Sedeonic Equations of Ideal Fluid

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In the present paper we propose the generalized equations for ideal fluid based on space-time algebra of sixteen-component sedeons. It is shown that the dynamics of isentropic fluid can be described by sedeonic first-order wave equation for fluid potentials. The key features of the proposed formalism are illustrated on the problem of the sound waves propagation. We consider the plane wave solution of linearized sedeonic wave equation and derive the second-order relations for the sound potentials analogues to the Pointing theorem in electrodynamics. The generalization of proposed sedeonic equations for the description of viscous fluid is also discussed.

1. Introduction

The analogy between equations of hydrodynamics and electrodynamics is widely discussed for a long time. In particular, there are several attempts to describe the fluid dynamics by the vector fields (analogous to electric and magnetic fields) satisfying the Maxwell-like equations and corresponding second-order wave equation [1-8]. This approach enables the application of useful electrodynamics' relations to the description of fluid behavior. On the other side, the analogy between hydrodynamics and electrodynamics has inspired the reformulation of equations for ideal fluid on the basis of noncommutative hypercomplex algebras [9-11].

The application of different hypercomplex numbers for the reformulation of electromagnetic field equations has a long history. Originally, the equations for electromagnetic field were formulated by J.C. Maxwell in terms of quaternions [12] and subsequently this approach was generalized in the works [13-15]. The algebraic structure of quaternions with four components (scalar and vector) corresponds to the relativistic four-vector concept that allows formulating all equations of electrodynamics in terms of quaternionic algebra. However, the essential disadvantage of quaternions is that they do not include pseudoscalar and pseudovector components. The full account of the space-inversion symmetry leads to the eight-component structures (enclosing scalar, pseudoscalar, vector and pseudovector) such as biquaternions (complex quaternions) octonions and octons [16-24]. However, natural generalization of this approach is the incorporation of the time-inversion symmetry in algebraic structure that requires the consideration of the extended sixteen-component algebras. One of them is the algebra of sedenions obtaining from octonions by Cayley-Dickson extension procedure [25-28] but the essential disadvantage of sedenions is their non-associativity.

Recently we proposed the associative algebra of sixteen-component sedeons, which takes into account the properties of physical values with respect to the space-time inversion and realizes the scalar-vector representation of Poincare group [29,30]. This formalism was successfully applied for the description of classical and quantum fields [31-33]. In particular, we have demonstrated the possibility to formulate Maxwell-like equations for the fields with nonzero mass of quantum [31,32] and the unification of equations for electromagnetic field and weak gravity in the frames of gravitoelectromagnetism theory [33]. In present paper we develop the description of ideal fluid using the sedeonic first-order wave equation for the fluid potentials.

2. Preliminaries: symmetric equations of isentropic fluid

As is well known [34], in terms of traditional vector algebra the dynamics of ideal fluid is described by a system of equations, which includes the Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla p = 0 , \tag{2.1}$$

continuity equation

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{v}) = 0 , \tag{2.2}$$

and adiabatic equation

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \nabla)s = 0 . \tag{2.3}$$

Here $\boldsymbol{v}$ is a local fluid velocity, ρ is a density, p is a pressure, s is specific entropy per unit mass, ∇ is the Hamilton nabla-operator. In the case of isentropic fluid the equations (2.1)-(2.3) can be reduced to a system of two symmetric equations [5]. Let us use the well-known thermodynamic relation:

$$dw = Tds + Vdp , \tag{2.4}$$

where w is a thermal function (enthalpy) of fluid unit mass, T is a temperature, V is the volume of fluid unit mass ($V = 1/\rho$). For the isentropic case $s = const$ we obtain the following relation for differentials

$$dw = \frac{1}{\rho} dp = \frac{c^2}{\rho} d\rho , \tag{2.5}$$

where c is the speed of sound ($c^2 = (\partial p/\partial \rho)_s$). It follows that

$$\frac{1}{\rho}\nabla p = \nabla w , \tag{2.6}$$

$$\frac{\partial \rho}{\partial t} = \frac{\rho}{c^2}\frac{\partial w}{\partial t}, \tag{2.7}$$

$$\nabla \rho = \frac{\rho}{c^2}\nabla w . \tag{2.8}$$

Then the system of equations (2.1) - (2.3) is rewritten in the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla w = 0 , \tag{2.9}$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v} \cdot \nabla)w + c^2(\nabla \cdot \boldsymbol{v}) = 0 . \tag{2.10}$$

These symmetric equations are suitable for the application in hydrodynamics, aerodynamics and plasma physics [5,8]. In contrast to the equations of electromagnetism, the equations (2.9)-(2.10) contain nonlinear terms. However despite the nonlinearity these symmetric equations can be easy incorporated in the frames of sedeonic algebra.

