Number Pi , Collection of Formulas

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abstract. In this paper we show some formulas for constant Pi.

resumen. En este paper mostramos una colección de fórmulas para la constante Pi.

Introducción. Recordamos una representación integral para la constante Pi:

$$\pi = \frac{8}{3} + \frac{8}{3} \int_{0}^{1/4} \sqrt{\frac{\sqrt{x}}{1-\sqrt{x}}}\, dx = 3.1415926535 \ldots$$

en esta nota mostramos una colección de fórmulas para la constante Pi.

1. Fórmulas

(1) $$\pi = 2 + 2 \sum_{n=1}^{\infty} \left((n+1) \sin\left(\frac{\pi}{2n+2}\right) - n \sin\left(\frac{\pi}{2n}\right) \right)$$

(2) $$\pi = 2\sqrt{2} + \sum_{n=1}^{\infty} 2^{n} \left(2\, s_{n+1} - s_{n} \right)$$

donde

$$s_n = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{n-radicales}$$

(3)

$$\pi = 6\sqrt{3}\, t \sum_{n=0}^{\infty} (1+3t^2)^{-n-1} \frac{2^n n!}{(2n+1)!} c_n$$

$$\pi = 8\left(\sqrt{2}+1\right) t \sum_{n=0}^{\infty} \left(1+\left(\sqrt{2}+1\right)^2 t^2\right)^{-n-1} \frac{2^n n!}{(2n+1)!} c_n$$

$$\pi = 8\left(\sqrt{3}+2\right) t \sum_{n=0}^{\infty} \left(1+\left(\sqrt{3}+2\right)^2 t^2\right)^{-n-1} \frac{2^n n!}{(2n+1)!} c_n$$

donde

$$t = \tanh 1 = \frac{e^2-1}{e^2+1}$$

$$c_n = t(1-t^2)^n \frac{(2n)!}{2^n n!} + 2n\, c_{n-1}\,, n \in \mathbb{N}, c_0 = t$$

$$c_n = 2^n n!\, t + \sum_{k=1}^{n} \binom{2k}{k} 2^{n-2k} n!\, t\, (1-t^2)^k$$

(4) $$\pi = 3\sqrt{3} \sum_{n=0}^{\infty} a^{3n+1} \left(\frac{1}{3n+1} - \frac{a}{3n+2}\right)$$

$$+ 3\sqrt{3}\left(\frac{1-a}{1+a+a^2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} A^n \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} B^k$$

$$A = \frac{1+a-2a^2}{1+a+a^2} \ , B = \frac{1+a+a^2}{(1+2a)^2}$$

$$\frac{\sqrt{3}-1}{2} < a < 1$$

(5) $$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} \left(\frac{3}{4}\right)^{k+1} \sum_{m=0}^{[k/2]} \binom{k+1}{2m+1} (-3)^m$$

(6) $$\pi = 8 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} \left(\frac{2+\sqrt{2}}{4}\right)^{k+1} \sum_{m=0}^{[k/2]} \binom{k+1}{2m+1} (-1)^m \left(\sqrt{2} - 1\right)^{2m+1}$$

(7) $$\pi = 12 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} \left(\frac{2-\sqrt{3}}{4}\right)^{k+1} \sum_{m=0}^{[k/2]} \binom{k+1}{2m+1} (-1)^m (2 - \sqrt{3})^{2m+1}$$

(8)

$$\pi = 6\sqrt{6} - 3\sqrt{3} - 6 - 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{-3n}}{n+1} c_n$$

$$\pi = 6 + 3\sqrt{3} - 6\sqrt{2} + 6 \sum_{n=1}^{\infty} \frac{2^{-3n}}{n+1} c_n$$

donde

$$c_n = \sum_{k=1}^{n} \binom{2n-2k}{n-k} \binom{2k-2}{k-1} \frac{(-1)^{k-1}}{k} \ , n \in \mathbb{N}$$

