Some Series for the Constant Pi

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Abstract

In this note we show a collection of series for the constant Pi:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415926535 \ldots$$

Resumen

En esta nota mostramos una colección de series para la constante Pi:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415926535 \ldots$$

Introducción

En esta nota mostramos algunas series para la constante Pi , las cuales están relacionadas con la sencilla fórmula:

$$(1) \qquad \pi = 24 \int_0^{\sqrt{6}-\sqrt{3}+\sqrt{2}-2} \frac{1}{1+x^2} dx = 24 \sum_{n=0}^{\infty} c_n \left(\sqrt{6}-\sqrt{3}+\sqrt{2}-2\right)^{2n+1}$$

donde

$$c_n = \frac{(-1)^n}{2n+1} \ , n = 0,1,2,3, \ldots$$

Mediante métodos elementales del análisis se pueden obtener muchas series relacionadas con la fórmula (1) . Por el momento indiquemos que la serie de (1) se puede escribir de varias formas distintas:

(2) $$\pi = 24 \sum_{n=0}^{\infty} c_n \left(\sqrt{2}-1\right)^{2n+1} \left(\sqrt{3}-\sqrt{2}\right)^{2n+1}$$

(3) $$\pi = 24 \sum_{n=0}^{\infty} c_n \left(\sqrt{2}+1\right)^{-2n-1} \left(\sqrt{3}+\sqrt{2}\right)^{-2n-1}$$

(4) $$\pi = 24 \sum_{n=0}^{\infty} c_n \left(\sqrt{6}+\sqrt{3}+\sqrt{2}+2\right)^{-2n-1}$$

(5) $$\pi = 24 \sum_{n=0}^{\infty} c_n \left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right)^{2n+1}$$

(6) $$\pi = 24 \sum_{n=0}^{\infty} c_n \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}\right)^{2n+1}$$

Algunas Series Para La Constante Pi

Sea $r = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$, se tiene:

(7) $$\pi = \frac{4}{5}\left(-50+23\sqrt{2}-23\sqrt{3}+25\sqrt{6}\right) + \frac{12}{5}\sum_{n=0}^{\infty} r^{4n+5}\left(\frac{10}{4n+5}-\frac{3-2\sqrt{2}}{4n+7}-\frac{3+2\sqrt{2}}{4n+3}\right)$$

(8) $$\pi = \frac{4}{33}\left(-390+207\sqrt{2}-199\sqrt{3}+191\sqrt{6}\right) + \frac{4}{11}\sum_{n=0}^{\infty} r^{4n+5}\left(\frac{66}{4n+5}-\frac{15-8\sqrt{3}}{4n+7}-\frac{15-8\sqrt{3}}{4n+3}\right)$$

(9) $$\pi = \frac{4}{5}\left(386 - 413\sqrt{2} + 333\sqrt{3} - 153\sqrt{6}\right) + \frac{12}{5}\sum_{n=0}^{\infty} r^{4n+5}\left(\frac{10}{4n+5} - \frac{3 - 100\sqrt{2} + 80\sqrt{3}}{4n+7} - \frac{51 + 36\sqrt{2}}{4n+3}\right)$$

(10) $$\pi = \frac{4}{1785}\left(-5917648 + 4173707\sqrt{2} - 3409482\sqrt{3} + 2417619\sqrt{6}\right) + \frac{1}{85}\sum_{n=0}^{\infty} r^{4n+9}\left(\frac{1}{4n+3} - \frac{3366}{4n+7} + \frac{2040}{4n+9} - \frac{1280\sqrt{6} - 3101}{4n+11}\right)$$

(11) $$\pi = \frac{4}{231 \cdot 165} A + \frac{1}{330}\sum_{n=0}^{\infty} r^{4n+13}\left(\frac{7920}{4n+13} - \frac{133}{4n+15} - \frac{14223}{4n+11} + \frac{3333}{4n+7} - \frac{1}{4n+3}\right)$$

donde

$$A = 32749247168 - 23157181201\sqrt{2} + 18907724626\sqrt{3} - 13369787449\sqrt{6}$$

