

We see that the sum in (4.1) can be computed without regard to which of the indices $m$ and $n$ is greater. The empty sum here corresponds to $m = n + 1$.

## 4.2 Sum of consecutive Fibonacci numbers

The generalized Fibonacci sequence $(F_i)$, $i \in \mathbb{Z}$, is defined through the recurrence relation $F_{i+2} = F_{i+1} + F_i$, where two boundary terms (usually $F_0$ and $F_1$) need to be specified.

Consider the sum

$$S = \sum_{i=m}^{n} F_i \,, \quad m, n \in \mathbb{Z} \,. \tag{4.3}$$

Using the recurrence relation we can write

$$\sum_{i=m}^{n} F_{i+2} = \sum_{i=m}^{n} F_{i+1} + \sum_{i=m}^{n} F_i \,. \tag{4.4}$$

Assuming $n > m$ and shifting the summation index, we have

$$\sum_{i=m}^{n} F_{i+2} = \sum_{i=m+2}^{n+2} F_i = -F_m - F_{m+1} + \sum_{i=m}^{n} F_i + F_{n+1} + F_{n+2}$$

and

$$\sum_{i=m}^{n} F_{i+1} = \sum_{i=m+1}^{n+1} F_i = -F_m + \sum_{i=m}^{n} F_i + F_{n+1} \,.$$

Substituting these in (4.4), we obtain

$$S = \sum_{i=m}^{n} F_i = F_{n+2} - F_{m+1} \,. \tag{4.5}$$

If we had assumed that $m > n$ in (4.3) we would have used (3.2) and the result would have been

$$S = \sum_{i=m}^{n} F_i = -\sum_{i=n+1}^{m-1} F_i = -\left(F_{m+1} - F_{n+2}\right) = F_{n+2} - F_{m+1} \,,$$

which, of course, would be the same result as (4.5).

# 5 Conclusion

We have extended the interpretation of the sigma summation notation to allow the evaluation of a sum in which the lower limit is greater than the upper limit of summation. The scheme is

$$\boxed{\sum_{i=a}^{c} f_i \equiv -\sum_{i=c+1}^{a-1} f_i, \quad a, c \in \mathbb{Z} \text{ and } a > c} \,.$$

In particular

$$\sum_{i=b+1}^{b} f_i = 0, \quad b \in \mathbb{Z} \,,$$

and

$$\sum_{i=a+1}^{a-1} f_i \equiv -f_a \,.$$

## References


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