Clifford Algebraic Unification of Conformal Gravity with an Extended Standard Model

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Abstract

A brief review of the basics of the Clifford $Cl(5, C)$ Unified Gauge Field Theory formulation of Conformal Gravity and $U(4) \times U(4) \times U(4)$ Yang-Mills in $4D$ is presented. A physically relevant subgroup is $SU(2, 2) \times SU(4)_C \times SU(4)_L \times SU(4)_R$ and which is compatible with the Coleman-Mandula theorem (in the absence of a mass gap). This proposal for a Clifford Algebraic Unification of Conformal Gravity with an Extended Standard Model deals mainly with models of $four$ generations of fermions. Mirror fermions can be incorporated as well. Whether these mirror fermions are dark matter candidates is an open question. There are also residual $U(1)$ groups within this Clifford group unification scheme that should play an important in Cosmology in connection to dark matter particles coupled to gravity via a Bimetric extension of General Relativity. Other four generation scenarios based on $Cl(6, R), Cl(8, R)$ algebras, Supersymmetric Field Theories and Quaternions are discussed.

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1 Introduction : Clifford Algebraic Unification

A proposal for a $Cl(5, C)$ unified gauge field theory formulation of conformal Gravity and $U(4) \times U(4) \times U(4)$ Yang-Mills in $4D$, and its implications for the Pati-Salam group $SU(4) \times SU(2)_L \times SU(2)_R$, the Trinification GUT models of 3 fermion generations (based on the group $SU(3)_C \times SU(3)_L \times SU(3)_R$) and the Standard Model group $SU(3) \times SU(2) \times U(1)$ was analyzed in detail in [1], [2]. One can also obtain a $U(2, 2) \times U(2, 2) \times U(4) \times U(4)$ Yang-Mills gauge theory from a $Cl(5, C)$ gauge theory [1].

Recently in a short note we provided further evidence why this tentative proposal for a unification model of gravity and the Standard Model in four dimensions, that is based on the complexified Clifford algebra $Cl(5,C)$, has precisely the right number of $4 \times 64 = 256$ degrees of freedom [1] to accommodate the vielbein field e^a_μ of gravity, the 12 gauge bosons and 48 fermions (three generations) of the Standard Model. Clifford algebras are very natural to use because spinors are just the left/right ideal elements of the Clifford algebra [4].

The complex Clifford $Cl(5,C)$ algebra [3], [4] admits the decomposition

$$Cl(5,C) \;=\; Cl(4,C) \oplus Cl(4,C) \tag{1.1}$$

and each complex $Cl(4,C)$ algebra is isomorphic to the matrix algebra $M(4,C)$ consisting of 4×4 matrices with complex entries. Consequently, one has

$$Cl(4,C) \;\sim\; M(4,C) \;\sim\; Cl(p,q,R) \oplus \;\mathbf{i}\; Cl(p,q,R) \tag{1.2}$$

where (p,q) represents the chosen signature of the 4-dimensional tangent space metric subjected to the condition $p + q = 4$. Hence, from the isomorphism described in eq-(1.2) one can construct many different (pesudo) unitary algebras, $u(4), u(3,1), u(2,2), u(1,3)$ in terms of the $Cl(p,q,R)$ algebra generators [1].

For example, the Hermitian generators of the $su(4)$ algebra associated with the compact $U(4) = SU(4) \times U(1)$ unitary group can be expressed in terms of the $Cl(3,1)$ algebra generators as [1], [2]

$$M_a \;=\; \frac{1}{2}\Gamma_a\;(1\;+\;\Gamma_5); \quad N_a \;=\; \frac{1}{2}\Gamma_a\;(1\;-\;\Gamma_5); \quad \mathcal{D} \;=\; \frac{i}{2}\;\Gamma_5, \;\; \mathcal{L}_{ab} \;=\; -\;\frac{i}{2}\;\Gamma_{ab}. \tag{1.3}$$

where $\Gamma_5 \equiv -\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and $(\Gamma_5)^2 = -\mathbf{1}$. The above generators lead to the algebra $so(6) \sim su(4)$ whose commutators are given by

$$[M_a,\;\mathcal{D}] \;=\; i\;N_a; \quad [N_a,\;\mathcal{D}] \;=\; -\,i\;M_a; \quad [M_a,\;N_b] \;=\; -\,2i\;g_{ab}\;\mathcal{D}$$

$$[M_a,\;M_b] \;= [N_a,\;N_b] \;=\; \frac{1}{2}\;\Gamma_{ab} \;=\; i\;\mathcal{L}_{ab}; \;\; \ldots \tag{1.4}$$

In the most general case, any pseudo-unitary algebra $u(p,q)$ can be obtained from the unitary one $u(p+q)$ via the Weyl unitary trick [5] which maps the anti-Hermitian generators of the compact group $U(p+q;C)$ to the anti-Hermitian *and* Hermitian generators of the noncompact group $U(p,q;C)$ as follows :

The $(p+q) \times (p+q)$ $U(p+q;C)$ complex matrix generator is comprised of the diagonal blocks of $p \times p$ and $q \times q$ complex anti-Hermitian matrices $M_{11}^\dagger = -M_{11}$; $M_{22}^\dagger = -M_{22}$, respectively. The off-diagonal blocks are comprised of the $q \times p$ complex matrix M_{12} and the $p \times q$ complex matrix $-M_{12}^\dagger$, i.e. the off-diagonal blocks are the anti-Hermitian complex conjugates of each other. In this fashion the $(p+q) \times (p+q)$ $U(p+q;C)$ complex matrix generator $\mathbf{M}$ is anti-Hemitian $\mathbf{M}^\dagger = -\mathbf{M}$ such that upon an exponentiation $U(t) = e^{t\mathbf{M}}$ it

[1] Not to be confused with the actual *physical* degrees of freedom

generates a unitary group element obeying the condition $U^\dagger(t) = U^{-1}(t)$ for $t = real$.

