The $2(2S+1)$– Formalism and Its Connection with Other Descriptions

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Abstract

In the framework of the Joos-Weinberg $2(2S+1)$– theory for massless particles, the dynamical invariants have been derived from the Lagrangian density which is considered to be a 4– vector. A la Majorana interpretation of the 6–component "spinors'", the field operators of $S=1$ particles, as the left– and right–circularly polarized radiation, leads us to the conserved quantities which are analogous to those obtained by Lipkin and Sudbery. The scalar Lagrangian of the Joos-Weinberg theory is shown to be equivalent to the Lagrangian of a free massless field, introduced by Hayashi. As a consequence of a new "gauge" invariance this skew-symmetric field describes physical particles with the longitudinal components only. The interaction of the spinor field with the Weinberg's $2(2S+1)$- component massless field is considered. New interpretation of the Weinberg field function is proposed.

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In the beginning of the sixties Joos [1], Weinberg [2] and Weaver, Hammer, and Good [3] developed free particle theories for arbitrary spins $S = 0, \frac{1}{2}, 1, \frac{3}{2} \cdots$ on using the Wigner's ideas [4] of construction of the quantum field theory. Following this description, the spin-one case [5]-[7] as well as the spin-$\frac{3}{2}$ case [8] have been presented. The formulas for the Hamiltonian for any spin have also been obtained [9, 10][1] The field functions in this approach form the basis of the $(S, 0) \oplus (0, S)$ representation of the Lorentz group. They are presented by the $2(2S + 1)$– component "spinor":

$$\Psi = \begin{pmatrix} \chi_\sigma \\ \\ \phi_\sigma \end{pmatrix}, \tag{1}$$

The transformation rules

$$\begin{cases} \chi_\sigma(\vec{p}) = exp\left(+\theta\hat{\vec{p}}\hat{\vec{J}}\right)\chi_\sigma(0), \\ \phi_\sigma(\vec{p}) = exp\left(-\theta\hat{\vec{p}}\hat{\vec{J}}\right)\phi_\sigma(0) \end{cases} \tag{2}$$

(with θ is the boost parameter, $tanh\theta = \frac{|\vec{p}|}{E}$, $\hat{\vec{p}} = \frac{\vec{p}}{|\vec{p}|}$, $\hat{\vec{J}}$ is the angular momentum operator) represent the generalizations of the well–known Lorentz boosts for the Dirac particle. It was noted in Ref. [2b, p. 888] that the equation for this "spinor":

$$(\gamma_{\mu\nu} p_\mu p_\nu + m^2)\Psi = 0 \tag{3}$$

can be transformed to the equations for left– and right–circularly polarized radiation when the massless $S = 1$ field being considered. The $\gamma_{\mu\nu}$ matrices are covariantly defined $6 \otimes 6$- matrices [14], $\mu, \nu = 1 \ldots 4$.

Thus, we come to the Maxwell's free-space equations (Eqs. (4.21) and (4.22) of

[1] I would also like to mention the following earlier articles concerning with this formalism [11]-[13].

Ref. [2b]):

$$\begin{cases} \vec{\nabla} \times [\vec{E} - i\vec{H}] + i(\partial/\partial t)[\vec{E} - i\vec{H}] = 0, \\ \vec{\nabla} \times [\vec{E} + i\vec{H}] - i(\partial/\partial t)[\vec{E} + i\vec{H}] = 0, \end{cases} \tag{4}$$

in vacuum provided that we consider (1) as the "bivector" [2] which can be decomposed as, e.g. [16]:

$$\begin{cases} \chi = \quad \vec{E} + i\vec{H}, \\ \phi = -\vec{E} + i\vec{H} \end{cases} \tag{5}$$

($\vec{E}$ and $\vec{H}$ are the 3-vectors). In fact, this is the formulation which is similar to [17]-[19][3].