(9) $$\pi = 3 \ln\left(\frac{9}{10}\right) + \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \, 3^{-2n} \left(\frac{3}{2n+1} + \frac{1/\sqrt{3}}{2n+2} + \frac{1/3}{2n+3} + \frac{1/(3\sqrt{3})}{2n+4}\right) \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-3)^k \, 2^{2n-4k}$$

(10) $$\pi = 6\alpha_k \sqrt{3} \sum_{n=0}^{\infty} \frac{c_n(\beta_k)}{(2n+1)2^{2n+1}} \left(\ln \frac{\gamma_k + 1}{\gamma_k - 1}\right)^{2n+1}$$

donde

$$k \in \mathbb{N}$$

$$c_n(\beta_k) = -\beta_k^2 \sum_{m=1}^{n} \frac{2^{2m-1}}{(2m)!} c_{n-m}(\beta_k) \ , n \in \mathbb{N}, c_0(\beta_k) = 1$$

$$\alpha_{k+2} = 4\alpha_{k+1} - \alpha_k \ , \alpha_1 = 1, \alpha_2 = 4$$

$$\beta_{k+2} = 4\beta_{k+1} - \beta_k \ , \beta_1 = 2, \beta_2 = 7$$

$$\gamma_{k+2} = 4\gamma_{k+1} - \gamma_k \ , \gamma_1 = 3, \gamma_2 = 12$$

(11)

$$\pi \ln\left(1 - \frac{\sqrt{3}}{2}\right) = 24 \sum_{n=1}^{\infty} c_n \, f\left(n, \frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2}\right)$$

$$\pi \ln\left(\frac{2+\sqrt{3}}{32}\right) = 24 \sum_{n=1}^{\infty} c_n \, f\left(n, 2-\sqrt{3}, \sqrt{3}-1\right)$$

$$\pi \ln\left(\frac{9\left(2-\sqrt{3}\right)}{128}\right) = 24 \sum_{n=1}^{\infty} c_n \, f\left(n, \frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}\right)$$

$$\pi\left(2\ln\left(8\left(24 - 5\sqrt{2} - 14\sqrt{3} + 3\sqrt{6}\right)\right) - 3\ln\left(16\left(3-2\sqrt{2}\right)\left(2-\sqrt{3}\right)\right)\right)$$
$$= 24 \sum_{n=1}^{\infty} c_n \, f\left(n, \frac{\left(2-\sqrt{2}\right)\left(\sqrt{3}-1\right)}{2}, \frac{2}{3+\sqrt{2}+\sqrt{3}}\right)$$

$$\pi\left(3\ln\left(3-2\sqrt{2}\right) - 4\ln\left(2-\sqrt{2}\right) + \ln\left(2\left(5 - 3\sqrt{2} - \sqrt{3} + \sqrt{6}\right)\right)\right)$$
$$= 24 \sum_{n=1}^{\infty} c_n \, f\left(n, 1 - \frac{\sqrt{6}+\sqrt{3}-3}{\sqrt{2}}, 1 - \sqrt{\frac{3}{2} + \frac{1}{\sqrt{2}}}\right)$$

$$\pi \ln 2 = -\frac{16}{3} \sum_{n=1}^{\infty} c_n \, f\left(n, \sqrt{2}-1, 2-\sqrt{2}\right)$$

donde

$$c_n = \left(H_n - H_{2n} - \frac{1}{2n}\right)\frac{1}{n} = \frac{1}{n}\sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \ \ , H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$f(n,a,b) = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} a^{2n-2k-1} b^{2k+1}$$

