Hausdorff Dimension of the Sierpinski Triangle

Edgar Valdebenito

08-01-2016 15:32:10

Abstract

A collection of formulas involving constant $s = \log_2 3 = \frac{\ln 3}{\ln 2} = 1.5849625 \ldots$, is shown.

Introduction

In theory of fractals , the number $s = \frac{\ln 3}{\ln 2}$, is known as the hausdorff dimension of de sierpinski triangle.for details see references (2),(3),(5). In this paper is a collection of representations for constant s shows.

Formulas

Recurrences for s :

(1) $s_{n+1} = 3 s_n 2^{-s_n}$, $s_1 = 1$, $\lim_{n\to\infty} s_n = s$

(2) $s_{n+1} = s_n - \frac{1}{\ln 2} + \frac{3}{\ln 2} 2^{-s_n}$, $s_1 = 1$, $\lim_{n\to\infty} s_n = s$

(3) $s_{n+1} = (2 s_n + 3 - 2^{-s_n})/2$, $s_1 = 1$, $\lim_{n\to\infty} s_n = s$

(4) $t_{n+1} = \frac{3}{2} + t_n - 2^{t_n}$, $t_1 = 0$, $\lim_{n\to\infty} t_n = s - 1$

(5) $s_{n+1} = \left(1 - 3^{-2n-1}(2n+1)^{-1}\left(\sum_{k=0}^{n+1} 3^{-2k-1}(2k+1)^{-1}\right)^{-1}\right) s_n + 2^{-2n-1}(2n+1)^{-1}$, $s_0 = 3/2$, $\lim_{n\to\infty} s_n = s$

Sequences for s :

(6) $s_n = \frac{\sum_{k=0}^{n} 2^{-2k-1}(2k+1)^{-1}}{\sum_{k=0}^{n} 3^{-2k-1}(2k+1)^{-1}}$, $\lim_{n\to\infty} s_n = s$

(7) $$s_n = 2\frac{\sum_{k=1}^{n}(n+2k)^{-1}}{\sum_{k=1}^{n}(n+k)^{-1}} \quad , \lim_{n\to\infty} s_n = s$$

(8) $$s_n = \frac{3^{2^{-n}}-1}{2^{2^{-n}}-1} \quad , \lim_{n\to\infty} s_n = s$$

(9) $$s_n = \left(3^{2^{-n}}-1\right)\prod_{k=1}^{n}\left(1+2^{2^{-k}}\right) \quad , \lim_{n\to\infty} s_n = s$$

A limit :

(10) $$s = \frac{\ln 3}{\ln 2} = \lim_{x\to 0}\frac{3^x-1}{2^x-1}$$

On infinite product for s :

(11) $$s = 2\prod_{n=1}^{\infty}\frac{1+2^{2^{-n}}}{1+3^{2^{-n}}}$$

Representation of s as division series :

(12) $$s = \frac{\sum_{n=0}^{\infty} 2^{-2n-1}(2n+1)^{-1}}{\sum_{n=0}^{\infty} 3^{-2n-1}(2n+1)^{-1}} = \frac{2^{-1}+24^{-1}+160^{-1}+\cdots}{3^{-1}+81^{-1}+1215^{-1}+\cdots}$$

(13) $$s = 1+\frac{\sum_{n=0}^{\infty} 5^{-2n-1}(2n+1)^{-1}}{\sum_{n=0}^{\infty} 3^{-2n-1}(2n+1)^{-1}} = 1+\frac{5^{-1}+375^{-1}+15625^{-1}+\cdots}{3^{-1}+81^{-1}+1215^{-1}+\cdots}$$

(14) $$s = 2-\frac{\sum_{n=0}^{\infty} 7^{-2n-1}(2n+1)^{-1}}{\sum_{n=0}^{\infty} 3^{-2n-1}(2n+1)^{-1}} = 2-\frac{7^{-1}+1029^{-1}+84035^{-1}+\cdots}{3^{-1}+81^{-1}+1215^{-1}+\cdots}$$

(15) $$s = \frac{3}{2}+\frac{\sum_{n=0}^{\infty} 17^{-2n-1}(2n+1)^{-1}}{2\sum_{n=0}^{\infty} 3^{-2n-1}(2n+1)^{-1}} = \frac{3}{2}+\frac{17^{-1}+17^{-3}3^{-1}+\cdots}{2(3^{-1}+81^{-1}+\cdots)}$$

(16) $$s = 1+\frac{\sum_{n=1}^{\infty} 3^{-n}n^{-1}}{\sum_{n=1}^{\infty} 2^{-n}n^{-1}} = 1+\frac{3^{-1}+2^{-1}3^{-2}+3^{-1}3^{-3}+\cdots}{2^{-1}+2^{-1}2^{-2}+3^{-1}2^{-3}+\cdots}$$

Involving serial number $e = 2.718181$ ...

