# The logical self-reference inside the Fourier transform

**Steve Faulkner**

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**Abstract** I show that, in general, the Fourier transform is necessarily self-referent and logically circular.

**Keywords** self-reference, logical circularity, mathematical logic, Fourier transform, vector space, orthogonality, orthogonal, unitarity, unitary, imaginary unit, foundations of quantum theory, quantum mechanics, quantum indeterminacy, quantum information, prepared state, pure state, mixed state, wave packet, scalar product, tensor product.

## 1 Introduction

An orthogonal vector space (or tensor space ) can be thought of as a composite of information – consisting of – information that comprises a general, arbitrary vector space, plus additional information that renders that space orthogonal. More formally we might think of axioms imposing rules for vector spaces with additional axioms imposing orthogonality. However, the information of orthogonality need not originate in axioms; it can originate through *self-reference* or *logical circularity* [2]. This has logical implications for vector spaces used as isomorphic structures in quantum, in representing mixed states.

Before this self-reference may occur, a dual-pair of vector spaces forms into a closed system. Information of orthogonality does not enter from outside, but instead arises through the exchange of information, passed between the vector spaces in the dual-pair. This exchange is possible because the process is *logically independent* of axioms. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. Hence, no information in the system opposes the exchange. Specifically, there is no *syntactic* information in the system axioms (axioms of Linear Algebra or Elementary Algebra) that causes or prevents (implies or contradicts) the information transfer.

Logical independence is proved by the fact that the self-reference implies existence of the imaginary unit and that number's logical independence in Elementary Algebra is well-known [1].

The exchange of information occurs through a cross-substitution of this form:

$$\forall \mathsf{B} \exists \mathsf{A} \mid \mathsf{A} = f(\mathsf{B}) \qquad \text{AND} \qquad \forall \mathsf{A} \exists \mathsf{B} \mid \mathsf{B} = F(\mathsf{A}) \,. \tag{1}$$

Viewed as a single proposition, (1) asserts the cyclic flow of information where the outcome of $\mathsf{A}$ has dependency on $\mathsf{B}$ and the outcome of $\mathsf{B}$ has dependency on $\mathsf{A}$. It implies the single system:

$$\exists \mathsf{A} \exists \mathsf{B} \left| \begin{array}{l} \mathsf{A} = f(F(\mathsf{A})) \\ \\ \mathsf{B} = F(f(\mathsf{B})) \end{array} \right. \tag{2}$$

Steve Faulkner
*Logical Independence in Physics. Information flow and self-reference in Arithmetic.*
E-mail: StevieFaulkner@googlemail.com

This should be read as: there exists an $\mathsf{A}$ and there exists a $\mathsf{B}$ such that $\mathsf{A} = f(F(\mathsf{A}))$ and $\mathsf{B} = F(f(\mathsf{B}))$. It is important to note that both $\mathsf{A}$ and $\mathsf{B}$ are bound by *existential quantifiers*; there is no universal quantifier in this formula. That means that (2) has only *accidental* solutions, since there is no guarantee of any simultaneous, coinciding solutions. For that reason, this is not the kind of self-reference found in feed-back systems used in the engineering world. This proposition's condition might not be met at all. In fact, whether (2) is true depends on whether the domains of $f$ and $F$ are capable of orthogonality. If that condition can be met then the self-reference goes ahead, unopposed.

## 2 The Fourier transform

I use the notation $\int_{\mathsf{k}} f(\mathsf{x}) = \int_{-\infty}^{+\infty} f(\mathsf{x})\,\mathrm{d}\mathsf{x}$.

Consider the following pair of formulae.

$$\forall\eta\forall x\exists a\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} [\exp(+\eta x\mathsf{k})\, a(\mathsf{k})] \tag{3}$$

$$\forall\eta\forall k\exists b\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} [\exp(-\eta k\mathsf{x})\, b(\mathsf{x})] \tag{4}$$

These formulae contain only information deriving from Axioms of Elementary Algebra [See Appendix] (pending some caveats concerning limiting values and irrational numbers). In writing these, the san-serif notated $\mathsf{k}$ and $\mathsf{x}$ are the dummy (bound) variables over the integrals. The italicised variables $\eta, k, x, a, b$ are all bound variables over the existential quantifier $\exists$ and universal quantifier $\forall$. I have laid out the ordering of variables to mirror the convention of repeated dummy indices used in summations of discrete quantities.

Note that these formulae do not assert equality, they assert existence. Also note that the integrals exist, and the pair of propositions are true, only if $a$ and $b$ are functions restricted to the Banach space $L^1$ (at least).

I now explore the possibility of (3) and (4) accepting information, circularly, from one another, through a mechanism where $a(\mathsf{k})$ feeds off $\Phi(k)$ and $b(\mathsf{x})$ feeds off $\Psi(x)$. There is no *cause* implying this self-reference; the idea is that it is *prevented* by nothing. Indeed, the fact of this self-reference is information, logically independent of all algebraic rules in operation.