3. Space-time sedeons

Let us briefly recall the basic properties of sedeons [30]. The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{3.1}$$

Let us introduce a scalar-vector basis $\mathbf{a}_0$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a}_0$ is an absolute scalar unit

($\mathbf{a_0} \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. We also introduce the four space-time units $\mathbf{e_0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e_0}$ is an absolute scalar unit ($\mathbf{e_0} \equiv 1$); $\mathbf{e_1}$ is a time scalar unit ($\mathbf{e_1} \equiv \mathbf{e_t}$); $\mathbf{e_2}$ is a space scalar unit ($\mathbf{e_2} \equiv \mathbf{e_r}$); $\mathbf{e_3}$ is a space-time scalar unit ($\mathbf{e_3} \equiv \mathbf{e_{tr}}$). Using space-time basis $\mathbf{e}_{\boldsymbol{\alpha}}$ and scalar-vector basis $\mathbf{a}_{\boldsymbol{\beta}}$ (Greek indexes $\boldsymbol{\alpha}, \boldsymbol{\beta} = 0, 1, 2, 3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$
\begin{aligned}
V &= \mathbf{e_0} V_{00} \mathbf{a_0}, \\
\vec{V} &= \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right), \\
V_{\mathbf{t}} &= \mathbf{e_1} V_{10} \mathbf{a_0}, \\
\vec{V}_{\mathbf{t}} &= \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right), \\
V_{\mathbf{r}} &= \mathbf{e_2} V_{20} \mathbf{a_0}, \\
\vec{V}_{\mathbf{r}} &= \mathbf{e_2} \left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right), \\
V_{\mathbf{tr}} &= \mathbf{e_3} V_{30} \mathbf{a_0}, \\
\vec{V}_{\mathbf{tr}} &= \mathbf{e_3} \left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{3.2}
$$

Then sedeon (3.1) can be written in the following expanded form:

$$
\begin{aligned}
\tilde{\mathbf{V}} = & \mathbf{e_0} \left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) \\
& + \mathbf{e_1} \left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) \\
& + \mathbf{e_2} \left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) \\
& + \mathbf{e_3} \left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{3.3}
$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will omit units $\mathbf{a_0}$ and $\mathbf{e_0}$ for the simplicity. The important property of sedeons is that the equality of two sedeons means the equality of all sixteen components $V_{\alpha\beta}$.

Let us consider the multiplication rules for the basis elements $\mathbf{a_n}$ and $\mathbf{e_k}$ (Latin indexes **n**, **k** = 1, 2, 3). The unit vectors $\mathbf{a_n}$ have the following multiplication and commutation rules:

$$\mathbf{a_n a_n} = \mathbf{a_n}^2 = 1, \tag{3.4}$$

$$\mathbf{a_n a_k} = -\,\mathbf{a_k a_n} \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}, \tag{3.5}$$

$$\mathbf{a_1 a_2} = i\mathbf{a_3}, \quad \mathbf{a_2 a_3} = i\mathbf{a_1}, \quad \mathbf{a_3 a_1} = i\mathbf{a_2}, \tag{3.6}$$

while the space-time units $\mathbf{e_k}$ satisfy the following rules:

$$\mathbf{e_k e_k} = \mathbf{e_k}^2 = 1, \tag{3.7}$$

$$\mathbf{e_n e_k} = -\,\mathbf{e_k e_n} \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}, \tag{3.8}$$

$$\mathbf{e_1 e_2} = i\mathbf{e_3}, \quad \mathbf{e_2 e_3} = i\mathbf{e_1}, \quad \mathbf{e_3 e_1} = i\mathbf{e_2}. \tag{3.9}$$

Here and further the value i is imaginary unit ($i^2 = -1$). The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_n}$ and space-time units $\mathbf{e_k}$ can be presented for obviousness as the tables 1 and 2. Note that units $\mathbf{e_k}$ commute with vectors $\mathbf{a_n}$:

$$\mathbf{a_n e_k} = \mathbf{e_k a_n} \tag{3.10}$$

for any $\mathbf{n}$ and $\mathbf{k}$.