Conformal Gravity can be constructed as a gauge field theory based on the $su(2,2)$ algebra. By applying the Weyl unitary trick to the $u(4)$ algebra, one can obtain the generators of the $su(2,2)$ conformal algebra in terms of the $Cl(3,1)$ algebra as follows

$$P_a = \frac{1}{2}\Gamma_a (1 - i\,\Gamma_5); \quad K_a = \frac{1}{2}\Gamma_a (1 + i\,\Gamma_5); \quad D = -\frac{i}{2}\Gamma_5, \quad L_{ab} = \frac{1}{2}\Gamma_{ab}. \tag{1.5}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (1.5), the explicit evaluation of the commutators yields

$$[P_a,\ D] = P_a; \quad [K_a,\ D] = -K_a; \quad [P_a,\ K_b] = -2g_{ab}\,D + 2\,L_{ab}$$

$$[P_a,\ P_b] = 0; \quad [K_a, K_b] = 0; \ \ldots \tag{1.6}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations.

The complex extension of $U(p+q, C)$ is $GL(p+q; C)$. Since the algebras $u(p+q; C), u(p,q; C)$ differ only by the Weyl unitary trick, they both have identical complex extensions $gl(p+q; C)$ [5]. $gl(N,C)$ has $2N^2$ generators whereas $u(N,C)$ has N^2. As a direct result of the Weyl unitary trick, from the complex Clifford algebra $Cl(5,C)$ on can extract many $different$ algebras given by the direct sum of unitary and/or pseudo-unitary algebras by choosing the appropriate basis of generators.

In particular, the algebras $u(2,2) \oplus u(2,2) \oplus u(4) \oplus u(4)$, and $u(2,2) \oplus u(4) \oplus u(4) \oplus u(4)$. Setting aside the $u(1)$ subalgebras, from the algebra $u(2,2) \oplus u(2,2) \oplus u(4) \oplus u(4)$ one can construct a complexified conformal gravity (or a bi-conformal gravity model with two gravitons) and a $SU(4) \times SU(4)$ Yang-Mills theory [1]. The complexified conformal algebra is $sl(4,C) \sim su(2,2) \oplus i\ su(2,2)$. A $SU(4)_C \times SU(4)_F$ color-flavor unification model was proposed long ago by [14] inspired by the Pati-Salam model [15]. The right-handed multiplets are $flipped$ with respect to the left-handed ones and which allows to gauge *both* the right and left-handed multiplets under the *same* flavor group $SU(4)_F$ [14]. There are $four$ generations of fermions in this model.

The complexified conformal gravity model (bi-conformal gravity with two gravitons) combined with the GUT group $SU(4)_C \times SU(4)_F$ [14] also deserves further investigation. Bimetric theories of gravity has been an active research topic in recent years as a promising avenue to modify General Relativity at large distances in order to explain the accelerating expansion of the Universe (Dark Energy). A review of Bimetric gravity with a large number of references can be found in [7]. The fact that conformal gravity is encoded in the $Cl(5,C)$algebra unification program might be relevant to the recently introduced Mimetic gravity (a Weyl-symmetric extension of the General Relativity) which can play the role of an imperfect fluid-like Dark Matter [8].

In this $Cl(5, C)$ unification scheme, there are residual $U(1)$ groups which should have important physical consequences. For example, in cosmology a $U(1)$ vector field has been recently introduced to link together two different species of dark matter particles coupled to gravity via a bimetric extension of general relativity. The rich phenomenology and physical consequences of this model were studied with great detail in [9]. The impact of the extra $U(1)$ vector fields in cosmology and particle physics is beyond the scope of this work at the moment, and should be addressed in the future.

Let us focus for now on the subalgebra $su(2,2) \oplus su(4) \oplus su(4) \oplus su(4)$ of the algebra $u(2,2) \oplus u(4) \oplus u(4) \oplus u(4)$ which is physically relevant for our purposes. It corresponds to the group $SU(2,2) \times SU(4)_C \times SU(4)_L \times SU(4)_R$ that is the *direct* product of the Conformal group $SU(2,2)$ (spacetime symmetry) with the gauge internal group $[SU(4)]^3$, and consequently is compatible with the Coleman-Mandula theorem in the absence of a mass gap. If there is a mas gap, then the spacetime symmetry must be given by the Poincare group. The first factor of the internal group $SU(4)_C$ reflects the extended color symmetry, and the last two factors $SU(4)_L \times SU(4)_R$ are the gauge internal groups associated with the left, right handed fermions, respectively.

The group $SU(4)_C \times SU(4)_L \times SU(4)_R$ is the natural extension of the group $SU(3)_C \times SU(3)_L \times SU(3)_R$ associated with the Trinification gauge model of Glashow [6] involving 3 generations of fermions. The group is combined with a discrete symmetry group Z_3 exchanging left, right and color symmetries. A breaking of $SU(3)_C \times SU(3)_L \times SU(3)_R \rightarrow SU(3)_C \times SU(2)_W \times U(1)_Y$ furnishes the Standard Model gauge group.