To proceed, the strategy followed will be to posit a hypothesis that such self-reference does occur, then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

**Hypothesised coincidence:**

$$\forall a\exists\Phi \mid \quad a = \Phi; \tag{5}$$

$$\forall b\exists\Psi \mid \quad b = \Psi. \tag{6}$$

When these assumptions are substituted into (3) and (4) we get:

$$\forall\eta\forall x\exists\Phi\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} [\exp(+\eta x\mathsf{k})\, \Phi(\mathsf{k})] \tag{7}$$

$$\forall\eta\forall k\exists\Psi\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} [\exp(-\eta k\mathsf{x})\, \Psi(\mathsf{x})] \tag{8}$$

**Simultaneous propositions**
Illustrating. Taking the two propositions:

$$\forall x : \mathsf{y} = \mathsf{a}x + \mathsf{b}$$
$$\forall x : \mathsf{y} = \mathsf{c}x + \mathsf{d}$$

If these are to be solved simultaneously, the *repeated* $\forall x$ must lost, with instances of $x$ from each formulae, being particularised first. Their joint solution then:

$$\mathsf{ax} + \mathsf{b} = \mathsf{cx} + \mathsf{d}$$

where $\mathsf{x}$ is the *particular value* variable.

and that allows cross-substitution of $\Phi$ and $\Psi$, invoking a simultaneous pair of propositions, which together, will force particular values on $\eta$. Before the pair can be considered as simultaneous, in order to preserve validity, the repeated $\forall\eta$ quantifier must be lost, leaving the particularised (bold) $\boldsymbol{\eta}$. Substituting (8) into (7), and (7) into (8), we get:

$$\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp(+\boldsymbol{\eta} x\mathsf{k}) \int_{\mathsf{x}} [\exp(-\boldsymbol{\eta}\mathsf{k}\mathsf{x})\, \Psi(\mathsf{x})]\right] \tag{9}$$

$$\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp(-\boldsymbol{\eta} k\mathsf{x}) \int_{\mathsf{k}} [\exp(+\boldsymbol{\eta}\mathsf{x}\mathsf{k})\, \Phi(\mathsf{k})]\right] \tag{10}$$

Taking the integral signs outside and reversing their order, these tidy up to become:

$$\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{x}}\int_{\mathsf{k}} \exp[+\boldsymbol{\eta}(x - \mathsf{x})\,\mathsf{k}]\, \Psi(\mathsf{x}) \tag{11}$$

$$\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{k}}\int_{\mathsf{x}} \exp[-\boldsymbol{\eta}(k - \mathsf{k})\,\mathsf{x}]\, \Phi(\mathsf{k}) \tag{12}$$

These integrals, over the exponentials, exist only when $\boldsymbol{\eta}$ is pure imaginary. And therefore this pair of propositions is true, and – the **Hypothesised coincidence** is guaranteed – only for imaginary $\boldsymbol{\eta}$.

Up to this point, all the above is derivable under the Axioms of Elementary Algebra (pending some caveats concerning limiting values and irrational numbers). And it is well-known that collectively or otherwise, those Axioms contain no information asserting existence of the imaginary or complex scalars – yet nor do they contain any information denying existence of the imaginary or complex scalars, either [1]. Indeed, up to this point, no imaginary information exists in the system. In order to validate the pair of integrals, new information must be introduced. This information must be assumed. To properly document this assumption, the hypothesis is formally declared, thus:

**Hypothesised existence:**

$$\exists i \mid \quad i^2 = -1$$

Setting the *particular* number $\mathsf{i} = \sqrt{-1}$ and also $\boldsymbol{\eta} = \mathsf{is}$, where $\mathsf{s}$ is real or rational. I may write:

$$\forall x \exists \Psi \mid \quad \Psi(x) = \int_{\mathsf{x}} \int_{\mathsf{k}} \exp\left[+\mathsf{is}\,(x - \mathsf{x})\,\mathsf{k}\right] \Psi(\mathsf{x}) \tag{13}$$

$$\forall k \exists \Phi \mid \quad \Phi(k) = \int_{\mathsf{k}} \int_{\mathsf{x}} \exp\left[-\mathsf{is}\,(k - \mathsf{k})\,\mathsf{x}\right] \Phi(\mathsf{k}) \tag{14}$$

and in conclusion, claim that this pair of formulae are true, providing they are allowed self-referential information.

## References


1. Steve Faulkner, *Logical independence of imaginary and complex numbers in elementary algebra. [context: Theory of indeterminacy and quantum randomness]*, http://vixra.org/abs/1512.0286 (2015).
2. Elemér E Rosinger and Gusti van Zyl, *Self-referential definition of orthogonality*, arXiv:90904.0082v2 (2009).


## Appendix

**Axioms of Elementary Algebra**

| | | |
|---|---|---|
| | ADDITIVE GROUP | |
| A0 | $\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$ | CLOSURE |
| A1 | $\exists 0 \forall\alpha \mid \alpha + 0 = \alpha$ | IDENTITY 0 |
| A2 | $\forall\alpha\exists\beta \mid \alpha + \beta = 0$ | INVERSE |
| A3 | $\forall\alpha\forall\beta\forall\gamma \mid (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ | ASSOCIATIVITY |
| A4 | $\forall\alpha\forall\beta \mid \alpha + \beta = \beta + \alpha$ | COMMUTATIVITY |
| | MULTIPLICATIVE GROUP | |
| M0 | $\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta \times \gamma$ | CLOSURE |
| M1 | $\exists 1 \forall\alpha \mid \alpha \times 1 = \alpha$ | IDENTITY 1 |
| M2 | $\forall\beta\exists\alpha \mid \alpha \times \beta = 1 \wedge \beta \neq 0$ | INVERSE |
| M3 | $\forall\alpha\forall\beta\forall\gamma \mid (\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ | ASSOCIATIVITY |
| M4 | $\forall\alpha\forall\beta \mid \alpha \times \beta = \beta \times \alpha$ | COMMUTATIVITY |
| AM | $\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$ | DISTRIBUTIVITY |
| C0 | $0 \neq 1; \quad 0 \neq p, \quad p =$ any prime | CHARACTERISTIC 0 |

**Table 1** Axioms of Elementary Algebra. These are written as sentences in *first-order logic*. They comprise the *standard field axioms* with added axioms that exclude modulo arithmetic.