In sedeonic algebra we assume the Clifford multiplication of vectors. The sedeonic product of two vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B} \right) + \left[\vec{A} \times \vec{B} \right]. \tag{3.11}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol " $\cdot$ " and round brackets

$$\left(\vec{A}\cdot\vec{B}\right)=A_1B_1+A_2B_2+A_3B_3\,, \tag{3.12}$$

and sedeonic vector multiplication (external product) by symbol “$\times$” and square brackets

$$\left[\vec{A}\times\vec{B}\right]=i\mathbf{a_1}\left(A_2B_3-A_3B_2\right)+i\mathbf{a_2}\left(A_3B_1-A_1B_3\right)+i\mathbf{a_3}\left(A_1B_2-A_2B_1\right). \tag{3.13}$$

Table 1. Multiplication rules for absolute unit vectors $\mathbf{a_n}$.

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2. Multiplication rules for space-time units $\mathbf{e_k}$.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

Note that in sedeonic algebra the expression for the vector product differs from analogous expression in Gibbs-Heaviside vector algebra. For the transition to the common used vector algebra the change $i\left[\vec{\nabla}\times\vec{A}\right]\Rightarrow-[\boldsymbol{\nabla}\times\boldsymbol{A}]$ should be made in all vector expressions.

4. Generalized sedeonic equation of ideal liquid

Let us assume that speed of sound is independent of the coordinates and time ($c = const$). Then, if we introduce the vector potential $\vec{A}$

$$\vec{A}=c\vec{v}\,, \tag{4.1}$$

the equations (2.9) and (2.10) can be represented in the following symmetric form:

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+\vec{\nabla}w=0\,, \tag{4.2}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)w+(\vec{\nabla}\cdot\vec{A})=0\,. \tag{4.3}$$

In sedeon algebra, the system of equations (4.2) - (4.3) can be written as a single first-order wave equation:

$$\left\{i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)+\mathbf{e_r}\vec{\nabla}\right\}\tilde{\mathbf{W}}=0\,, \tag{4.4}$$

where the sedeonic wave function $\tilde{\mathbf{W}}$ is

$$\tilde{\mathbf{W}}=w+\mathbf{e_{tr}}\vec{A}\,, \tag{4.5}$$

so, in expanded form the equation (4.4) can be written as

$$\left\{i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)+\mathbf{e_r}\vec{\nabla}\right\}\left(w+\mathbf{e_{tr}}\vec{A}\right)=0\,. \tag{4.6}$$

Indeed, after the act of operator on the wave function in (4.6) we get

$$\mathbf{e}_{\mathrm{r}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+i\mathbf{e}_{\mathrm{t}}\left(\vec{\nabla}\cdot\vec{A}\right)+i\mathbf{e}_{\mathrm{t}}\left[\vec{\nabla}\times\vec{A}\right]$$
$$+i\mathbf{e}_{\mathrm{t}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)w+\mathbf{e}_{\mathrm{r}}\vec{\nabla}w=0. \tag{4.7}$$

Separating the values with different space-time properties we obtain the following system of equations:

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)w+\left(\vec{\nabla}\cdot\vec{A}\right)=0, \tag{4.8}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+\vec{\nabla}w=0, \tag{4.9}$$

$$\left[\vec{\nabla}\times\vec{A}\right]=0. \tag{4.10}$$

As can be seen the equations (4.8) and (4.9) coincide with equations (4.2) and (4.3). The equation (4.8) is the law of enthalpy conservation and vector $\vec{A}$ is the enthalpy flux. The equation (4.9) describes the change of enthalpy flux. The equation (4.10) means that the circulation of enthalpy flux along the closed contour (in the simply connected geometry) is equal to zero.

5. Maxwell equations for ideal fluid

Of course, the potentials w and $\vec{A}$ satisfy also the second order wave equation. By analogy with electrodynamics we can also introduce the appropriate field strengths, which satisfy the equations similar to Maxwell's equations.

Let us introduce the operator

$$D=\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right). \tag{5.1}$$

Then the sedeonic first-order wave equation (4.6) is rewritten as

$$\left(i\mathbf{e}_{\mathrm{t}}D+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(w+\mathbf{e}_{\mathrm{tr}}\vec{A}\right)=0. \tag{5.2}$$

Applying the operator $\left(i\mathbf{e}_{\mathrm{t}}D+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)$ to the equation (5.2), we obtain the second-order wave equation

$$\left(i\mathbf{e}_{\mathrm{t}}D+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathrm{t}}D+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(w+\mathbf{e}_{\mathrm{tr}}\vec{A}\right)=0. \tag{5.3}$$