As a reminder, the Clifford $Cl(p, q, \mathbf{R})$ algebra generators Γ_a obey the anti-commutator relations

$$\{ \Gamma_a, \ \Gamma_b \} \ \equiv \ \Gamma_a \ \Gamma_b \ + \ \Gamma_b \ \Gamma_a \ = \ 2 \ g_{ab} \ \mathbf{1} \qquad (1.7a)$$

where $\mathbf{1}$ is the unit element and the metric g_{ab} has (p, q) signature. In four dimensions, the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined by a signed-permutation sum of the anti-symmetrizated products (exterior algebra) of the gammas Γ_a such that

$$[\Gamma_a, \Gamma_b] \ = \ 2 \ \Gamma_{ab}, \ \ldots, \ \ \Gamma_5 \ = \ - \ \Gamma_0 \ \Gamma_1 \ \Gamma_2 \ \Gamma_3, \ \ \{\Gamma_5, \Gamma_a\} \ = \ 0 \qquad (1.7b)$$

$$(\Gamma_5)^2 \ = \ (-1)^{(p-q)(p-q-1)/2} \qquad (1.7c)$$

which equals 1 when $p - q = 0, 1$, and -1 when $p - q = 2, 3$. There are also the important relations

$$\Gamma_{abcd} \ = \ \epsilon_{abcd} \ \Gamma_5, \ \ \Gamma_{abc} \ = \ \epsilon_{abcd} \ \Gamma_5 \ \Gamma^d, \ \ \frac{1}{4!} \ \epsilon^{abcd} \ \Gamma_{abc} \ = \ \Gamma_5 \ \Gamma^d \qquad (1.8)$$

One can represent each element of the 16-dim Clifford $Cl(4)$ algebra as Γ_A where the polyvector valued index A ranges from $1, 2, \ldots, 16$ and spans the identity element $\mathbf{1}$, the four vectors Γ_a, the six bivectors $\Gamma_{ab} = \Gamma_a \wedge \Gamma_b$, the four trivectors $\Gamma_{abc} = \Gamma_a \wedge \Gamma_b \wedge \Gamma_c$, and the pseudo-scalar Γ_{abcd}. As a result, the above relations (1.6-1.8) will then allow a gauge field $\mathbf{A}_\mu$ valued in the $su(2,2) \oplus su(4)_C \oplus su(4)_L \oplus su(4)_R$ algebra to be decomposed into a linear combination of the $Cl(5,C)$ algebra generators (excluding the identity element) as follows

$$\mathbf{A}_\mu \;=\; \Omega^i_\mu \; T_i + \; \mathcal{G}^i_\mu \; T_{iC} + \; \mathcal{W}^i_{\mu L} \; T_{iL} \; + \; \mathcal{W}^i_{\mu R} \; T_{iR} \;=$$

$$^{(1)}A^A_\mu \; {}_{(1)}\Gamma_A \; + \; ^{(2)}A^A_\mu \; {}_{(2)}\Gamma_A \; + \; ^{(3)}A^A_\mu \; {}_{(3)}\Gamma_A \; + \; ^{(4)}A^A_\mu \; {}_{(4)}\Gamma_A \tag{1.9}$$

where ${}_{(1)}\Gamma_A, \; {}_{(2)}\Gamma_A, \; {}_{(3)}\Gamma_A, \; {}_{(4)}\Gamma_A$ are the respective generators of the $four$ copies of the $Cl(4)$ algebras associated with the decomposition of the $Cl(5,C)$ algebra (excluding the identity element). The indices i range from $1, 2, \ldots, 15$. More precisely, in eq-(1.5) one has the expression for the generators of the $su(2,2)$ conformal algebra given in terms of a linear combination of the $Cl(3,1)$ algebra generators. Writing the generators of eq-(1.5) in the form $T_i \;=\; {}^{(1)}c^A_i \; {}_{(1)}\Gamma_A$, where ${}^{(1)}c^A_i$ are numerical coefficients, gives

$$\Omega^i_\mu \; T_i \;=\; \Omega^i_\mu \; ({}^{(1)}c^A_i \; {}_{(1)}\Gamma_A) \;=\; {}^{(1)}A^A_\mu \; {}_{(1)}\Gamma_A \;\Rightarrow\; {}^{(1)}A^A_\mu \;=\; {}^{(1)}c^A_i \; \Omega^i_\mu \tag{1.10}$$

and from which one can read the linear relationship between ${}^{(1)}A^A_\mu$ and Ω^i_μ. In similar fashion, the expressions in eq-(1.3) furnishing the $su(4)$ generators in terms of the $Cl(3,1,R)$ algebra ones, yields the corresponding relations among the other fields ${}^{(2)}A^A_\mu, \; {}^{(3)}A^A_\mu, \; {}^{(4)}A^A_\mu$ and $\mathcal{G}^i_\mu, \mathcal{W}^i_{\mu L}, \mathcal{W}^i_{\mu R}$, involving the coefficients ${}^{(2)}c^A_i, \; {}^{(3)}c^A_i, \; {}^{(4)}c^A_i$, respectively.