(17) $$s = \frac{1+\left(\frac{3-e}{e}\right)^1-\frac{1}{2}\left(\frac{3-e}{e}\right)^2+\frac{1}{3}\left(\frac{3-e}{e}\right)^3-\cdots}{1-\left(\frac{e-2}{e}\right)^1-\frac{1}{2}\left(\frac{e-2}{e}\right)^2-\frac{1}{3}\left(\frac{e-2}{e}\right)^3-\cdots}$$

Serie for s :

(18) $$s = 4\sum_{n=1}^{\infty} c_n\, 2^{-n} \quad , c_n = \left\{1,-\frac{1}{2},\frac{1}{4},-\frac{5}{24},\frac{7}{48},-\frac{191}{1440},\frac{33}{320},\ldots\right\}$$

$$c_n = \frac{1-(-1)^n}{2n} - \sum_{k=1}^{n}\frac{c_{n-k}}{k+1} \quad , \qquad c_0 = 0$$

Integrals for s :

(19) $s = 1 + \int_1^{\infty} \frac{1}{1+2^x} dx$

(20) $s = \int_{-1}^{\infty} \frac{1}{1+2^x} dx$

(21) $s = 2 - \frac{1}{2} \int_0^3 \frac{1}{1+2^x} dx$

(22) $\frac{1}{s} = m \int_0^{\infty} \frac{1}{1+3^{m\,x}} dx \ \ , m > 0$

(23) $\frac{1}{1+s} = \int_0^{\infty} \frac{1}{1+6^x} dx$

(24) $s = 2 - \int_0^1 \frac{1}{1+2^x} dx$

(25) $\frac{1}{s} = \int_0^{\infty} \frac{1}{1+3^x} dx$

(26) $s = \int_0^1 \int_0^1 \frac{2^{x+1}}{1+2y} dx\, dy$

(27) $\frac{1}{s} = \frac{1}{2} \int_0^1 \int_0^1 \frac{3^x}{1+y} dx\, dy$

(28) $s = \ln 3 + \int_2^e \int_2^e \frac{2}{x(\ln x)^2(2y+e-6)} dx\, dy$

(29) $\frac{1}{s} = \ln 2 - \int_e^3 \int_e^3 \frac{1}{x(\ln x)^2(y-2e+3)} dx\, dy$

(30) $s = 1 - \int_0^{1/2} \left(\frac{1}{1+x} - \frac{\ln(1+x)}{x \ln x}\right) \frac{dx}{\ln x}$

(31) $s = 1 - \int_2^{\infty} \left(\frac{1}{1+x} - \frac{\ln(1+x)}{x \ln x}\right) \frac{dx}{\ln x}$

(32) $s = \ln(1+e) + \int_2^e \left(\frac{\ln(1+e)}{x \ln x} - \frac{1}{1+x}\right) \frac{dx}{\ln x}$

(33) $s = \frac{1}{\ln(e-1)} - \int_e^3 \left(\frac{\ln x}{(x-1)\ln(x-1)} - \frac{1}{x}\right) \frac{dx}{\ln(x-1)}$

Other formulas

(34) $s_n = \left(\frac{2}{3}\right)^{n(n+1)/2} \frac{\sum_{m=1}^{n} (m-1)!(m+1)_{n-m}\, 3^{\frac{n(n+1)}{2}-m}}{\sum_{m=1}^{n} (m-1)!(m+1)_{n-m}\, 2^{\frac{n(n+1)}{2}-m}} \quad , s = 1 + \lim_{n\to\infty} s_n$

(35) $s_n = \left(\frac{3}{2}\right)^{2n+1} \frac{\sum_{m=0}^{n} 2^{2n-2m}\,(2m)!(2m+2)_{2n-2m}}{\sum_{m=0}^{n} 3^{2n-2m}\,(2m)!(2m+2)_{2n-2m}} \quad , s = \lim_{n\to\infty} s_n$

In formulas (34) and (35) shows the symbol Pochhammer:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1)\,, (a)_0 = 1$$

(36) $$s_{n+1} = s_n + 1 - \frac{1}{3} 2^{s_n} \quad , s_1 = 0 \, , \lim_{n\to\infty} s_n = s$$

(37) $$s_{n+1} = s_n - 1 + 3 \cdot 2^{-s_n} \quad , s_1 = 0 \, , \lim_{n\to\infty} s_n = s$$

(38) $$s_{n+1} = s_n - \left(trunc\left(\frac{2}{2^{s_n}-3}\right)\right)^{-1} \quad , s_1 = 1 \, , \lim_{n\to\infty} s_n = s$$

The function $trunc(x)$ gives the integer part of x , ignoring the fractional part . Formula (38) produces the series :

$$s = 1 + \frac{1}{2} + \frac{1}{11} - \frac{1}{161} + \frac{1}{3635} - \frac{1}{91484} + \cdots$$

(39) $$s_{n+1} = \frac{6+s_n-2^{1+s_n}}{4-2^{s_n}} \quad , s_1 = 1, \lim_{n\to\infty} s_n = s$$