We can introduce the scalar-vector field strengths in accordance with the following definitions:

$$\begin{aligned}
\vec{E}&=-D\vec{A}-\vec{\nabla}w,\\
\vec{H}&=-i\left[\nabla\times\vec{A}\right],\\
\varepsilon&=Dw+\left(\vec{\nabla}\cdot\vec{A}\right).
\end{aligned} \tag{5.4}$$

Then the second-order wave equation (5.3) can be represented as

$$\left(i\mathbf{e}_{\mathrm{t}}D+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathrm{t}}\varepsilon-\mathbf{e}_{\mathrm{r}}\vec{E}-\mathbf{e}_{\mathrm{t}}\vec{H}\right)=0. \tag{5.5}$$

The field strengths ε, $\vec{E}$ and $\vec{H}$ satisfy the system of equations, similar to the Maxwell equations in electrodynamics [5,6]. Indeed, performing the action of the operators in the equation (5.5) and separating the variables with different spatial and temporal properties, we get

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}\right)+D\varepsilon=0,\\
&\left(\vec{\nabla}\cdot\vec{H}\right)=0,\\
&D\vec{E}+i\left[\vec{\nabla}\times\vec{H}\right]+\vec{\nabla}\varepsilon=0,\\
&D\vec{H}-i\left[\vec{\nabla}\times\vec{E}\right]=0.
\end{aligned} \tag{5.6}$$

If we require the calibrating condition similar to the Lorentz gauge in electrodynamics

$$Dw+\left(\vec{\nabla}\cdot\vec{A}\right)=0\,, \tag{5.7}$$

then the scalar filed ε can be eliminated ($\varepsilon=0$). In this case the wave equation (5.5) is rewritten as

$$\left(i\mathbf{e_t}D+\mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_r}\vec{E}+\mathbf{e_t}\vec{H}\right)=0\,, \tag{5.8}$$

and the system (5.6) takes the following form:

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}\right)=0,\\
&\left(\vec{\nabla}\cdot\vec{H}\right)=0,\\
&D\vec{E}+i\left[\vec{\nabla}\times\vec{H}\right]=0,\\
&D\vec{H}-i\left[\vec{\nabla}\times\vec{E}\right]=0.
\end{aligned} \tag{5.9}$$

The analogy with electrodynamics is obvious. However the second-order wave equation (5.3) has the superfluous solutions, since initially wave function satisfies the first-order equation (4.6). In fact, the initial equation (4.6) (and corresponding equation (5.2)) means that corresponding field strengths ε, $\vec{E}$ and $\vec{H}$ are equal to zero:

$$i\mathbf{e_t}\varepsilon-\mathbf{e_r}\vec{E}-\mathbf{e_t}\vec{H}=0\,. \tag{5.10}$$

6. Linearized wave equation

Let us consider the weak disturbance ($|\vec{v}|<<c$) of laminar constant fluid flow. In this case convective derivative is simplified and operator (5.1) can be represented as

$$D_0=\frac{1}{c}\left(\frac{\partial}{\partial t}+\left(\vec{v}_0\cdot\vec{\nabla}\right)\right), \tag{6.1}$$

where $\vec{v}_0=const$ is the velocity of fluid flow [34], which does not depend on time and coordinates. Then D_0 is the linear operator and corresponding wave equation

$$\left(i\mathbf{e_t}D_0+\mathbf{e_r}\vec{\nabla}\right)\left(w+\mathbf{e_{tr}}\vec{A}\right)=0 \tag{6.2}$$

is gauge-invariant. Indeed it is simple to see that this equation is not changed under the following replacement

$$w(\vec{r},t)\Rightarrow w(\vec{r},t)+D_0\alpha(\vec{r},t)\,, \tag{6.3}$$

$$\vec{A}(\vec{r},t)\Rightarrow \vec{A}(\vec{r},t)-\vec{\nabla}\alpha(\vec{r},t)\,, \tag{6.4}$$

where $\alpha(\vec{r},t)$ is arbitrary function satisfying the homogeneous second order wave equation

$$\left(-D_0^2+\Delta\right)\alpha(\vec{r},t)=0 \tag{6.5}$$

in accordance with Lorentz gauge condition (5.7).