The algebra of Grand Unified theories, related to the $SO(10), SU(5)$ and Pati-Salam group, was analyzed from a different perspective than the Clifford algebraic one presented here by [12]. It is interesting to note that the number of gauge fields associated with the $SO(10)$ group is the same as the number in $SU(4)_C \times SU(4)_L \times SU(4)_R : \; \frac{10\times 9}{2} = 3 \times 15 = 45$.

2 Fermionic Kinetic Terms and 4 Generations

In this section we shall study the fermionic kinetic terms corresponding to the $SU(4)_C \times SU(4)_L \times SU(4)_R$ multiplets and explore other possibilities based on $Cl(6,R), Cl(8,R)$ algebras, Supersymmetric Field Theories and Quaternions. In most of these cases, the models with $four$ generations are the most natural.

The 16 fermions of each generation of the Standard Model can be assembled into the entries of a 4×4 matrix representation of the $Cl(4)$ algebra whose 16 generators are $\Gamma^A, A = 1, 2, 3, ..., 16$. The latter generators can be represented in terms of 4×4 matrices $(\Gamma^A)_{ij}$ whose indices are $i, j = 1, 2, 3, 4$. A fermion field Ψ^A_α carries double indices, A represents an internal $Cl(4)$-valued gauge index, while α represents a $Cl(3,1)$ spinor index associated with the four-dim spacetime. The ordinary 16 fermions of the first generation of the Standard Model (assuming a massive neutrino) can can be assembled into the entries of a 4×4 matrix as [1], [2]

$$\sum_A \Psi^A_\alpha \ (\Gamma_A)_{ij} \ \equiv \ \begin{pmatrix} \nu_{eL} & u_{rL} & u_{bL} & u_{gL} \\ e_L & d_{rL} & d_{bL} & d_{gL} \\ e^+_R & \bar{d}^{\bar{r}}_R & \bar{d}^{\bar{b}}_R & \bar{d}^{\bar{g}}_R \\ \bar{\nu}_{eR} & \bar{u}^{\bar{r}}_R & \bar{u}^{\bar{b}}_R & \bar{u}^{\bar{g}}_R \end{pmatrix} \tag{2.1}$$

where both left and right handed particles are lumped together into a single matrix. There are many ways to *extend* the Standard Model. One way is by doubling the number of fermions in such a way that left and right handed particles are *not* lumped together inside the same matrix. The number of 16 fermions can be doubled to 32 such that the the left handed sector can be assembled as

$$\sum_A \Psi^A_{\alpha,L} \ (\Gamma_A)_{ij} \ \equiv \ \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \\ e^+ & \bar{d}^{\bar{r}} & \bar{d}^{\bar{b}} & \bar{d}^{\bar{g}} \\ \bar{\nu}_e & \bar{u}^{\bar{r}} & \bar{u}^{\bar{b}} & \bar{u}^{\bar{g}} \end{pmatrix}_L \tag{2.2}$$

and the right handed sector as

$$\sum_A \Psi^A_{\alpha,R} \ (\Gamma_A)_{ij} \ \equiv \ \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \\ e^+ & \bar{d}^{\bar{r}} & \bar{d}^{\bar{b}} & \bar{d}^{\bar{g}} \\ \bar{\nu}_e & \bar{u}^{\bar{r}} & \bar{u}^{\bar{b}} & \bar{u}^{\bar{g}} \end{pmatrix}_R \tag{2.3}$$

We have arranged the entries of the above 4×4 matrix in order to accommodate the chiral fermions into representations of the Pati-Salam (PS) $SU(4) \times SU(2)_L \times SU(2)_R$ group [15] such that the above 4×4 matrix entries admit the following $SU(4) \times SU(2)_L \times SU(2)_R$ decomposition. The left-handed fermions are displayed in the following representation of the Pati-Salam group

$$(\mathbf{4}, \mathbf{2}, \mathbf{1}) \ : \ \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \end{pmatrix}_L \tag{2.4}$$

Since the right-handed antiparticles feel the left-handed weak $SU(2)_L$ force [12] one has

$$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}) \ : \ \begin{pmatrix} e^+ & \bar{d}^{\bar{r}} & \bar{d}^{\bar{b}} & \bar{d}^{\bar{g}} \\ \bar{\nu}_e & \bar{u}^{\bar{r}} & \bar{u}^{\bar{b}} & \bar{u}^{\bar{g}} \end{pmatrix}_R \tag{2.5}$$

Since the left-handed antiparticles feel the right-handed weak $SU(2)_R$ force [12] one has

$$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \ : \ \begin{pmatrix} e^+ & \bar{d}^{\bar{r}} & \bar{d}^{\bar{b}} & \bar{d}^{\bar{g}} \\ \bar{\nu}_e & \bar{u}^{\bar{r}} & \bar{u}^{\bar{b}} & \bar{u}^{\bar{g}} \end{pmatrix}_L \tag{2.6}$$

and, finally, the right-handed fermions are displayed in the representation

$$(\mathbf{4}, \mathbf{1}, \mathbf{2}) \ : \ \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \end{pmatrix}_R \tag{2.7}$$

where we have omitted the spacetime spinorial indices $\alpha = 1,2,3,4$ in each one of the entries of the above matrices. In particular, e, ν_e denote the electron and its neutrino. The subscripts r, b, g denote the red, blue, green color of the up and down quarks, u, d. The subscripts $\bar{r}, \bar{b}, \bar{g}$ denote the anti-red, anti-blue, anti-green color of the up and down antiquarks, $\bar{u}, \bar{d}$. The anti-particles are denoted by $\bar{e}, \bar{\nu}_e, \bar{u}, \bar{d}$. The remaining chiral fermions (Weyl spinors) of the second and third generation have identical decomposition as the one displayed in eqs-(2.4-2.7). One simply replaces e for the muon and tau μ, τ particles; the neutrino ν_e for the neutrinos ν_μ, ν_τ, and the u, d quarks for the charm, strange c, s and top, bottom t, b quarks, respectively.