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Sea $t = \sqrt{6} - \sqrt{3} - \sqrt{2} + 1$, se tiene:

(12) $$\pi = 24 \sum_{n=0}^{\infty} (-1)^n \left(\frac{t}{2}\right)^{n+1} \sum_{k=0}^{n} \left(\frac{t}{2}\right)^k \binom{n+k}{n-k} \frac{1}{2k+1}$$

(13) $$\pi = 24 \sum_{n=0}^{\infty} (-1)^n \left(\frac{t}{2}\right)^{n+1} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{n-2k} \frac{1}{2k+1}$$

Sea $w = 3 - \sqrt{2} + \sqrt{3} - \sqrt{6}$, se tiene:

(14) $$\pi = 12 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^{-k}}{n+k+1} \left(1 - w^{n+k+1}\right)$$

(15) $$\pi = 12 \sum_{n=0}^{\infty} \frac{1 - w^{n+1}}{n+1} \sum_{k=0}^{[n/2]} (-2)^{-k} \binom{n-k}{k}$$

Sea $v = \sqrt{3} - \sqrt{2}$, se tiene:

(16) $$\pi = 24(\sqrt{2} - 1) \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{n}{k} 6^{n-k} v^{2n+2k+1} \left(\frac{1}{2n+2k+1} + \frac{v^2 (\sqrt{2}+1)^2}{2n+2k+3} \right)$$

(17) $$\pi = 24(\sqrt{2} - 1) \sum_{n=0}^{\infty} (-1)^n v^{2n+1} \left(\frac{1}{2n+1} + \frac{v^2 (\sqrt{2}+1)^2}{2n+3} \right) \sum_{k=0}^{[n/2]} (-1)^k 6^{n-2k} \binom{n-k}{k}$$

Sea $u = \sqrt{6} + \sqrt{3} + \sqrt{2} + 3$, se tiene:

(18) $$\pi = 24 \sum_{n=0}^{\infty} u^{-n-1} \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k u^{-k}}{2k+1}$$

(19) $$\pi = 24 \sum_{n=0}^{\infty} u^{-n-1} \sum_{k=0}^{n} \binom{n}{n-2k} \frac{(-1)^k}{2k+1}$$

Sea $r = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$, $R = 14 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}$, se tiene:

(20) $$\pi = 24r \left(1 - \sum_{n=0}^{\infty} (-1)^n R^{-n-1} \sum_{k=0}^{n} \binom{n}{n-k} \frac{1}{2k+3} \right)$$

Sea $z = \sqrt{6} + \sqrt{3} + \sqrt{2}$, se tiene:

(21) $$\pi = 24 \sum_{n=0}^{\infty} (-1)^n z^{-n-1} \sum_{k=0}^{[n/2]} \binom{n}{n-2k} \frac{(-1)^k 2^{n-2k}}{2k+1}$$

Sea $x = \sqrt{2} - 1$, $y = \sqrt{3} - 1$, se tiene:

(22) $$\pi = 24 \sum_{n=0}^{\infty} (-1)^n\, y^n \sum_{k=[n/2]}^{\infty} \binom{2k+1}{n} \frac{(-1)^{k+1} x^{4k+2-n}}{2k+1}$$

(23) $$\pi = 24 \sum_{n=1}^{\infty} (-1)^{n-1}\, x^n \sum_{k=0}^{2[(n-1)/2]} \binom{2k+1}{n-2k-1} \frac{(-1)^k y^{4k+2-n}}{2k+1}$$

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(24) $$\pi = 12 \sum_{n=0}^{\infty} (-1)^n \left(\sqrt{3}-\sqrt{2}\right)^{2n+1} \left(1-\left(\sqrt{2}-1\right)^{2n+2}\right)\left(\frac{1}{2n+1}+\frac{5-2\sqrt{6}}{2n+3}\right)$$

