Euler (1781)

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Abstract

In this note we show a collection of identities involving the constant Pi , identities are inspired by a formula of the mathematician L. Euler ,1781.

Resumen

En esta nota mostramos una colección de identidades que involucran a la constante Pi , las identidades están inspiradas en una fórmula del matemático L. Euler, 1781.

Introducción

La constante Pi se define por la serie:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535\ldots$$

Euler (1781) :

$$(1)\qquad \frac{\pi}{4} = \left(\int_0^1 \frac{1}{\sqrt{1-x^4}}dx\right)\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx\right) = \left(\frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}\right)\left(\frac{\Gamma(3/4)^2}{\sqrt{2\pi}}\right)$$

En esta nota mostramos una colección de fórmulas, algunas relacionadas con la identidad (1), otras similares.

Notación:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\ ,\ (a)_n = a(a+1)\ldots(a+n-1)\ ,\ (a)_0 = 1$$

Fórmulas

(2) $$\pi = \frac{1}{4}\left(\int_0^1 \frac{1}{\sqrt[4]{x^3}\,\sqrt{1-x}}\,dx\right)\left(\int_0^1 \frac{1}{\sqrt[4]{x}\,\sqrt{1-x}}\,dx\right)$$

(3) $$\pi = \frac{1}{4}\left(\int_0^1 \frac{1}{\sqrt{x}\,\sqrt[4]{(1-x)^3}}\,dx\right)\left(\int_0^1 \frac{1}{\sqrt{x}\,\sqrt[4]{1-x}}\,dx\right)$$

(4) $$\pi = \left(\int_0^1 \frac{1}{\sqrt{x(1-x^2)}}\,dx\right)\left(\int_0^1 \sqrt{\frac{x}{1-x^2}}\,dx\right)$$

(5) $$\pi = \frac{4}{9}\left(\int_0^1 \frac{1}{\sqrt[3]{x^2}\,\sqrt{1-x\sqrt[3]{x}}}\,dx\right)\left(\int_0^1 \frac{1}{\sqrt{1-x\sqrt[3]{x}}}\,dx\right)$$

(6) $$\pi = 4n^2\left(\int_0^1 \frac{x^{n-1}}{\sqrt{1-x^{4n}}}\,dx\right)\left(\int_0^1 \frac{x^{3n-1}}{\sqrt{1-x^{4n}}}\,dx\right),\, n \in \mathbb{N}$$

(7) $$\pi = 4nm\left(\int_0^1 \frac{x^{n-1}}{\sqrt{1-x^{4n}}}\,dx\right)\left(\int_0^1 \frac{x^{3m-1}}{\sqrt{1-x^{4m}}}\,dx\right),\, n, m \in \mathbb{N}$$

(8) $$\pi = 4\left(\int_0^1 \frac{1}{\sqrt{x(2-x)(2-2x+x^2)}}\,dx\right)\left(\int_0^1 \frac{(1-x)^2}{\sqrt{x(2-x)(2-2x+x^2)}}\,dx\right)$$

(9) $$\pi = \frac{1}{4}\left(\int_1^\infty \frac{1}{\sqrt[4]{x^7(x-1)^3}}\,dx\right)\left(\int_1^\infty \frac{1}{\sqrt[4]{x^9(x-1)}}\,dx\right)$$

(10) $$\pi = 4\left(\int_1^\infty \frac{1}{\sqrt{x^4-1}}\,dx\right)\left(\int_1^\infty \frac{1}{x^2\,\sqrt{x^4-1}}\,dx\right)$$

(11) $$\pi = \left(\int_1^\infty \frac{1}{\sqrt{x(x^2-1)}}\,dx\right)\left(\int_1^\infty \frac{1}{\sqrt{x^3(x^2-1)}}\,dx\right)$$

(12) $$\pi = \left(\int_0^1 \frac{1}{\sqrt[4]{(1-x^2)^3}}\,dx\right)\left(\int_0^1 \frac{1}{\sqrt[4]{1-x^2}}\,dx\right)$$

(13) $$\pi = \left(\int_1^2 \frac{1}{\sqrt{x(2-x)(x-1)}}\,dx\right)\left(\int_1^2 \sqrt{\frac{x-1}{x(2-x)}}\,dx\right)$$

(14) π

$$= 4\left(\int_{-1}^{1} \frac{1}{\sqrt{(1-x)(3+x)(5+2x+x^2)}}\,dx\right)\left(\int_{-1}^{1} \frac{(1+x)^2}{\sqrt{(1-x)(3+x)(5+2x+x^2)}}\,dx\right)$$

(15) $$\pi = 4\left(\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx\right)\left(\int_0^1 \sqrt{\frac{1+x^2}{1-x^2}}\,dx\right) - 4\left(\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx\right)^2$$

(16) $$\pi = 4\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right)\left(\int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\,dx\right) + 4\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right)^2$$

(17) $$\pi = 2\left(\int_0^1 \sqrt{\frac{1+x^2}{1-x^2}}\,dx\right)\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right) + 2\left(\int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\,dx\right)\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right)$$