In linear approximation the field strengths (5.4) is rewritten as:

$$\begin{aligned}
&\vec{E}=-D_0\vec{A}-\vec{\nabla}w,\\
&\vec{H}=-i\left[\nabla\times\vec{A}\right],
\end{aligned} \tag{6.7}$$

and the Maxwell equations (5.9) take the following form:

$$\begin{aligned}&\left(\vec{\nabla}\cdot\vec{E}\right)=0,\\&\left(\vec{\nabla}\cdot\vec{H}\right)=0,\\&D_0\vec{E}+i\left[\vec{\nabla}\times\vec{H}\right]=0,\\&D_0\vec{H}-i\left[\vec{\nabla}\times\vec{E}\right]=0.\end{aligned}\tag{6.6}$$

If we multiply last two equations in (6.6) on corresponding field strengths (scalar multiplication) and add them together we obtain the following relation:

$$\frac{1}{2}D_0\left(\vec{E}^2+\vec{H}^2\right)-i\left(\vec{\nabla}\left[\vec{E}\times\vec{H}\right]\right)=0\,,\tag{6.8}$$

which is the analog of Pointing theorem in electrodynamics [21,32].

7. Sound waves

Let us consider the oscillatory motion of a fluid with speed $|\vec{v}|<<c$. In this case, the operator corresponding to convective derivative $(\vec{v}\cdot\nabla)$ can be neglected [34] and sedeonic wave equation (4.4) is simplified as

$$\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\tilde{\mathbf{W}}=0\,.\tag{7.1}$$

The first-order wave equation (7.1) has the solution in the form of plane wave:

$$\tilde{\mathbf{W}}=\tilde{\mathbf{U}}\exp\left\{-i\omega t+i\left(\vec{k}\cdot\vec{r}\right)\right\}.\tag{7.2}$$

Here ω is a frequency, $\vec{k}$ is a wave vector and the wave amplitude $\tilde{\mathbf{U}}$ does not depend on coordinates and time. For equation (7.1) the dependence of the frequency on the wave vector has two branches:

$$\omega_{\pm}=\pm ck\,,\tag{7.3}$$

where k is the modulus of wave vector ($k=|\vec{k}|$). In general, the solution of equation (7.1) can be written as a plane wave of the following form [31]:

$$\tilde{\mathbf{W}}=\left(\mathbf{e}_{\mathbf{t}}\frac{\omega_{\pm}}{c}+i\mathbf{e}_{\mathbf{r}}\vec{k}\right)\tilde{\mathbf{M}}\exp\left\{-i\omega_{\pm}t+i\left(\vec{k}\cdot\vec{r}\right)\right\},\tag{7.4}$$

where $\tilde{\mathbf{M}}$ is arbitrary sedeon with constant components, which do not depend on coordinates and time. Indeed the expression

$$\left(\mathbf{e}_{\mathbf{t}}\frac{\omega_{\pm}}{c}+i\mathbf{e}_{\mathbf{r}}\vec{k}\right)\tag{7.5}$$

is so-called zero divisor since taking into account (7.3) we have

$$\left(\mathbf{e}_{\mathbf{t}}\frac{\omega_{\pm}}{c}+i\mathbf{e}_{\mathbf{r}}\vec{k}\right)\left(\mathbf{e}_{\mathbf{t}}\frac{\omega_{\pm}}{c}+i\mathbf{e}_{\mathbf{r}}\vec{k}\right)\equiv 0\,.\tag{7.6}$$

In simplest case $\tilde{\mathbf{M}}$ can be chosen as constant sedeon $\tilde{\mathbf{M}}=\mathbf{e}_{\mathbf{t}}\alpha$, then we have

$$\tilde{\mathbf{W}}=w+\mathbf{e}_{\mathbf{tr}}\vec{A}=\left(\frac{\omega_{\pm}}{c}+\mathbf{e}_{\mathbf{tr}}\vec{k}\right)\alpha\exp\left\{-i\omega_{\pm}t+i\left(\vec{k}\cdot\vec{r}\right)\right\}.\tag{7.7}$$

Thus this plane wave solution describes the scalar wave of enthalpy w and longitudinal vector wave of sound potential $\vec{A}$.