The uphsot of having the $Cl(4)$-algebraic description of the 16 left/right handed fermions (Weyl spinors) in eqs-(2.2-2.3) is that it is consistent with the $SU(4)$ color symmetry (force) of the Pati-Salam model [15]. The leptons are seen as the carriers of the white "fourth" color. Furthermore, one is confined to the observed four-spacetime dimensions.

Another way to *extend* the Standard Model is by adding a $fourth$ generation of fermions such that now the fermionic arrangement of entries of the 4×4 matrices are assigned into representations of $SU(4)_C \times SU(4)_L \times SU(4)_R$, instead of the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$. Henceforth, we shall have one multiplet Ψ_{1L} (omitting spinorial indices) which belongs to the $(\mathbf{4}, \mathbf{4}, \mathbf{1})$ representation given by

$$(\mathbf{4}, \mathbf{4}, \mathbf{1}) \ : \ \mathbf{\Psi_{1L}} \ \equiv \ \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \\ \nu_\mu & c_r & c_b & c_g \\ \mu & s_r & s_b & s_g \end{pmatrix}_{\mathbf{L}} \tag{2.8}$$

The antiparticles (charge conjugation is denoted now by the superscript c) of the above multiplet $(\mathbf{\Psi_{1L}})$ belong to the right handed antiparticle multiplet $(\mathbf{\Psi^c_{1R}})$ and which feels the left-handed $SU(4)_L$ force. Therefore one has

$$(\bar{\mathbf{4}}, \mathbf{4}, \mathbf{1}) \ : \ \mathbf{\Psi^c_{1R}} \ \equiv \ \begin{pmatrix} \nu^c_e & u^c_r & u^c_b & u^c_g \\ e^c & d^c_r & d^c_b & d^c_g \\ \nu^c_\mu & c^c_r & c^c_b & c^c_g \\ \mu^c & s^c_r & s^c_b & s^c_g \end{pmatrix}_{\mathbf{R}} \tag{2.9}$$

One has another multiplet $(\mathbf{\Psi_{2L}})$ which also belongs to the $(\mathbf{4}, \mathbf{4}, \mathbf{1})$ representation, and containing the third and fourth generation of fermions given by

$$(\mathbf{4}, \mathbf{4}, \mathbf{1}) \ : \ \mathbf{\Psi_{2L}} \ \equiv \ \begin{pmatrix} \nu_\tau & t_r & t_b & t_g \\ \tau & b_r & b_b & b_g \\ \nu_E & T_r & T_b & T_g \\ E & B_r & B_b & B_g \end{pmatrix}_{\mathbf{L}} \tag{2.10}$$

where the new leptons are E, ν_E (its neutrino) and the new quarks are T, B in three colors, r = red, b = blue, g = green.

The antiparticles (charge conjugation) of the above multiplet ($\mathbf{\Psi_{2L}}$) belong to the right handed antiparticle multiplet ($\mathbf{\Psi^c_{2R}}$) and which feels the left-handed $SU(4)_L$ force. Therefore one has

$$(\bar{\mathbf{4}}, \mathbf{4}, \mathbf{1}) \ : \ \mathbf{\Psi^c_{2R}} \ \equiv \ \begin{pmatrix} \nu^c_\tau & t^c_r & t^c_b & t^c_g \\ \tau^c & b^c_r & b^c_b & b^c_g \\ \nu^c_E & T^c_r & T^c_b & T^c_g \\ E^c & B^c_r & B^c_b & B^c_g \end{pmatrix}_{\mathbf{R}} \tag{2.11}$$

If one wishes one could add mirror fermions, although it is *not* necessary. The mirror fermions multiplets are obtained under the exchange $\mathbf{L} \leftrightarrow \mathbf{R}$ above so that now ($\mathbf{\Psi_{1R}}$) and ($\mathbf{\Psi_{2R}}$) belong to the $(\mathbf{4}, \mathbf{1}, \mathbf{4})$ representation, and ($\mathbf{\Psi^c_{1L}}$) and ($\mathbf{\Psi^c_{2L}}$) belong to the $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{4})$ representation. This would amount to having 4 generations and doubling the number of fermions in each generation if, and only if, one were to introduce mirror fermions.

