A short note on why the imaginary unit is inherent in physics

Steve Faulkner

23rd December 2015

Abstract I write out a proof adapted from the work of W E Baylis et al showing how existence of the square root of minus one is inherent in physical theories demanding orthogonal mathematics.

Keywords imaginary numbers, complex numbers, foundations of quantum theory, quantum physics, quantum mechanics, wave mechanics, Canonical Commutation Relation unitary, non-unitary, unitarity, elementary algebra, quantum indeterminacy, quantum randomness.

1 The imaginary-*i* implied by orthogonality

The following is a proof adapted from the work of W E Baylis, J Huschilt and Jiansu Wei [1]. It shows that the square-root of minus one arises in logical consequence of orthogonality in any vector space of dimension greater than 2.

Assume existence of a vector space with independent basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, ... and further assume orthogonality embodied in their products, thus:

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = \mathbb{0} \tag{1}$$

$$\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 = \mathbb{0} \tag{2}$$

$$\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 = \mathbb{0} \tag{3}$$

$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_3 = \mathbb{1} \tag{4}$$

where $\mathbb{0}$ and $\mathbb{1}$ are linear operators such that $\mathbf{e}_i + \mathbb{0} = \mathbf{e}_i$ and $\mathbf{e}_i\mathbb{1} = \mathbf{e}_i$.

By (1) and (4):

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 \\ \Rightarrow \mathbf{e}_1\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 \\ \Rightarrow \mathbf{e}_1\mathbf{e}_3 &= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \ . \end{aligned} \tag{5}$$

And similarly, by (2) and (4),

$$\begin{aligned} \mathbf{e}_3\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 \ . \end{aligned} \tag{6}$$

Adding (5) and (6) gives:

$$\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 = -\left(\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right) .$$

And substituting (3) results in:

$$\begin{aligned} \mathbb{0} &= \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbb{1} \\ \Rightarrow \left(\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\right)^2 &= (-1)\,\mathbb{1} \qquad (7) \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= \pm i\mathbb{1} \ . \qquad (8) \end{aligned}$$

Steve Faulkner
Logical Independence in Physics. Information flow and self-reference in Elementary Algebra.
E-mail: StevieFaulkner@googlemail.com

This means that any (ANY) vector space with three or more basis vectors, mutually orthogonal, will imply existence of the imaginary unit. This number then finds its way from the Linear Algebra into the general algebra (Elementary Algebra or School Algebra) as complex numbers and the whole complex field. With this number, there is no longer closure over the rational numbers or reals and the algebra is extended to cover the complex plane.

Introduction of $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6, \ldots$, adds nothing more of interest.

References

1. W E Baylis, J Hushilt, and Jiansu Wei, *Why i?*, American Journal of Physics **60** (1992), no. 9, 788–797.