8. The second-order relations for sound potentials

Multiplying the equation (7.1) on the sedeon ($w-\mathbf{e}_{\mathbf{tr}}\vec{A}$) from the left, we have the following equation:

$$\left(w-\mathbf{e}_{\mathbf{tr}}\vec{A}\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(w+\mathbf{e}_{\mathbf{tr}}\vec{A}\right)=0\,.\tag{8.1}$$

Performing the sedeonic multiplication and separating the quantities with different spatial and temporal properties, we get the relations for the sound potentials:

$$\frac{1}{2c}\frac{\partial}{\partial t}\left(w^2+\vec{A}^2\right)+\left(\vec{A}\cdot\vec{\nabla}w\right)+w\left(\vec{\nabla}\cdot\vec{A}\right)=0\,, \tag{8.2}$$

$$\frac{1}{2}\vec{\nabla}\left(w^2+\vec{A}^2\right)+\frac{1}{c}\frac{\partial}{\partial t}\left(w\vec{A}\right)+\vec{A}\left(\vec{\nabla}\cdot\vec{A}\right)-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{A}+2\left[\vec{A}\times\left[\vec{\nabla}\times\vec{A}\right]\right]=0\,, \tag{8.3}$$

$$w\left[\vec{\nabla}\times\vec{A}\right]+\left[\vec{A}\times\vec{\nabla}w\right]+\frac{1}{c}\left[\vec{A}\times\frac{\partial\vec{A}}{\partial t}\right]=0\,, \tag{8.4}$$

$$\left(\vec{A}\cdot\left[\vec{\nabla}\times\vec{A}\right]\right)=0\,. \tag{8.5}$$

The expressions (8.2) and (8.3) are the "Poynting" theorem for the sound potentials satisfying the first-order wave equation (7.1).

9. Highly symmetric sedeonic wave equation

To take into account the effects associated with the fluid entropy the equation (4.6) can be extended in the following highly symmetric form:

$$\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)+\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\left(w+\mathbf{e}_{\mathbf{tr}}\vec{A}+\mathbf{e}_{\mathbf{tr}}s+\vec{B}\right)=0\,. \tag{9.1}$$

Here following to the symmetry requirement we included the entropy s and the additional vector potential $\vec{B}$ characterizing the entropy flux as it will be shown below. Performing sedeonic multiplication in (9.1) we obtain:

$$\begin{aligned}
&\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)+\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\left(w+\mathbf{e}_{\mathbf{tr}}s+\mathbf{e}_{\mathbf{tr}}\vec{A}+\vec{B}\right)=\\
&i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)w+\mathbf{e}_{\mathbf{r}}\vec{\nabla}w\\
&+\mathbf{e}_{\mathbf{r}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)s+i\mathbf{e}_{\mathbf{t}}\vec{\nabla}s\\
&+\mathbf{e}_{\mathbf{r}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+i\mathbf{e}_{\mathbf{t}}\left(\vec{\nabla}\cdot\vec{A}\right)+i\mathbf{e}_{\mathbf{t}}\left[\vec{\nabla}\times\vec{A}\right]\\
&+i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{B}+\mathbf{e}_{\mathbf{r}}\left(\vec{\nabla}\cdot\vec{B}\right)+\mathbf{e}_{\mathbf{r}}\left[\vec{\nabla}\times\vec{B}\right].
\end{aligned} \tag{9.2}$$

Separating the values with different space-time properties, we get the following symmetric system of equations:

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)w+\left(\vec{\nabla}\cdot\vec{A}\right)=0\,, \tag{9.3}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)s+\left(\vec{\nabla}\cdot\vec{B}\right)=0\,. \tag{9.4}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+\vec{\nabla}w=0\,, \tag{9.5}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{B}+\vec{\nabla}s=0\,, \tag{9.6}$$

$$\left[\vec{\nabla}\times\vec{A}\right]=0\,, \tag{9.7}$$

$$\left[\vec{\nabla}\times\vec{B}\right]=0\,. \tag{9.8}$$

The equations (9.3) - (9.6) are the conservation laws for enthalpy, entropy and corresponding fluxes. The equations (9.7) and (9.8) mean that the circulation of enthalpy flux and entropy flux along the closed contour (in the simply connected geometry) is equal to zero.

10. Sedeonic equations for viscous fluid

The symmetric equations (9.1) and (9.3)-(9.6) can be expanded for the description on viscous fluid by addition of operator describing the dissipative processes caused by internal friction [4]. Let us introduce the operator

$$D_{\upsilon} = \frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right) \tag{10.1}$$

where υ is viscosity of fluid. Then the wave equation (9.1) can be reformulated as generalized Navier-Stokes equation

$$\left(i\mathbf{e}_{\mathbf{t}}D_{\upsilon} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(w + \mathbf{e}_{\mathbf{tr}}s + \mathbf{e}_{\mathbf{tr}}\vec{A} + \vec{B}\right) = 0\,, \tag{10.2}$$

or in expanded form

$$\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right) + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\left(w + \mathbf{e}_{\mathbf{tr}}s + \mathbf{e}_{\mathbf{tr}}\vec{A} + \vec{B}\right) = 0\,. \tag{10.3}$$