The fermionic kinetic terms (omitting spacetime spinorial indices) associated with the $(\mathbf{4}, \mathbf{4}, \mathbf{1})$ representation are of the form

$$\mathcal{L}_m \ = \ \sum_{i=1}^{i=2} \ Trace \ \Big(\ \bar{\mathbf{\Psi}}_{\mathbf{iL}} \ \gamma^\mu \ (\ \partial_\mu \mathbf{\Psi_{iL}} \ + i g_2 \ \mathcal{G}_\mu \ \tilde{\mathbf{\Psi}}_{\mathbf{iL}} \ + i g_3 \ \mathcal{W}_{\mu L} \ \mathbf{\Psi_{iL}}) \ \Big) \tag{2.12}$$

the kinetic terms associated with the $(\bar{\mathbf{4}}, \mathbf{4}, \mathbf{1})$ representation are obtained by replacing $\mathbf{\Psi_{iL}} \rightarrow \mathbf{\Psi^c_{iR}}$ in eq-(2.12). g_2, g_3 are the respective coupling constants of $SU(4)_C$ and $SU(4)_L$. One may notice that the $SU(4)_C$ color group above acts on the *transpose matrix* $\tilde{\mathbf{\Psi}}_{\mathbf{i}}$ [2]. The Dirac adjoint of the above matrices $\bar{\mathbf{\Psi}}$ is obtained by taking the Dirac adjoint of each one of the entries of the 4×4 matrices $\mathbf{\Psi}$ (each entry represents a Weyl/chiral spinor) .

The 4×4 matrix representation of the $SU(4)_C$-Lie algebra valued gauge field is $\mathcal{G}_\mu = (G^a_\mu T_{aC})^{mn}$. The matrix representation of the $SU(4)_L$-Lie algebra valued gauge field is $\mathcal{W}_{\mu L} = (W^a_{\mu L} T_{aL})^{mn}$. The range of the matrices indices is $m, n = 1, 2, \ldots, 4$. T_{aC} $(a = 1, 2, \ldots, 15)$ are the 15 generators of $SU(4)_C$; and T_{aL} $(a = 1, 2, \ldots, 15)$ are the 15 generators of $SU(4)_L$. One has summed over the two multiplets $i = 1, 2$ in eq-(2.12) (encoding the four generations) and the trace is taken over the 4×4 matrix obtained from the products of 4×4 matrices in eq-(2.12).

The fermionic kinetic terms associated with the $(\mathbf{4}, \mathbf{1}, \mathbf{4})$ representation are of the form

$$\mathcal{L}_m \ = \ \sum_{i=1}^{i=2} \ Trace \ \Big(\ \bar{\mathbf{\Psi}}_{\mathbf{iR}} \ \gamma^\mu \ (\ \partial_\mu \mathbf{\Psi_{iR}} \ + i g_2 \ \mathcal{G}_\mu \ \tilde{\mathbf{\Psi}}_{\mathbf{iR}} \ + i g_4 \ \mathcal{W}_{\mu R} \ \mathbf{\Psi_{iR}} \ ) \ \Big) \tag{2.13}$$

the kinetic terms associated with the $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{4})$ representation are obtained by replacing $\mathbf{\Psi_{iR}} \rightarrow \mathbf{\Psi^c_{iL}}$ in eq-(2.13). g_4 is the $SU(4)_R$ coupling constant. The

[2] Instead of *reversing* the order of the matrix product $\mathbf{\Psi_{iL}} \mathcal{G}_\mu$ as in [14].

matrix representation of the $SU(4)_R$-Lie algebra valued gauge field is $\mathcal{W}_{\mu R} = (W^a_{\mu R} T_{aR})^{mn}$. T_{aR} $(a = 1, 2, \ldots, 15)$ are the 15 generators of $SU(4)_R$.

There other ways to incorporate 4 generations. Within the context of Clifford algebras, and supersymmetric field theories, the 4 generations of 16 fermions $(4 \times 16 = 64)$ fit into the 64-dimensions of the adjoint representation of the $Cl(6, R)$ algebra if one identifies the massless fermions with the *gauginos* ψ^A of a $Cl(6)$-valued vector *supermultiplet* comprised of the gauge fields $\mathcal{A}^A_\mu$, the gauginos ψ^A and auxiliary scalar fields D^A in the Wess-Zumino gauge [13]. The range of indices is $A = 1, 2, \ldots, 64$. In this case the fermionic kinetic terms encoding the 4 generations is

$$\mathcal{L}_m \;=\; \bar{\Psi}^A_\alpha \; \gamma^\mu_{\alpha\beta} \; (\; \delta_{AC} \; \partial_\mu \; + ig \; f_{ABC} \; \mathcal{A}^B_\mu \;) \; \Psi^C_\beta . \tag{2.14}$$

where the indices $A, B, C = 1, 2, 3, \ldots\ldots, 64$ run over the $2^6 = 64$-dimensions of the $Cl(6)$ algebra and f_{ABC} denote the structure constants of the $Cl(6)$ gauge algebra.

The 4 generations and Quaternions also fit into the $Cl(6, R)$ algebra picture. It is known that the $Cl(6, R)$ algebra is isomorphic to the matrix algebra $M(4, \mathbf{H})$ of Quaternionic-valued 4×4 matrices (whose real dimension is $4 \times 16 = 64$). Thus the 4 generations of 16 fermions can be collectively assembled into the entries of the Quaternionic valued 4×4 matrices $\mathbf{\Psi}$. The Lagrangian for the fermionic kinetic terms is in this case given by

$$\mathcal{L}_m \;=\; Trace \; \Big(\; \bar{\mathbf{\Psi}} \; \gamma^\mu \; (\; \partial_\mu \Psi \; + ig \; \mathcal{A}_\mu \; \mathbf{\Psi}) \; \Big) \tag{2.15}$$

where $\mathcal{A}_\mu$ is a matrix algebra $M(4, \mathbf{H})$ valued gauge field and $\mathbf{\Psi}$ comprises the fermionic entries of the 4×4 Quaternionic-valued matrices. A Lagrangian for Gravity and a $SU(3) \times SU(2) \times U(1)$ Yang-Mills in $4D$ can be obtained from a $8D$ Quaternionic Gravitational theory after a Kaluza-Klein compactification on an internal four-dimensional space CP^2 [16], [2]. Thus, we have shown that Quaternions can play an important role in building unification models as advanced by [11].