Attempts at describing the quantized electromagnetic field in the terms of electric and magnetic field vectors $\vec{E}, \vec{H}$ (but not potential) as independent variables, or, equivalently, antisymmetric strength tensors, have been undertaken previously [21]-[23]. For example, in Ref. [23] the 4–vector Lagrangian density:

$$\mathcal{L}_\alpha =^* F^{\mu\nu}\partial_\nu F_{\mu\alpha} - F^{\mu\nu}\partial_\nu {}^*F_{\mu\alpha} - 2\,{}^*F_{\alpha\mu}j^\mu \tag{6}$$

($F_{\mu\nu}$ is the electromagnetic field tensor, ${}^*F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is its dual, j^μ is the electromagnetic current 4–vector) has been used to determine the new conserved quantities analogous to those deduced from the Lipkin tensor [24]. The remarkable feature of this formulation is that the energy-momentum conservation is associated *not* with the translational invariance but with the invariance under duality rotations.

In the present article the similar properties are shown for the Lagrangian density of

[2]See also [15, p.149] for discussion about interpretations of components of the field transforming on the $(S,0)\oplus(0,S)$ representation of the Lorentz group.

[3]See [20] for discussion about connection of $2(2S+1)$– component multispinor $\Psi = (\Psi_{\alpha_1\ldots\alpha_{2S}})$, of the massless Bargmann-Wigner equations with the antisymmetric field tensor $F_{\mu_1\nu_1\ldots\mu_S\nu_S}$.

the Joos-Weinberg theory. Following [23], the Lagrangian is chosen to be the 4–vector[4]:

$$\mathcal{L}_\alpha = -i\bar{\Psi}\gamma_{\alpha\beta}\partial_\beta\Psi + i(\partial_\beta\bar{\Psi})\gamma_{\alpha\beta}\Psi. \tag{7}$$

On using the variational principle of the stationary action the above Lagrangian leads to the Euler–Lagrange equations:

$$\begin{cases} \gamma_{\alpha\beta}\partial_\beta\Psi = 0, \\ (\partial_\beta\bar{\Psi})\gamma_{\alpha\beta} = 0, \end{cases} \tag{8}$$

which are, in fact, the Eqs.$(4, 4')$ of Ref. [17]. When $\alpha = 4$ Eqs.(8) are rewritten to Eqs. (4), whereas when $\alpha = i = 1, 2, 3$ we come to:

$$\begin{cases} \epsilon_{ikl}\frac{\partial E_l}{\partial t} + \partial_k H_i - \partial_i H_k + (\partial_j H_j)\delta_{ik} = 0, \\ \epsilon_{ikl}\frac{\partial H_l}{\partial t} + \partial_i E_k - \partial_k E_i - (\partial_j E_j)\delta_{ik} = 0, \end{cases} . \tag{9}$$

The symmetric and antisymmetric parts give us the usual four Maxwell's equations. Let us mark the coincidence of these equations with Eqs. on p.L34 of Ref. [23] as well as with the system of equations (17) in Ref. [26, p.76]:

$$\begin{cases} \frac{\partial\hat{H}}{\partial t} + \vec{\nabla}\wedge\vec{E} - (\vec{\nabla}\vec{E})\delta_{ik} = 0, \\ \frac{\partial\hat{E}}{\partial t} - \vec{\nabla}\wedge\vec{H} + (\vec{\nabla}\vec{H})\delta_{ik} = 0. \end{cases} \tag{10}$$

Here hats above E and H designate volutors.

The use of the proposed Lagrangian (7) simplifies the calculations. It gives us the opportunity to obtain dynamical invariants:

1)The energy-momentum tensor has the following form:

$$T_\alpha^{\mu\nu} = \mathcal{L}_\alpha\delta_{\mu\nu} + i\bar{\Psi}\gamma_{\alpha\nu}\partial_\mu\Psi - i(\partial_\mu\bar{\Psi})\gamma_{\alpha\nu}\Psi. \tag{11}$$

[4]See [25] for the details of the vector Lagrangian description.