(18) $$\pi = 2\left(\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx\right)\left(\int_0^1 x^2\sqrt{\frac{1+x^2}{1-x^2}}\,dx\right) + 2\left(\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx\right)\left(\int_0^1 x^2\sqrt{\frac{1-x^2}{1+x^2}}\,dx\right)$$

(19) $$\pi = 2\left(\int_0^1 \frac{1}{\sqrt{x(1+x^2)}}\,dx\right)\left(\int_0^1 \frac{(1-x)^2}{(1+x)^2\sqrt{x(1+x^2)}}\,dx\right)$$

(20) $$\pi = 16\left(\int_0^1 \frac{1}{\sqrt{(2-x^2)(2-2x^2+x^4)}}\,dx\right)\left(\int_0^1 \frac{(1-x^2)^2}{\sqrt{(2-x^2)(2-2x^2+x^4)}}\,dx\right)$$

(21) $$\pi = \frac{1}{4}\left(\int_1^\infty \frac{1}{\sqrt[4]{x}\,\sqrt{x-1}}\,dx\right)\left(\int_1^\infty \frac{1}{\sqrt[4]{x^5}\,\sqrt{x-1}}\,dx\right)$$

(22) $$\pi = \frac{1}{4}\left(\int_0^\infty \frac{1}{\sqrt{x}\,\sqrt[4]{1+x}}\,dx\right)\left(\int_0^\infty \frac{1}{\sqrt{x}\,\sqrt[4]{(1+x)^5}}\,dx\right)$$

(23) $$\pi = \frac{1}{4}\left(\int_0^a \frac{1}{\sqrt[4]{x^3}\,\sqrt{1-x}}\,dx\right)\left(\int_0^a \frac{1}{\sqrt[4]{x}\,\sqrt{1-x}}\,dx\right) + \frac{1}{4}\left(\int_0^a \frac{1}{\sqrt[4]{x^3}\,\sqrt{1-x}}\,dx\right)\left(\int_0^{1-a} \frac{1}{\sqrt[4]{1-x}\,\sqrt{x}}\,dx\right) + \frac{1}{4}\left(\int_0^{1-a} \frac{1}{\sqrt[4]{(1-x)^3}\,\sqrt{x}}\,dx\right)\left(\int_0^a \frac{1}{\sqrt[4]{x}\,\sqrt{1-x}}\,dx\right) + \frac{1}{4}\left(\int_0^{1-a} \frac{1}{\sqrt[4]{(1-x)^3}\,\sqrt{x}}\,dx\right)\left(\int_0^{1-a} \frac{1}{\sqrt[4]{1-x}\,\sqrt{x}}\,dx\right),\ 0 \le a \le 1$$

(24) $$\pi = \frac{1}{4}\left(\int_0^{1/2}\left(\frac{1}{\sqrt[4]{x^3}\,\sqrt{1-x}} + \frac{1}{\sqrt{x}\,\sqrt[4]{(1-x)^3}}\right)dx\right)\left(\int_0^{1/2}\left(\frac{1}{\sqrt[4]{x}\,\sqrt{1-x}} + \frac{1}{\sqrt{x}\,\sqrt[4]{1-x}}\right)dx\right)$$

(25) $$\pi = \left(\int_0^{1/\sqrt[4]{2}}\left(\frac{1}{\sqrt{1-x^4}} + \frac{x}{\sqrt[4]{(1-x)^3}}\right)dx\right)\left(\int_0^{1/\sqrt[4]{2}}\left(\frac{x^2}{\sqrt{1-x^4}} + \frac{x}{\sqrt[4]{1-x^4}}\right)dx\right)$$

(26) $$\pi = \frac{1}{4}\left(2\sqrt{2}\,\sqrt[4]{2}\,\sum_{n=0}^{\infty}\frac{(1/2)_n\,2^{-n}}{n!\,(4n+1)} + \sqrt{2}\,\sum_{n=0}^{\infty}\frac{(3/4)_n\,2^{-n}}{n!\,(2n+1)}\right)\times\left(2\,\sqrt[4]{2}\,\sum_{n=0}^{\infty}\frac{(1/2)_n\,2^{-n}}{n!\,(4n+3)} + \sqrt{2}\,\sum_{n=0}^{\infty}\frac{(1/4)_n\,2^{-n}}{n!\,(2n+1)}\right)$$

(27) $$\pi = 4\left(\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{4n+1}\right)\left(\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{4n+3}\right)$$

(28) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{2^{-2n}}{(4n-4k+1)(4k+3)}$$

(29) $$\pi = 4\left(1 + \int_1^{\infty}\left(1 - \sqrt[4]{1-x^{-2}}\right)dx\right)\left(\int_0^{\infty}\left(1 - \sqrt[4]{\frac{x^2}{1+x^2}}\right)dx\right)$$

(30) $$\pi = 4\left(\int_1^{\infty}\left(1 + x^{-2} - \sqrt[4]{1-x^{-2}}\right)dx\right)\left(\int_0^{\infty}\left(1 - \sqrt[4]{\frac{x^2}{1+x^2}}\right)dx\right)$$