(12) $$\pi = 2\left(\sqrt{2}+1\right)\sqrt{\sqrt{2}-1} - 2\sqrt{2}\sum_{n=2}^{\infty}\frac{2^{-n}}{2n-1}c_n - \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-4n}}{(2n+1)^2}\binom{4n}{2n}$$

donde

$$c_n = \sqrt{\sqrt{2}+1}\ Im((1-i)^n) + \sqrt{\sqrt{2}-1}\,Re((1-i)^n)$$

(13) $$\pi = 24\sum_{n=0}^{\infty}\binom{2n}{n}(-4)^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}\left(\frac{2^{-k}}{\sqrt{2}} - \left(1-\frac{1}{\sqrt{2}}\right)^k\sqrt{1-\frac{1}{\sqrt{2}}}\right)$$

(14) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{2+\sqrt{2}}\right)^{-n}a_n = 4\sum_{n=1}^{\infty}\frac{2^{-n}}{n}b_n$$

donde

$$a_{n+2} = \sqrt{2+\sqrt{2}}\,a_{n+1} - a_n\ ,\ a_1 = \sqrt{2-\sqrt{2}}\,,\ a_2 = \sqrt{2}$$

$$b_{n+2} = 2\,b_{n+1} - \left(4-2\sqrt{2}\right)b_n\ ,\ b_1 = 2\sqrt{2}-2\ ,\ b_2 = 4\sqrt{2}-4$$

(15) $$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-3)^{-n}\zeta(2mn+2m+1)}{2n+1} - 6\sum_{n=2}^{\infty}\frac{1}{n^{m+1}}\tan^{-1}\left(\frac{1}{\sqrt{3}\,n^m}\right)$$

$$m \in \mathbb{N}$$

$\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$, función zeta de Riemann.

(16) $$\pi = 4\sum_{n=1}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{(-1)^{n-k-1}}{(n-k)2^{n-k}}\right)\sin\left(\frac{n\,\pi}{2}\right)$$

$$= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{2n-1}}\sum_{k=0}^{[(2n-1)/2]}\binom{2n-k-1}{k}\frac{(-2)^{k}}{2n-k-1}$$

(17) $$\pi = 8\sum_{n=1}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{(-1)^{n-k-1}}{(n-k)}\left(\frac{2-\sqrt{2}}{2}\right)^{n-k}\right)\sin\left(\frac{n\,\pi}{4}\right)$$

(18) $$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}\frac{(-3)^{n}}{2n+1}\left(\frac{e-1}{e+1}\right)^{2n+1}$$

$$+\frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{e^{-3n+2}}{3n-2}+\frac{e^{-3n+1}}{3n-1}\right)$$

(19) $$\pi = 4\tan^{-1}\left(\frac{e^{x}}{e^{x}+1}\right)+2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n}e^{-(n+k+1)x}}{2^{k}(n+k+1)}$$

$$x > -\ln\left(\sqrt{3}-1\right)$$

(20) $$\pi = 4\tan^{-1}\left(\frac{e^{x}}{e^{x}-1}\right)-2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}e^{-(n+k+1)x}}{2^{k}(n+k+1)}$$

$$x \geq 0$$

Si $x = 0$, se tiene:

$$\pi = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{2^{k}(n+k+1)}$$

(21) $$\pi = 2\tan^{-1}\left(\frac{e^{x}}{e^{x}-1}\right)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}\left(1-e^{-(n+k+1)x}\right)}{2^{k}(n+k+1)}$$

$$x > 0$$

(22) $$\pi = 3\tan^{-1}\left(\sqrt{3}\ \tanh x\right) + \frac{3\sqrt{3}}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{e^{-(6n-4)x}}{3n-2} + \frac{e^{-(6n-2)x}}{3n-1}\right)$$

$$x \geq 0$$

(23) $$\pi = 6\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-3)^k}{(2k+1)(3^{2k+1}+1)^{n-k+1}}$$