2)The angular momentum tensor is

$$\mathcal{M}_{\alpha}^{\nu,\mu\beta} = x_\mu T_\alpha^{\nu\beta} - x_\beta T_\alpha^{\nu\mu} + \\ + \quad i\bar{\Psi}\gamma_{\alpha\nu} A_{\mu\beta}^{\Psi}\Psi - i\bar{\Psi} A_{\mu\beta}^{\bar{\Psi}}\gamma_{\alpha\nu}\Psi \tag{12}$$

(with $A_{\mu\beta}^{\Psi}$ and $A_{\mu\beta}^{\bar{\Psi}}$ are the generators of the Lorentz transformations).

And, finally,

3)the current tensor is equal to

$$J_\alpha^\mu = -2\bar{\Psi}\gamma_{\alpha\mu}\Psi\,. \tag{13}$$

It is obtained as the consequence of gradient transformations:

$$\begin{cases} \Psi = e^{i\theta}\Psi, \\ \bar{\Psi} = \bar{\Psi}e^{-i\theta} \end{cases}, \tag{14}$$

where $\bar{\Psi} = \Psi^+\gamma_{44}$. It corresponds to the duality rotations:

$$\begin{cases} F_{\mu\nu} \rightarrow F_{\mu\nu} cos\theta +^* F_{\mu\nu} sin\theta, \\ ^*F_{\mu\nu} \rightarrow -F_{\mu\nu} sin\theta +^* F_{\mu\nu} cos\theta, \end{cases} \tag{15}$$

implemented by Sudbery [23].

Considering the Weinberg "spinor" in accordance with Eq. (5) and restricting oneself by the first term of Lagrangian (7) [5], we get the following conserved quantities:

$$T_{\{i}{}^{4\}4} = (\vec{E}\vec{\nabla})\vec{H} - (\vec{H}\vec{\nabla})\vec{E} + \vec{E}(\nabla\vec{H}) - \vec{H}(\vec{\nabla}\vec{E}), \tag{16}$$

$$T_{\{4}{}^{4\}4} = \vec{E}[\vec{\nabla}\times\vec{H}] - \vec{H}[\vec{\nabla}\times\vec{E}], \tag{17}$$

$$T_{\{i}{}^{j\}4} = \vec{\nabla}\vee\left[\vec{E}\times\vec{H}\right], \tag{18}$$

[5]It is possible because in terms of $\vec{E}$ and $\vec{H}$ the both of Eqs. (8), obtained from the first and second terms of (7), lead to the same motion equations.

$$T_{[i}{}^{4]4} = -i[(\vec{E}\vec{\nabla})\vec{E} + (\vec{H}\vec{\nabla})\vec{H} + \vec{E}(\vec{\nabla}\vec{E}) + \vec{H}(\vec{\nabla}\vec{H})], \qquad (19)$$

$$\tilde{T}_i = \frac{1}{2}\epsilon_{ijk}T_{[j}{}^{k]4} = \left[(\vec{E}\vec{\nabla})\vec{H} - (\vec{H}\vec{\nabla})\vec{E} + \vec{H}(\vec{\nabla}\vec{E}) - \vec{E}(\vec{\nabla}\vec{H})\right]. \qquad (20)$$

The value of $A^{\Psi}_{\mu\beta}$ is shown in [5] to be $A^{\Psi}_{\mu\beta} = -\frac{1}{6}\gamma_{5,\mu\beta}$ and, correspondingly, $A^{\bar{\Psi}}_{\mu\beta} = \frac{1}{6}\gamma_{5,\mu\beta}$, (the $S = 1$ case). As opposed to [23] we obtained

$$S_4^{4,ij} = 0, \qquad (21)$$

but

$$S_4^{4,4i} = -4\left[\vec{E}\times\vec{H}\right]_i. \qquad (22)$$

At last, we have the same expressions for J^{μ}_{α} as in Ref. [23]:

$$J_{44} = -2(\vec{E}^2 + \vec{H}^2), \qquad (23)$$

$$J_{4i} = 4i\epsilon_{ijk}E_jH_k, \qquad (24)$$

$$J_{ij} = 2[(\vec{E}^2 + \vec{H}^2)\delta_{ij} - E_iE_j - H_iH_j], \qquad (25)$$

which are the components of energy-momentum tensor in the common-used formulation of QED. Thus, the gauge transformations of the first kind lead to the energy-momentum conservation and the “charge” is identified with the energy density of the field.