Since the fermions of the Standard Model belong to the *fundamental* representation of the gauge groups, instead of identifying the fermions with the *gauginos* of a $Cl(6, R)$-valued vector superfield, one may instead recur to a $Cl(8, R)$ algebra such that the 16 fermions of each generation are accommodated into the 16 entries of a *column* matrix ψ_m, $m = 1, 2, \ldots, 16$, and the fermionic kinetic terms, summed over n_f generations, are of the form

$$\sum_{i=1}^{n_f} \bar{\psi}_{mi} \, \gamma^\mu \; \Big(\; \partial_\mu \psi_{mi} \; + ig \; (\mathcal{A}^A_\mu \; \Gamma_A)^{mn} \; \psi_{ni} \; \Big) , \;\; m, n = 1, 2, \ldots, 16, \;\; i = 1, 2, \ldots, n_f \tag{2.16}$$

the index A ranges over the 256 generators of the 2^8-dimensional $Cl(8)$ algebra, $A = 1, 2, \ldots, 256$ and the $(\Gamma_A)^{mn}$'s are 16×16 square matrices corresponding to the fundamental representation of $Cl(8)$. A $Cl(8)$ algebraic approach to unification in $D = 8$ has been advanced by Smith [10]. Its relation to E_8 has also been studied by Smith [10] and analyzed and reviewed in detail by us in [2]. An explicit realization of E_8 in terms of $Cl(16) = Cl(8) \otimes Cl(8)$ generators can be found in [17].

It remains now to include gravity into the picture. The covariant derivative of a Lorentz spinor is defined via the introduction of the spin connection ω_μ^{ab} and given in terms of the tangent space Clifford algebra generators as follows

$$\nabla_\mu \psi^\alpha \ = \ \partial_\mu \psi^\alpha \ + \frac{1}{2} \ \omega_\mu^{ab} \gamma_{ab}^{\alpha\beta} \ \psi_\beta, \quad \gamma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \tag{2.17}$$

The fermionic kinetic terms are $\bar{\psi}\gamma^\mu \nabla_\mu \psi$, and the spacetime Clifford algebra generators γ^μ are given in terms of the tangent spacetime ones γ^a by $\gamma^\mu = e_a^\mu \gamma^a$, with e_a^μ being the inverse vielbein field. The inclusion of conformal gravity requires using the $SU(2,2)$ covariant derivatives acting on the spinor fields. As explained in the derivation of eqs-(1.9, 1.10), after expressing the conformal algebra generators T_i in terms of the Clifford algebra ones as displayed in eqs-(1.5), it allows to generalize the expression of the fermionic kinetic terms in eq-(2.17) from the Lorentz group $SO(3,1)$ to the Conformal group $SO(4,2)$ case and given by (omitting spacetime spinorial indices)

$$\bar{\psi} \ e_a^\mu \ \gamma^a \ (\partial_\mu \psi \ + \ \Omega_\mu^i \ T_i \ \psi) \ = \ \bar{\psi} \ e_a^\mu \ \gamma^a \ (\partial_\mu \psi \ + \ {}^{(1)}A_\mu^A \ {}_{(1)}\Gamma_A \ \psi) \tag{2.18}$$

where the ${}_{(1)}\Gamma_A$ span the generators of the $Cl(3,1)$ algebra (excluding the identity element) living inside $Cl(5,C)$.

If one wishes to implement full covariance under $SU(2,2)$ and $SU(4)_C \times SU(4)_L \times SU(4)_R$ it requires to use the derivative operator in (2.18)

$$\mathcal{D}_\mu \ = \ \nabla_\mu \ + \ \mathbf{A}_\mu \ = \ \partial_\mu \ + \ \Omega_\mu \ + \ \mathcal{G}_\mu \ + \ \mathcal{W}_{\mu L} \ + \ \mathcal{W}_{\mu R} \tag{2.19}$$

Eq-(2.19) can be rewritten in terms of the $Cl(5,C)$ algebra generators as explained in the derivation of eqs-(1.9, 1.10). The $SU(2,2)$-valued gauge field $\Omega_\mu = \Omega_\mu^i \ T_i = ({}^{(1)}A_\mu^A \ {}_{(1)}\Gamma_A)^{\alpha\beta}$ acts on each one of the 16 spinorial entries of the 4×4 matrices $\mathbf{\Psi}$, explicitly displayed in eqs-(2.8-2.11), as shown in eq-(2.18). The kinetic terms for the Lie-algebra valued gauge field strengths $\mathbf{F}_{\mu\nu}$ corresponding to the group $SU(2,2) \times SU(4)_C \times SU(4)_L \times SU(4)_R$, after absorbing the four coupling constants into the definition of the gauge fields, are given as usual by

$$\mathcal{L}_F \ = \ - \ \sum_{i=1}^{4} \ \frac{1}{4g_i^2} \ {}_{(i)}F_{\mu\nu J} \ {}_{(i)}F^{\mu\nu J}, \quad J = 1, 2, \ldots, 15 \tag{2.20}$$

at the grand unification point all the values of the gauge couplings coincide $g_1 = g_2 = g_3 = g_4$.