(24) $$\pi = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}^{-1}\left(\frac{2}{29}\right)^{2n}(21\cdot 20^{2n-1} + 20\cdot 21^{2n-1})$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}^{-1}\left(\frac{2}{29}\right)^{2n} c_n$$

donde

$$c_{n+2} = 841c_{n+1} - 176400c_n\ \ , c_1 = 840, c_2 = 353220$$

(25) $$\pi = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}^{-1}\left(\frac{2}{169}\right)^{2n}(120\cdot 119^{2n-1} + 119\cdot 120^{2n-1})$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}^{-1}\left(\frac{2}{169}\right)^{2n} c_n$$

donde

$$c_{n+2} = Ac_{n+1} - Bc_n\ \ , c_1 = 28560, c_2 = 407851080$$

$$A = 119^2 + 120^2\ , B = 119^2\cdot 120^2$$

(26)

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(\left(\frac{1}{15}\right)^{n-k}\left(\frac{8}{15}\right)^{2k+1} + \left(\frac{1}{56}\right)^{n-k}\left(\frac{15}{56}\right)^{2k+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(\left(\frac{1}{21}\right)^{n-k}\left(\frac{10}{21}\right)^{2k+1} + \left(\frac{1}{40}\right)^{n-k}\left(\frac{13}{40}\right)^{2k+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(\left(\frac{1}{24}\right)^{n-k}\left(\frac{11}{24}\right)^{2k+1}+\left(\frac{1}{35}\right)^{n-k}\left(\frac{12}{35}\right)^{2k+1}\right)$$

(27) $$\pi = \frac{24}{5}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(-2+2\sqrt{2}+\sqrt{3}-\sqrt{6}\right)^{n-k}\left(1+\sqrt{2}-\sqrt{3}\right)^{2k+1}$$

(28) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(6+5\sqrt{2}-3\sqrt{5}-2\sqrt{10}\right)^{n-k}\left(-5+\sqrt{5}+\sqrt{10}\right)^{2k+1}$$

(29) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}\left(-\frac{\left(\sqrt{2}-1\right)^3}{2}\right)^{n-k}\left(\frac{15-10\sqrt{2}}{2}\right)^{2k+1}$$

(30)

$$\pi = 3\sqrt{3}\,F\left(\frac{1}{3},1;\frac{4}{3};-\frac{1}{8}\right)-\sqrt{3}\ln 3$$

$$\pi = \frac{6\sqrt{3}}{\sqrt[3]{9}}\,F\left(\frac{1}{3},\frac{1}{3};\frac{4}{3};\frac{1}{9}\right)-\sqrt{3}\ln 3$$

$$\pi = \frac{8\sqrt{3}}{3}\,F\left(1,1;\frac{4}{3};\frac{1}{9}\right)-\sqrt{3}\ln 3$$

$F(a,b;c;x)$ es la clásica función hipergeométrica de Gauss.

(31) $$\pi = 2\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1/144)^{n-k}}{2k+1}\left(\left(\frac{2}{3}\right)^{2k+1}+\left(\frac{3}{4}\right)^{2k+1}\right)$$

(32) $$\pi = 2\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1/289)^{n-k}}{2k+1}\left(\left(\frac{11}{17}\right)^{2k+1}+\left(\frac{13}{17}\right)^{2k+1}\right)$$

(33) $$\pi = \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\, 24^{-2n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{(-1)^k}{2k+1}\left(\frac{144}{145}\right)^k\left(8^{2k+1}+9^{2k+1}\right)$$

(34) $$\pi = \frac{2}{17}\sum_{n=0}^{\infty}(-1)^n\, 34^{-2n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{(-1)^k}{2k+1}\left(\frac{289}{290}\right)^k\left(11^{2k+1}+13^{2k+1}\right)$$

(35) $$\pi = 3\sum_{n=0}^{\infty}\binom{2n}{n}2^{-4n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{(-1)^k 2^{2k}}{2k+1}$$

(36) $$\pi = 3\sum_{n=0}^{\infty}2^{-2n}\sum_{k=0}^{n}\frac{(-1)^k\left(k+\frac{1}{2}\right)_{n-k}}{(2k+1)(n-k)!}$$