The scalar Lagrangian of the Joos-Weinberg's $2(2S+1)-$ theory was presented in [11, 13] :

$$\mathcal{L}^{JW} = \partial_\mu\bar{\Psi}\gamma_{\mu\nu}\partial_\nu\Psi + m^2\bar{\Psi}\Psi. \qquad (26)$$

Let us note, implying the interpretation of the Weinberg's 6-“spinor” as in (5), we can rewrite the Lagrangian (26) in the following form:

$$\mathcal{L}^{JW} = (\partial_\mu F_{\nu\alpha})(\partial_\mu F_{\nu\alpha}) - 2(\partial_\mu F_{\mu\alpha})(\partial_\nu F_{\nu\alpha}) + 2(\partial_\mu F_{\nu\alpha})(\partial_\nu F_{\alpha\mu}). \qquad (27)$$

It leads to the Euler-Lagrange equation:

$$\Box F_{\alpha\beta} - 2(\partial_\beta F_{\alpha\mu,\mu} - \partial_\alpha F_{\beta\mu,\mu}) = 0, \tag{28}$$

where $\Box = \partial_\nu \partial_\nu$. The Lagrangian (27) is found out here to be equivalent to the Lagrangian of the free massless skew-symmetric field given in [27, 28] [6]:

$$\mathcal{L}^H = \frac{1}{8} F_k F_k, \tag{29}$$

with $F_k = i\epsilon_{kjmn} F_{jm,n}$. It can be rewritten

$$\begin{aligned}\mathcal{L}^H &= \frac{1}{4}(\partial_\mu F_{\nu\alpha})(\partial_\mu F_{\nu\alpha}) - \frac{1}{2}(\partial_\mu F_{\nu\alpha})(\partial_\nu F_{\alpha\mu}) = \\ &= -\frac{1}{4}\mathcal{L}^{JW} - \frac{1}{2}(\partial_\mu F_{\alpha\mu})(\partial_\nu F_{\alpha\nu}), \end{aligned} \tag{30}$$

which confirms the above statement, taking into account the possibility of the Fermi method *mutatis mutandis* as in Ref. [28]. The second term in (27) can be excluded by means of the generalized Lorentz condition (which is formally similar to the well-known Maxwell equations within normalizations of the field functions) [7].

In turn the Lagrangian (29) is invariant under new "gauge" transformations:

$$F_{\mu\nu} \to F_{\mu\nu} + A_{[\mu\nu]} = F_{\mu\nu} + \partial_\nu \Lambda_\mu - \partial_\mu \Lambda_\nu \tag{31}$$

The cited paper [28] proves that the Lagrangian describes massless particles having the longitudinal physical components only. The transversal components are removed by means of the "gauge" transformation (31). If we implement this "gauge" transfor-

[6] See also description of closed strings on the base of this Lagrangian in [29, 30].

[7] Let us mention some analogy with the potential formulation of QED. In some sense the Lagrangian (27) corresponds to the choice of "gauge-fixing" parameter $\xi = -1$, $\mathcal{L}^H$ of Ref. [28, formula (5)] corresponds to the "Landau gauge" ($\xi = 0$), and $\mathcal{L}^H$ (formula (9) of cited paper) is in the "Feynman gauge" ($\xi = 1$) for the antisymmetric tensor fields.

mations to the "bivector" [8]:

$$F \to F + e_4 \wedge A_{[4k]} e_k + \frac{i}{2} A_{[jk]} e_j \wedge e_k = F + \frac{1}{2} A_{[\mu\nu]} e_\mu \wedge e_\nu \tag{32}$$

we can obtain the same result. It is surprising in the point of view of the Weinberg theorem about connection between the helicity λ and the Lorentz group representation (A, B), namely, $B - A = \lambda$.