Performing sedeonic multiplication in (10.3) we obtain the following generalized equations for viscous fluid:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right)w + \left(\vec{\nabla}\cdot\vec{A}\right) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right)s + \left(\vec{\nabla}\cdot\vec{B}\right) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right)\vec{A} + \vec{\nabla}w = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \upsilon\Delta\right)\vec{B} + \vec{\nabla}s = 0,\\
&\left[\vec{\nabla}\times\vec{A}\right] = 0,\\
&\left[\vec{\nabla}\times\vec{B}\right] = 0.
\end{aligned} \tag{10.4}$$

On the other hand, applying the operator $\left(i\mathbf{e}_{\mathbf{t}}D_{\upsilon} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)$ to the equation (10.2), we obtain the following second-order wave equation for viscous fluid

$$\left(i\mathbf{e}_{\mathbf{t}}D_{\upsilon} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}D_{\upsilon} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(w + \mathbf{e}_{\mathbf{tr}}s + \mathbf{e}_{\mathbf{tr}}\vec{A} + \vec{B}\right) = 0\,. \tag{10.5}$$

Similar to (5.5) we can introduce the complex scalar-vector field strengths in accordance with the following definitions:

$$\begin{aligned}
&\vec{E} = -D_{\upsilon}\vec{A} - \vec{\nabla}w - \left[\vec{\nabla}\times\vec{B}\right],\\
&\vec{H} = -iD_{\upsilon}\vec{B} - i\vec{\nabla}s - i\left[\nabla\times\vec{A}\right],\\
&\varepsilon = D_{\upsilon}w + \left(\vec{\nabla}\cdot\vec{A}\right),\\
&h = -iD_{\upsilon}s - i\left(\vec{\nabla}\cdot\vec{B}\right).
\end{aligned}, \tag{10.6}$$

Then the wave equation (10.5) takes the following form:

$$\left(i\mathbf{e}_{\mathbf{t}}D_{\upsilon} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\varepsilon - i\mathbf{e}_{\mathbf{r}}h - \mathbf{e}_{\mathbf{r}}\vec{E} - \mathbf{e}_{\mathbf{t}}\vec{H}\right) = 0\,. \tag{10.7}$$

Performing the action of operator in (10.7) we get the following system of the Maxwell equations:

$$
\begin{aligned}
&D_\upsilon \varepsilon + \left(\vec{\nabla}\cdot\vec{E}\right) = 0,\\
&D_\upsilon h + \left(\vec{\nabla}\cdot\vec{H}\right) = 0,\\
&D_\upsilon \vec{E} + i\left[\vec{\nabla}\times\vec{H}\right] + \vec{\nabla}\varepsilon = 0,\\
&D_\upsilon \vec{H} - i\left[\vec{\nabla}\times\vec{E}\right] + \vec{\nabla}h = 0.
\end{aligned}
\tag{10.8}
$$

If we require the Lorentz gauge conditions

$$
\begin{aligned}
&D_\upsilon w + \left(\vec{\nabla}\cdot\vec{A}\right) = 0,\\
&D_\upsilon s + \left(\vec{\nabla}\cdot\vec{B}\right) = 0,
\end{aligned}
\tag{10.9}
$$

then the scalar fields ε and h can be eliminated and the system (10.8) takes the following form:

$$
\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}\right) = 0,\\
&\left(\vec{\nabla}\cdot\vec{H}\right) = 0,\\
&D_\upsilon \vec{E} + i\left[\vec{\nabla}\times\vec{H}\right] = 0,\\
&D_\upsilon \vec{H} - i\left[\vec{\nabla}\times\vec{E}\right] = 0.
\end{aligned}
\tag{10.10}
$$

These are the Maxwell equations for viscous fluid.

11. Sedeonic equations for sound in viscous fluid

In case of $|\vec{v}| << c$ the operator corresponding to convective derivative $(\vec{v}\cdot\nabla)$ can be neglected and sedeonic wave equation (10.3) takes the following form:

$$
\left\{ i\mathbf{e}_t\left(\frac{1}{c}\frac{\partial}{\partial t} - \upsilon\Delta\right) + \mathbf{e}_r\vec{\nabla} \right\}\tilde{\mathbf{W}} = 0\,. \tag{11.1}
$$