(37) $$\pi = 4\sum_{n=0}^{\infty}(-3)^{-n}\left(\frac{1}{2n+1}-\sum_{k=1}^{n}\binom{2k-2}{k-1}\frac{(-3)^k 2^{-4k+1}}{k(2n-2k+1)}\right)$$

(38)

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}x^{3n+1}\left(\frac{1}{3n+1}-\frac{x}{3n+2}\right) + \sqrt{3}\sum_{n=0}^{\infty}(-1)^n\, 3^{-3n}y^{6n+1}\left(\frac{1}{6n+1}+\frac{y}{6n+2}+\frac{(2/3)y^2}{6n+3}+\frac{(1/3)y^3}{6n+4}+\frac{(1/9)y^4}{6n+5}\right)$$

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}x^{3n+1}\left(\frac{1}{3n+1}-\frac{x}{3n+2}\right)+\sqrt{3}\sum_{n=0}^{\infty}\frac{y^{n+1}}{n+1}\sum_{k=0}^{[n/2]}\binom{n-k}{k}(-3)^{-k}$$

donde

$$x = \frac{\sqrt{3}-1}{2}\ ,y = \frac{3-\sqrt{3}}{2}$$

(39)

$$\pi = 2A\left(\rho\, a + \sum_{n=1}^{\infty} \frac{(-1)^n a^{n+1}}{n+1}(\rho\, c_n - c_{n-1})\right)$$

$$\pi = 3A\left(\rho\, b + \sum_{n=1}^{\infty} \frac{(-1)^n b^{n+1}}{n+1}(\rho\, c_n - c_{n-1})\right)$$

$$\pi = 4A\left(\rho\, c + \sum_{n=1}^{\infty} \frac{(-1)^n c^{n+1}}{n+1}(\rho\, c_n - c_{n-1})\right)$$

$$\pi = 6A\left(\rho\, d + \sum_{n=1}^{\infty} \frac{(-1)^n d^{n+1}}{n+1}(\rho\, c_n - c_{n-1})\right)$$

donde

$$c_n = (-1)^n \sum_{k=0}^{[n/3]} (-1)^k \binom{n-2k}{k} \;, n = 0,1,2,3,\ldots$$

$$\rho = \frac{1}{3}\left(\frac{27-3\sqrt{69}}{2}\right)^{1/3} + \left(\frac{9+\sqrt{69}}{18}\right)^{1/3}$$

$$\rho = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}$$

$$A = \sqrt{3\rho^2 - 4}$$

$$a = \frac{2(\rho^2-1)}{\rho + A} \;, \qquad b = \frac{2(\rho^2-1)}{\rho + A\sqrt{3}}$$

$$c = \frac{2(\rho^2-1)\left(\sqrt{2}-1\right)}{\rho\left(\sqrt{2}-1\right)+A} \;, \qquad d = \frac{2(\rho^2-1)\left(2-\sqrt{3}\right)}{\rho\left(2-\sqrt{3}\right)+A}$$

(40) $$\pi = \frac{e^3}{4}\prod_{n=1}^{\infty}\left\{\left(\frac{n}{n+2}\right)^{n+1} e^2\right\}$$

(41) $$(a-b)\pi = \sum_{n=0}^{\infty} \ln\left(\frac{n^2+a^2}{n^2+b^2}\right) - \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m}{m+1}\left(\left(\frac{1}{(n-m)^2+b^2}\right)^{m+1} - \left(\frac{1}{(n-m)^2+a^2}\right)^{m+1}\right)\left(\frac{1}{2m+3} + \sum_{k=1}^{m+1}\binom{m+1}{k}\frac{(2n-2m)^k}{2m-k+3}\right)$$

donde

$$a > b > 0\,, a^2 > b^2 \geq 2$$

Observación: $\prod_{n=0}^{\infty}\left(\frac{n^2+a^2}{n^2+b^2}\right) = \frac{a\ \sinh(a\pi)}{b\ \sinh(b\pi)}$