Now we turn to the interaction of the $S = 1$ partcile in the Joos-Weinberg formalism. In Ref. [2a, p.B1323] and Ref. [31, p.361] the following invariant (the interaction Hamiltonian) for interaction of 3-"bispinors" (e.g.. two particles of the spin $S = 1/2$ and one particle of the spin $S = 1$) has been constructed:

$$\mathcal{H}_{\Psi\psi\psi} = g \sum_{\mu_1\,\mu_2\,\mu_3} \begin{pmatrix} S_1 & S_2 & S_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \Phi^{\mu_1}_{(S_1)} \phi^{\mu_2}_{(S_2)} \phi^{\mu_3}_{(S_3)} \pm \begin{pmatrix} S_1 & S_2 & S_3 \\ \dot\mu_1 & \dot\mu_2 & \dot\mu_3 \end{pmatrix} \Xi^{\dot\mu_1}_{(S_1)} \chi^{\dot\mu_2}_{(S_2)} \chi^{\dot\mu_3}_{(S_3)}, \tag{33}$$

where $\begin{pmatrix} S_1 & S_2 & S_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$ are the Wigner $3j$- symbols.

Assuming the interpretation of the Weinberg's spinor as the sum of vector and pseudovector[9, 10]:

$$\begin{cases} \chi_k = C_k + iA_k, \\ \phi_k = C_k - iA_k\,. \end{cases} \tag{34}$$

In the case of the massless helicity-1 particles (photons) we get the following invariant

[8] See Ref. [26, p.244] for discussion of Clifford algebra in the Minkowski space.

[9] In Ref. [32, 33] the importance of the pseudovector potential C_k in QED has been discussed. In the Singleton papers [34] as well.

[10] As shown in my previous papers the interpretation $\Psi^{(S=1)}$ according to [2b, p.B888] leads to the contradiction with the theorem about connection between the (A, B) representation of the Lorentz group and the helicity of a particle with the field function which transforms according to this representation $(B - A = \lambda)$. Moreover, the Weinberg's massless equations [2b, formulas (4.21) and (4.22)] admit the acausal $(E \neq \pm p)$ solutions.

for interaction of two spinor particles with the generalized electromagnetic field (the spinor representation is used) :

$$\mathcal{H}_{\Psi\psi\psi} = g \sum_{k\,\mu_2\,\mu_3} \left\{ \left[\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ k & \mu_2 & \mu_3 \end{pmatrix} \phi^{\mu_2}_{(\frac{1}{2})} \phi^{\mu_3}_{(\frac{1}{2})} + \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ k & \dot{\mu}_2 & \dot{\mu}_3 \end{pmatrix} \chi^{\dot{\mu}_2}_{(\frac{1}{2})} \chi^{\dot{\mu}_3}_{(\frac{1}{2})} \right] C_k + \right.$$
$$\left. + \;\; i \left[\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ k & \mu_2 & \mu_3 \end{pmatrix} \phi^{\mu_2}_{(\frac{1}{2})} \phi^{\mu_3}_{(\frac{1}{2})} - \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ k & \dot{\mu}_2 & \dot{\mu}_3 \end{pmatrix} \chi^{\dot{\mu}_2}_{(\frac{1}{2})} \chi^{\dot{\mu}_3}_{(\frac{1}{2})} \right] A_k \right\} . \qquad (35)$$

In (33) we choose the sign " + ". The question of the Lorentz transformation rules of the pseudovector is related to the transformation rules of 3-rank antisymmetric tensor. Taken into account the relation between the Pauli $\sigma-$ matrices and the Clebsh-Gordon coefficients (formula on the p. 65 in [35])

$$\sigma^{\mu}_{\alpha\beta} = -\sqrt{3} C^{\frac{1}{2}\alpha}_{1\mu\frac{1}{2}\beta} \qquad (36)$$

one can rewrite the previous expression (35) as follows:

$$\mathcal{H}_{\Psi\bar{\psi}\psi} = \frac{g}{\sqrt{6}} \left\{ -\bar{\psi}\alpha_k\gamma_5\psi C_k + i\bar{\psi}\alpha_k\psi A_k \right\} . \qquad (37)$$