We will find the solution in the form of plane wave:

$$
\tilde{\mathbf{W}} = \exp\left\{-i\omega t + i\left(\vec{k}\cdot\vec{r}\right)\right\}\tilde{\mathbf{U}}\,, \tag{11.2}
$$

where the wave amplitude $\tilde{\mathbf{U}}$ does not depend on coordinates and time. The dispersion relation for the equation (11.1) is

$$
\omega^2 + i2c\upsilon k^2\omega - c^2\upsilon^2k^4 - c^2k^2 = 0\,, \tag{11.3}
$$

The solution of this equation is

$$
\omega_\pm = -i\upsilon ck^2 \pm ck\,, \tag{11.4}
$$

In general, the solution of equation (11.1) can be written as a plane wave of the following form [31]:

$$
\tilde{\mathbf{W}} = \left(\mathbf{e}_t\left(\frac{\omega_\pm}{c} + i\upsilon k^2\right) + i\mathbf{e}_r\vec{k}\right)\exp\left\{-i\omega_\pm t + i\left(\vec{k}\cdot\vec{r}\right)\right\}\tilde{\mathbf{M}}\,, \tag{11.5}
$$

where $\tilde{\mathbf{M}}$ is arbitrary sedeon with constant components, which do not depend on coordinates and time. Indeed the expression

$$
\left(\mathbf{e}_t\left(\frac{\omega_\pm}{c} + i\upsilon k^2\right) + i\mathbf{e}_r\vec{k}\right) \tag{11.6}
$$

is the zero divisor since taking into account (11.4) we have

$$
\left(\mathbf{e}_t\left(\frac{\omega_\pm}{c} + i\upsilon k^2\right) + i\mathbf{e}_r\vec{k}\right)\left(\mathbf{e}_t\left(\frac{\omega_\pm}{c} + i\upsilon k^2\right) + i\mathbf{e}_r\vec{k}\right) \equiv 0\,. \tag{11.7}
$$

In simplest case $\tilde{\mathbf{M}}$ can be chosen as constant sedeon $\tilde{\mathbf{M}} = \mathbf{e}_t\alpha - i\mathbf{e}_r\beta$, then we have

$$\tilde{\mathbf{W}} = w + \mathbf{e}_{\mathbf{tr}} s + \mathbf{e}_{\mathbf{tr}} \vec{A} + \vec{B} = \left\{ \alpha \left(\left(\frac{\omega_{\pm}}{c} + i\upsilon k^2 \right) + \mathbf{e}_{\mathbf{tr}} \vec{k} \right) + \beta \left(\mathbf{e}_{\mathbf{tr}} \left(\frac{\omega_{\pm}}{c} + i\upsilon k^2 \right) + \vec{k} \right) \right\} \exp\left\{ -i\omega_{\pm} t + i\left(\vec{k} \cdot \vec{r} \right) \right\}. \quad (11.8)$$

This plane wave solution describes the scalar waves of enthalpy w and entropy s as well as the longitudinal vector waves of sound potentials $\vec{A}$ and $\vec{B}$.

13. Conclusion

Thus we have developed the description of ideal fluid on the basis of noncommutative algebra of space-time sedeons. We have shown that the equations describing the dynamics of ideal fluid can be reformulated in the compact symmetric form of single sedeonic first-order wave equation for fluid potentials w, $\vec{A}$, s, $\vec{B}$, which are analogs of electric φ_e, $\vec{A}_e$ and magnetic φ_m, $\vec{A}_m$ potentials in electrodynamics This representation allows one to introduce the corresponding field strengths and formulate the system of Maxwell-like equations as well as the second-order wave equation and derive the relations for the energy and momentum of hydrodynamic fields. However this procedure leads to an increase in the order of the equations and as a consequence to appearance of redundant solutions. So the sedeonic first-order wave equation is more appropriate. We have demonstrated that in linear approximation the sedeonic equation for sound has the plane wave solution for scalar and vector potentials with specific polarization properties. Also we derived the second-order relations for the sound potentials analogues to the Pointing theorem in an electrodynamics.

In addition, based on symmetric consideration we proposed the extended sedeonic first-order wave equation describing the viscous fluid. On the one hand, this equation can be also reformulated as the system of Maxwell equations for field strengths but with other complicated nonlinear differential operator. On the other hand, neglecting the convective derivative we obtained the sedeonic equation for the sound waves in viscous fluid. In this case the plane wave solution is based at the same sedeonic zero devisor properties but has more complicated space-time structure.

Of course, the extended sedeonic wave equation (10.3) is obtained from the symmetry considerations and should be specified in each concrete case. In particular, we propose that considered equations should be additionally investigated for the description of turbulent fluid motion.

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