(31) $$\pi = 6\left(\int_0^1 \sqrt{1-x^4}\,dx\right)\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right)$$

(32) $$\pi = 6\left(\int_0^1 \sqrt[4]{1-x^2}\,dx\right)\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx\right)$$

(33) $$\pi = 4\left(ab - \int_1^b \sqrt[4]{1-x^{-2}}\,dx + \int_0^{1-a} \frac{1}{\sqrt{x(2-x)(2-2x+x^2)}}\,dx\right)\left(cd - \int_0^d \sqrt[4]{\frac{x^2}{1+x^2}}\,dx + \int_0^{1-c} \frac{(1-x)^2}{\sqrt{x(2-x)(2-2x+x^2)}}\,dx\right)$$

$$0 < a < 1, b\sqrt{1-a^4} = 1, 0 < c < 1, d\sqrt{1-c^4} = c^2$$

(34) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n+2}{n+1}\binom{2n-2k}{n-k}2^{-4n+2k-2}\left(\frac{1}{(4n-4k+1)(4n+7)} + \frac{1}{(4n-4k+3)(4n+5)}\right) + 4\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{2^{-4n}}{(4n+1)(4n+3)}$$

(35) $$\pi = \frac{32}{5}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{2n}{n}\binom{n}{k}\binom{k}{m}\binom{m}{s}\left(-\frac{3}{20}\right)^{n-k}\left(\frac{3}{5}\right)^{k-m}\left(-\frac{2}{5}\right)^{m-s}\left(\frac{1}{10}\right)^{s} f(k,m,s)\right)$$
$$\times \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{2n}{n}\binom{n}{k}\binom{k}{m}\binom{m}{s}\left(-\frac{3}{20}\right)^{n-k}\left(\frac{3}{5}\right)^{k-m}\left(-\frac{2}{5}\right)^{m-s}\left(\frac{1}{10}\right)^{s} g(k,m,s)\right)$$

donde

$$f(k,m,s) = \frac{1}{2k+2m+2s+1}$$

$$g(k,m,s) = \frac{1}{2k+2m+2s+1} - \frac{2}{2k+2m+2s+3} + \frac{1}{2k+2m+2s+5}$$

(36) $$\pi = 4\left(\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}c_n\right)\left(\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}c_{n+2}\right)$$

donde

$$c_n = \frac{\sqrt{2}}{2n} - \frac{2n-1}{2n} c_{n-1}\,, c_0 = \ln\left(1+\sqrt{2}\right), n \in \mathbb{N}$$

(37) $$\pi = 4\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{k}{m}\frac{(-1)^m 2^{-3n+k-m}}{2n+2m+1}\right)\times$$
$$\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{k}{m}(-1)^m 2^{-3n+k-m}\left(\frac{1}{2n+2m+1}-\frac{2}{2n+2m+3}\right.\right.$$
$$\left.\left.+\frac{1}{2n+2m+5}\right)\right)$$

(38) $$\pi = \left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx\right)\left(\int_0^1 \frac{x^4}{(1+x^4)\sqrt{1-x^4}}dx\right)\left(\frac{1}{4} - a\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\right)^{-1}$$

donde

$$a = \frac{1}{8} + \frac{\sqrt{2}}{8}\sum_{n=0}^{\infty}\binom{2n}{n}^2 2^{-5n}$$

(39) $$\pi = 2^{4a-1}ab^2\left(\int_0^1 (1-x^b)^{a-1}x^{ab-1}\,dx\right)\left(\int_0^1 (1-x^b)^{a-\frac{1}{2}}x^{ab+\frac{a}{2}-1}\,dx\right)$$

$$a > 0, b > 0$$

(40) $$\pi = 4\left(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx\right)\left(\int_0^1 \sqrt{\frac{1-x}{1+x+x^2+x^3}}\,dx\right)\left(1-\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx\right)^{-1}$$

(41) $$\pi = 4\left(\int_0^1 \frac{1}{\sqrt{1-x^4}}dx\right)\left(\int_0^1 \sqrt{\frac{1-x}{1+x+x^2+x^3}}\;x\,dx\right)\left(-1+\int_0^1 \frac{1}{\sqrt{1-x^4}}dx\right)^{-1}$$

(42) $$\pi = 4\left(\frac{\pi}{4} + \int_0^1 \sqrt{\frac{1-x}{1+x+x^2+x^3}}\,dx\right)\left(\frac{\pi}{4} - \int_0^1 \sqrt{\frac{1-x}{1+x+x^2+x^3}}\;x\,dx\right)$$

(43) $$\pi = 4\left(\frac{1}{\sqrt{2}}\int_0^b \frac{1}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}dx + \sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}a^{4n+1}}{4n+1}\right)\times$$

$$\left(\sqrt{2}\int_0^b \frac{\sqrt{1-\frac{1}{2}x^2}}{\sqrt{1-x^2}}dx-\frac{1}{\sqrt{2}}\int_0^b \frac{1}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}dx+\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}a^{4n+3}}{4n+3}\right)$$

$$b=\sqrt{1-a^2}$$

Referencias

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