(25) $$\pi = 6\sqrt{2} \sum_{n=0}^{\infty} (-1)^n \left(\sqrt{2}-1\right)^{2n+1} \left(1-\left(\sqrt{3}-\sqrt{2}\right)^{2n+2}\right)\left(\frac{1}{2n+1}+\frac{3-2\sqrt{2}}{2n+3}\right)$$

(26) $$\pi = 4\sqrt{3} \sum_{n=0}^{\infty} \left(\sqrt{2}-1\right)^{2n+1} \left(1+(-1)^n\left(\sqrt{3}-\sqrt{2}\right)^{2n+2}\right)\left(\frac{1}{2n+1}-\frac{3-2\sqrt{2}}{2n+3}\right)$$

(27) $$\pi = 6\sqrt{2} \sum_{n=0}^{\infty} \left(\sqrt{3}-\sqrt{2}\right)^{2n+1} \left(1+(-1)^n\left(\sqrt{2}-1\right)^{2n+2}\right)\left(\frac{1}{2n+1}-\frac{5-2\sqrt{6}}{2n+3}\right)$$

(28) $$\pi = 3 \sum_{n=0}^{\infty} 2^{-3n} \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-8)^{-k}\left(6-\sqrt{2}-\sqrt{3}-\sqrt{6}\right)^{n-k}}{2k+1}$$

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Las fórmulas (5) y (6) se pueden escribir como:

(29) $$\pi = 24\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right)\sum_{n=0}^{\infty} c_n \left(\frac{\alpha_n - \beta_n\sqrt{2}}{\gamma_n + \delta_n\sqrt{6}}\right)$$
$$= 24\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}\right)\sum_{n=0}^{\infty} c_n \left(\frac{\gamma_n - \delta_n\sqrt{6}}{\alpha_n + \beta_n\sqrt{2}}\right)$$

donde

$$\alpha_{n+1} = 3\alpha_n + 4\beta_n \ , \beta_{n+1} = 2\alpha_n + 3\beta_n \ , \alpha_0 = 1, \beta_0 = 0$$

$$\gamma_{n+1} = 5\gamma_n + 12\delta_n \ , \delta_{n+1} = 2\gamma_n + 5\delta_n \ , \gamma_0 = 1, \delta_0 = 0$$

La fórmula (4) se puede escribir como:

(30) $$\pi = 24 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\left(\alpha_n + \beta_n\sqrt{2} + \gamma_n\sqrt{3} + \delta_n\sqrt{6}\right)}$$

donde

$$\alpha_{n+1} = 15\alpha_n + 20\beta_n + 24\gamma_n + 36\delta_n$$

$$\beta_{n+1} = 10\alpha_n + 15\beta_n + 18\gamma_n + 24\delta_n$$

$$\gamma_{n+1} = 8\alpha_n + 12\beta_n + 15\gamma_n + 20\delta_n$$

$$\delta_{n+1} = 6\alpha_n + 8\beta_n + 10\gamma_n + 15\delta_n$$

$$\alpha_0 = 2, \beta_0 = 1, \gamma_0 = 1, \delta_0 = 1$$

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Manipulando la integral de (1) , se obtiene la fórmula siguiente:

(31) $$\pi = 24\left(\sqrt{2}-1\right)\left(\sqrt{3}-\sqrt{2}\right)\left(I_1 + I_2\sqrt{2} + I_3\sqrt{3} + I_4\sqrt{6}\right)$$

donde

$$I_1 = \int_0^1 \frac{1 + 45x^2 + 67x^4 + 15x^6}{1 + 60x^2 + 134x^4 + 60x^6 + x^8}\,dx$$

$$I_2 = \int_0^1 \frac{2x^2(5+6x^2+5x^4)}{1+60x^2+134x^4+60x^6+x^8}\,dx$$

$$I_3 = \int_0^1 \frac{8x^2(-1+x^4)}{1+60x^2+134x^4+60x^6+x^8}\,dx$$

$$I_4 = \int_0^1 \frac{6x^2+20x^4+6x^6}{1+60x^2+134x^4+60x^6+x^8}\,dx$$

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