Momentum And Group Velocity As A Function Of The Wavefunction

Assuming a time independent Schrödinger equation, this paper shows the formulas for the wavelength, momentum and group velocity of a particle as a function of its wavefunction.

by R. A. Frino
Electronics Engineer
Degree from the National University of Mar del Plata
rodolfo_frino@yahoo.com.ar
March-December 2015

Keywords: *wavelength, momentum, group velocity, Schrödinger equation, wavefunction.*

1. Group Velocity as a Function of the Momentum and the Mass of the Particle

According to the definition of momentum of a particle of mass, *m*

$$p = m v_g \tag{1.1}$$

Then, the mass of the particle can be expressed as follows

$$m = \frac{p}{v_g} \tag{1.2}$$

This result can also be obtained from the definition of group velocity [2] for a non-relativistic particle. This definition is

$$v_g = \frac{dK}{dp} \tag{1.3}$$

the kinetic energy of a non-relativistic particle is given by

$$K = \frac{1}{2} m v_g^2 \tag{1.4}$$

Multiplying and dividing by the mass, *m*, of the particle, we get

$$K = \frac{1}{2m} p^2 \tag{1.5}$$

Differentiating with respect to the momentum, *p*, of the particle we get

$$v_g = \frac{dK}{dp} = \frac{1}{2\mathrm{m}} 2p \frac{dp}{dp} = \frac{p}{m} \tag{1.6}$$

or

$$m = \frac{p}{v_g} \tag{1.7}$$

Which is the same result given by equation (1.2).

2. Momentum, Group Velocity and the Wavefunction

Let's start with the time-independent Schrödinger equation [1]

(See note 1)

$$\nabla^2 \Psi + \frac{8\pi^2 m}{h^2} (E - U) \Psi = 0 \tag{2.1}$$

Because the total energy, *E*, of the particle is the kinetic energy, *K*, plus the potential energy, *U*, we may write

$$E = K + U \tag{2.2}$$

Hence

$$K = E - U \tag{2.3}$$

The last equation allow us to write the time-independent Schrödinger equation (equation 3.1) as follows

$$\nabla^2 \Psi + \frac{8\pi^2 m}{h^2} K \Psi = 0 \tag{2.4}$$

Now I shall substitute the kinetic energy with the second side of equation (2.5). This substitution yields

$$\nabla^2 \Psi + \frac{8\pi^2 m}{h^2} \frac{1}{2m} p^2 \Psi = 0 \tag{2.5}$$

And after some algebra we get

1. This is the original form of the Schrödinger equation published in reference [1] where Schrödinger used div grad instead of ∇^2 and *V* instead of *U* to denote the potential energy.

$$\nabla^2 \Psi + \frac{4\pi^2}{h^2} p^2 \Psi = 0 \tag{2.6}$$

And introducing the reduced Planck's constant $\hbar = h/2\pi$ we get

$$\nabla^2 \Psi + \frac{p^2}{\hbar^2} \Psi = 0 \tag{2.7}$$

Solving for p^2

$$p^2 = -\hbar^2 \frac{\nabla^2 \Psi}{\Psi} \tag{2.8}$$

And solving for p

$$p = i\hbar \sqrt{\frac{\nabla^2 \Psi}{\Psi}} \tag{2.9}$$

The above equation may be written as

Momentum of the particle as a function of the wavefunction

$$p = i \frac{\hbar}{\sqrt{\frac{\Psi}{\nabla^2 \Psi}}} \tag{2.10}$$

This is the equation of the momentum of the particle as a function of the wavefunction, Ψ. It is worthwhile to emphasize that the momentum of the particle will be a real positive number if and only if the quantity $\sqrt{\frac{\Psi}{\nabla^2 \Psi}}$ is a negative imaginary number (such as: - $i\,N$).

From equation (2.9) we may write the formula for the group velocity as follows

$$v_g = i \frac{\hbar}{m} \sqrt{\frac{\nabla^2 \Psi}{\Psi}} \tag{2.11}$$

Or, equivalently

Group velocity of the particle as a function of the wavefunction

$$v_g = \frac{i}{m} \frac{\hbar}{\sqrt{\frac{\Psi}{\nabla^2 \Psi}}} \tag{2.12}$$

This is the equation of the group velocity of the particle as a function of the wavefunction, Ψ. Here again, it is important to observe that the group velocity will be a real positive number if and only if the quantity $\sqrt{\frac{\Psi}{\nabla^2 \Psi}}$ is a negative imaginary number.

The quantity $\sqrt{\frac{\Psi}{\nabla^2\Psi}}$ must have the dimensions of a distance and therefore it should be a wavelength. I shall denote this wavelength with λ_Ψ

Wavelength of the particle as a function of the wavefunction

$$\lambda_\Psi \equiv \sqrt{\frac{\Psi}{\nabla^2\Psi}} \tag{2.13}$$

This is the wavelength of the wavefunction. Because the wavefunction, Ψ, is a function of the space coordinates: *x*, *y* and *z*, This is

$$\Psi = \Psi(x, y, z) \tag{2.14}$$

the wavelength, λ_Ψ, will also be a function of the space coordinates. Therefore we may write

$$\lambda_\Psi = \lambda_\Psi(x, y, z) \tag{2.15}$$

Finally, we may rewrite equations (2.10) and (2.12) in terms of the variable wavelength of the wavefunction. Thus we get the following two equations

$$p = i\frac{\hbar}{\lambda_\Psi} \tag{2.16}$$

and

$$v_g = i\frac{\hbar}{m\lambda_\Psi} \tag{2.17}$$

3. Summary

Table 1 summarises the above results

VARIABLE	FORMULA	FORMULA
Wavelength	$\lambda_\Psi \equiv \sqrt{\frac{\Psi}{\nabla^2\Psi}}$	(λ_Ψ)
Momentum	$p = i\frac{\hbar}{\sqrt{\frac{\Psi}{\nabla^2\Psi}}}$	$p = i\frac{\hbar}{\lambda_\Psi}$
Group velocity	$v_g = \frac{i}{m}\frac{\hbar}{\sqrt{\frac{\Psi}{\nabla^2\Psi}}}$	$v_g = i\frac{\hbar}{m\lambda_\Psi}$

TABLE 1: *Wavelength, momentum and group velocity formulas as a function of the wavefunction of the particle.*

4. Conclusions

In general, the wavelength λ_{ψ} of the wavefunction Ψ is not constant. It depends on the wavefunction which is a function of the space coordinates *x*, *y* and *z* (we have assumed a time independent Schrödinger equation). Therefore the wavelength of the wavefunction will be, in general, a variable quantity.

REFERENCES

[1] E. Schrödinger, *Quantisation as a Problem of Proper Values; Part I-IV*, Four Lectures on Wave Mechanics, Glasgow, (1928). Also published by S. W. Hawking in *The Dreams that Stuff is Made of.* Book published by Running Press in 2011.

[2] R. A. Frino, *Phase Velocity and Group Velocity for Beginners*, viXra.org: viXra 1501.0170, (2015).