(40) $$s_n = \frac{trunc\ (10^n \ln 3)}{trunc\ (10^n \ln 2)} \quad , n \in \mathbb{N}, s_n = \left\{\frac{5}{3}, \frac{109}{69}, \frac{122}{77}, \frac{10986}{6931}, \frac{109861}{69314}, \dots\right\}, \lim_{n\to\infty} s_n = s$$

(41) $$s = 1 + \left(1 + \left(1 + \frac{\ln(1+8^{-1})}{\ln(1+3^{-1})}\right)^{-1}\right)^{-1}$$

(42) $$s = (3 \cdot 2^{-1})\left(3 \cdot 2^{-3\cdot 2^{-1}}\right)\left(3 \cdot 2^{-9\cdot 2^{-1} - 2^{-3\cdot 2^{-1}}}\right)\left(3 \cdot 2^{-27\cdot 2^{-1} - 3\cdot 2^{-1} - 9\cdot 2^{-1} - 3\cdot 2^{-1}}\right)\cdots$$

(43) $$s = 2 \lim_{n\to\infty} \frac{\sum_{k=1}^{n} 2^{k/n}}{\sum_{k=1}^{n} 3^{k/n}}$$

(44) $$e^s = \lim_{n\to\infty} \prod_{k=1}^{n} \sqrt[n]{3^{\sqrt[n]{2^k}}}$$

(45) $$s_n = \frac{8\left(2+3\sum_{k=1}^{n-1}\frac{n}{n+2k}+6\sum_{k=1}^{n}\frac{n}{n+2k-1}\right)}{3\left(3+4\sum_{k=1}^{n-1}\frac{n}{n+k}+16\sum_{k=1}^{n}\frac{n}{2n+2k-1}\right)} \quad , n \in \mathbb{N}, \lim_{n\to\infty} s_n = s$$

(46) $$s = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{\ln(1+7153\cdot 2^{-19})}{\ln(1+13\cdot 3^{-5})}}}}}$$

(47) $$s = 1 + \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n(n+1)/2} \frac{b_n}{a_n}$$

$$a_{n+1} = 2^{n+1}(n+1)a_n + 2^{n(n+1)/2} n! \ , a_1 = 1$$

$$b_{n+1} = 3^{n+1}(n+1)b_n + 3^{n(n+1)/2} n! \ , b_1 = 1$$

(48) $$s\left(7 - 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1-2^{10}e^{-7}}{1+2^{10}e^{-7}}\right)^{2n+1}\right) = 11 - 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1-3^{10}e^{-11}}{1+3^{10}e^{-11}}\right)^{2n+1}$$

(49) $$s = \frac{3}{2}\left(1 - \cfrac{3^{-2}}{3-\cfrac{4\cdot 3^{-2}}{5-\cfrac{9\cdot 3^{-2}}{7-\cdots}}}\right)\left(\cfrac{1}{1-\cfrac{2^{-2}}{3-\cfrac{4\cdot 2^{-2}}{5-\cfrac{9\cdot 2^{-2}}{7-\cdots}}}}\right)$$

(50) $$s_{n+1} = s_n\left(1-\left(trunc\left(\frac{2s_n}{2^{s_n}-3}\right)\right)^{-1}\right)\ ,s_1 = 1\ ,\lim_{n\to\infty} s_n = s$$

Formula (50) produces the product :

$$s = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{17}\right)\left(1-\frac{1}{466}\right)\left(1+\frac{1}{11255}\right)\dots$$

(51) let $n \in \mathbb{N}$, $p_n \in \mathbb{N}$ such that $2^{p_n} < 3^n < 2^{1+p_n}$, then they have :

$$p_n = \{1,3,4,6,7,9,11,12,14,15,\dots\}$$

$$\frac{p_n}{n} < s < \frac{1+p_n}{n}$$

$$s \approx \frac{1}{n}\left(-1+p_n+\frac{3^n}{2^{p_n}}\right)$$

(52) let $n \in \mathbb{N}$, $q_n \in \mathbb{N}\cup\{0\}$ such that $3^{q_n} < 2^n < 3^{q_n}$, then they have :

$$q_n = \{0,1,1,2,3,3,4,5,5,6,\dots\}$$

$$\frac{q_n}{n} < \frac{1}{s} < \frac{1+q_n}{n}$$

$$\frac{1}{s} \approx \frac{1}{n}\left(-\frac{1}{2}+q_n+\frac{2^{n-1}}{3^{q_n}}\right)$$

(53) $$s = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cdots}}}}}}$$

References

1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York:Dover , 1965.
2. Falconer K. , The Geometry of Fractals Sets, Cambridge University Press 1985.

3. Falconer K. , Fractal Geometry: Mathematical Foundations and applications John Wiley and Sons, Ltd. 2003.
4. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals,Series, and Products (A.Jeffrey), Academic Press,New York, London, and Toronto, 1980.
5. Hutchinson, J.E. , Fractals and self-similarity,Indiana Univ.Math.J., 30 (1981),713-747.
6. M.R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York,1968.
7. E. Valdebenito, Pi Handbook, manuscript,unpublished,1989, (20000 formulas).