The Pati-Salam (PS) $SU(4)_C \times SU(2)_L \times SU(2)_R$ group arises very naturally from the symmetry breaking of $SU(4)_C \times SU(4)_L \times SU(4)_R$. It contains the Standard Model Group, which in turn, breaks down to the ordinary Maxwell

Electro-Magnetic (EM) $U(1)_{EM}$ and color (QCD) group $SU(3)_C$ after the following chain of symmetry breaking patterns

$$SU(2)_L \times SU(2)_R \times SU(4)_C \rightarrow SU(2)_L \times U(1)_R \times U(1)_{B-L} \times SU(3)_C \rightarrow$$
$$SU(2)_L \times U(1)_Y \times SU(3)_C \rightarrow U(1)_{EM} \times SU(3)_C. \tag{2.21}$$

where $B-L$ denotes the Baryon minus Lepton number charge; Y = hypercharge and the Maxwell EM charge is $Q = I_3 + (Y/2)$ where I_3 is the third component of the $SU(2)_L$ isospin.

The symmetry breaking scheme of $SU(4)_C \times SU(4)_F$ into $SU(3)_C \times U(1)_{EM}$ is quite elaborate [14]. The mass matrix for the gauge bosons (after symmetry breaking) was calculated by looking at the couplings of the Higgs scalars and gauge fields. The Yukawa couplings of fermions and Higgs scalars lead to the fermionic masses at tree level when the Higgs scalars acquire non-vanishing vacuum expectation values. The symmetry breaking scheme of $SU(4)_C \times SU(4)_L \times SU(4)_R$ into $SU(3)_C \times U(1)_{EM}$ is even more elaborate than the one of [14] and is outside the scope of this work.

To conclude, the Clifford Algebraic Unification of Conformal Gravity with an Extended Standard Model deals mainly with models of $four$ generations of fermions. Mirror fermions can be incorporated as well. Whether these mirror fermions are dark matter candidates is an open question. As mentioned earlier in the previous section, there are also residual $U(1)$ groups within this Clifford group unification scheme that should play an important in Cosmology in connection to dark matter particles coupled to gravity via a Bimetric extension of General Relativity [9], [7]. Our findings based on Clifford algebraic extensions of the Standard Model (SM) should be contrasted with the different background models of extended theories of gravity, which are minimally coupled to the SM fields, to explain the possibility of the genesis of dark matter without affecting the SM particle sector [18].

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References

[1] C. Castro, "A Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$" Advances in Applied Clifford Algebras **22**, no. 1 (2012), page 1-21.

[2] C. Castro, "A Clifford Algebra Based Grand Unification Program of Gravity and the Standard Model : A Review Study" Can. J. Phys. **92** (2014) 12, 1501-1527.

[3] https://en.wikipedia.org/wiki/Cliffordalgebra

[4] I. R. Porteous, *Clifford algebras and Classical Groups* (Cambridge Univ. Press, 1995).

[5] R. Gilmore, *Lie Groups, Lie Algebras and some of their Applications* (John Wiley and Sons Inc, New York, 1974).

[6] S. L. Glashow, "Trinification Of All Elementary Particle Forces", Print-84-0577 (BOSTON). A. Rizov, Bulg. J. Phys. **8** (1981) 461.

[7] A. Salomon, "Cosmology Beyond Einstein" arXiv : 1508.06859.

[8] A. H. Chamseddine and V. Mukhanov, Mimetic Dark Matter, JHEP **1311** (2013) 135.

Katrin Hammera and Alexander Vikmanb, "Many Faces of Mimetic Gravity" arXiv : 1512.09118

[9] L. Bernard and L. Blanchet (2014), "Phenomenology of Dark Matter via a Bimetric Extension of General Relativity" arXiv:1410.7708 [astro-ph].

[10] Frank (Tony) Smith, *The Physics of* E_8 *and* $Cl(16) = Cl(8) \otimes Cl(8)$ www.tony5m17h.net/E8physicsbook.pdf (Carterville, Georgia, June 2008, 367 pages).

Frank (Tony) Smith, "Cl(16) E_8 Lagrangian AQFT" vixra.org/1405.0030

[11] G. Dixon, *Division Algebras, Lattices, Physics, Windmill Tiling* (2010).

G. Dixon, *Division Algebras : Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Pjhysics* (Kluwer, 1994).

[12] J. Baez and J. Huerta, " The Algebra of Grand Unified Theories" arXiv : 0904.1556.

[13] S. Weinberg, *The Quantum Theory of Fields III* (Cambridge University Press 2000)

[14] S. Rajpot and M. Singer, "$SU(3) \times U(1)$ low-energy symmetry group within the unifying $SU(4)_F \times SU(4)_C$ gauge group" J. Phys. **G** : Nuc. Phys., **vol 5**, no. 7 (1979) 871.

[15] J. Pati and A. Salam, Phys. Rev. Lett **31** (1973) 661.

J. Pati and A. Salam, Phys. Rev. **D 8** (1973) 1240.

[16] C. Castro, Int. J. Theor. Phys, **Vol 51**, Issue 10 (2012) 3318.

[17] K. Koepsell, H. Nicolai and H. Samtleben, " An Exceptional Geometry of $d = 11$ Supergravity" hep-th/0006034.

[18] S. Choudhury, M. Sen and S. Sadhukhan, "Can Dark Matter be an artifact of extended theories of gravity?" arXiv: 1512.08176.