(42) $$\pi = 2^{k+1}\ln\left(2+\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k-radicales}\right) - 2^{k+1}\ln\left(\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k-radicales}\right) - \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{m-n}2^{-2kn-2k+2}}{(2m-2n+1)^{2n+3}(2n+3)}$$

$$k \in \mathbb{N}$$

(43)

$$\pi = 4\sum_{n=0}^{\infty}(-3)^{-n}\ c_n$$

$$\pi = 24\sqrt{2-\sqrt{3}}\sum_{n=0}^{\infty}(-1)^n\left(2-\sqrt{3}\right)^{2n+1}\ c_n$$

donde

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{(-1)^k 2^{-2k}}{2k+1} \quad , n = 0,1,2,3, \ldots$$

(44)

$$\pi = 4 \sum_{n=0}^{\infty} \left(\sqrt{2} - 1\right)^{2n+1} c_n$$

$$\pi = 6 \sum_{n=0}^{\infty} \left(2 - \sqrt{3}\right)^{2n+1} c_n$$

$$\pi = \sqrt{3} \sum_{n=0}^{\infty} 3^{-n} c_n$$

$$\pi = 4 \sum_{n=0}^{\infty} \left(\left(\sqrt{5} - 2\right)^{2n+1} + \left(\sqrt{10} - 3\right)^{2n+1}\right) c_n$$

donde

$$c_n = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k 2^{2k+1}}{2k+1} \quad , n = 0,1,2,3, \ldots$$

(45) $$\pi = 3\, exp\left\{\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{6^{-2k-2}\, 2^{-n+k-1}}{(k+1)(1-2^{-2k-1})} \sum_{m=0}^{n-k} \binom{n-k}{m} (-1)^m (m+1)^{-2k-2}\right\}$$

(46) $$\pi = \frac{3}{2} \prod_{n=1}^{\infty} \frac{9n^2(36n^2-4)}{(9n^2-1)(36n^2-1)} + 3 \prod_{n=1}^{\infty} \frac{36n^2(9n^2-4)}{(9n^2-1)(36n^2-1)}$$

(47) $$\pi = 2\sqrt{2}\, \Gamma(5/4) + \sum_{n=1}^{\infty} 2^n \left(2 s_{n+1} \Gamma(1 + 2^{-n-2}) - s_n \Gamma(1 + 2^{-n-1})\right)$$

donde s_n , se define como en (2) , y $\Gamma(x)$ es la función Gama.

(48) $$\pi = \left(\frac{2^{2k+1}\,k!}{(2k+1)!}\right)^2 \prod_{n=1}^{\infty} \frac{4(n+1)^{2k+1}}{(2n+2k+1)^2\,n^{2k-1}}$$

$$k \in \mathbb{N}$$

(49) $$\frac{216}{\pi^2}\left(\sqrt[3]{\frac{\sqrt{3}}{8}+\frac{3}{4}\sqrt[3]{\frac{\sqrt{3}}{8}+\frac{3}{4}\sqrt[3]{\frac{\sqrt{3}}{8}+\cdots}}}-\sqrt[3]{\frac{1}{8}+\frac{3}{4}\sqrt[3]{\frac{1}{8}+\frac{3}{4}\sqrt[3]{\frac{1}{8}+\cdots}}}\right)$$

$$= \prod_{n=1}^{\infty}\left(1-\left(\frac{1}{12n}\right)^2\right)\left(1-\left(\frac{1}{36n}\right)^2\right)$$

(50) $$\pi = 6e^a \sin b$$

donde

$$a = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{-n-1}}{2n+2} c_{2n+1} \,, \qquad b = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-n}}{2n+1} c_{2n}$$

$$c_0 = 1 \,, \qquad c_n = 1 - \sum_{k=1}^{n} \frac{c_{n-k}}{k} \,, n \in \mathbb{N}$$

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