In fact, the coupling constant g is equal to $ie\sqrt{6}$, e is electric charge in QED, $k = 1, 2, 3$. The matrix γ_5 has been chosen in the diagonal form:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad (38)$$

$$\beta = \alpha_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad (39)$$

and

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} . \qquad (40)$$

One can see that this interaction Hamiltonian leads to the following equations from the Hamiltonian (37):

$$i\hbar\frac{\partial\psi}{\partial t} = c\vec{\alpha}\cdot(\vec{p} - e\vec{A} - ie\gamma_5\vec{C})\psi + mc^2\beta\psi, \tag{41}$$

which is equivalent to the following system ($c = \hbar = 1$) for 2-spinors:

$$\begin{cases} \left[(\vec{\sigma}\vec{p}) - e(\vec{\sigma}\vec{A}) + ie(\vec{\sigma}\vec{C})\right]\xi + m\eta = E\xi, \\ \left[-(\vec{\sigma}\vec{p}) + e(\vec{\sigma}\vec{A}) + ie(\vec{\sigma}\vec{C})\right]\eta + m\xi = E\eta. \end{cases} \tag{42}$$

Therefore,

$$\begin{aligned}(E^2 - m^2)\xi &= \Big\{\vec{p}^{\,2} - e\left[(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{A}) + (\vec{\sigma}\vec{A})(\vec{\sigma}\vec{p})\right] + \\ &+ \quad ie\left[(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{C}) - (\vec{\sigma}\vec{C})(\vec{\sigma}\vec{p})\right] + e^2\vec{A}^{\,2} + e^2\vec{C}^{\,2} + 2ieE(\vec{\sigma}\vec{C})\Big\}\,\xi\,,\end{aligned} \tag{43}$$

and

$$\begin{aligned}(E^2 - m^2)\eta &= \Big\{\vec{p}^{\,2} - e\left[(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{A}) + (\vec{\sigma}\vec{A})(\vec{\sigma}\vec{p})\right] - \\ &- \quad ie\left[(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{C}) - (\vec{\sigma}\vec{C})(\vec{\sigma}\vec{p})\right] + e^2\vec{A}^{\,2} + e^2\vec{C}^{\,2} + 2ieE(\vec{\sigma}\vec{C})\Big\}\,\eta\,.\end{aligned} \tag{44}$$

We would like to mention that A_k, the vector potential, is the compensating field for the gauge transformation of the second kind, and C_k , the pseudovector potential, is the compensating field for the chirality gauge transformation[11]. Since we may assign $E_k = rot\, C_k$ we can see that $\vec{E} = \vec{0}$, and $\vec{H} = \vec{0}$ in the particular case [39]. However, the spectrum is influenced by the term $\vec{C}$.

We can implement the new $4\otimes 4$- matrix field corresponding to the electromagnetic field:

$$\Phi_k = \begin{pmatrix} A_k - iC_k & 0 \\ 0 & A_k + iC_k \end{pmatrix} \tag{45}$$

[11]See, e.g., Ref. [36] for discussion of the chirality (γ_5) symmetry of massless fields and neutrino theory of photons. As for the generalized gauge transformations, one can find them in [37, 38].

which is described by the Lagrangian:

$$\mathcal{L} = \bar{\Psi}^{(S=1)}\gamma_{\mu\nu}p_\mu p_\nu \Psi^{(S=1)} = i\bar{\Phi}_j \left\{ -i\epsilon_{ijk}p_4 p_i \otimes \gamma_5 + (\vec{p}^{\,2}\delta_{jk} - p_j p_k) \otimes I \right\} \Phi_k. \qquad (46)$$

The corresponding dynamical invariants are found from the energy-momentum tensor, which is written as following:

$$\begin{aligned}
T_{44} &= i\bar{\Phi}_j(\vec{p}^{\,2}\delta_{jk} - p_j p_k)\Phi_k, \\
T_{l4} &= i\epsilon_{ijk}\bar{\Phi}_j p_i p_l \otimes \gamma_5\, \Phi_k, \\
T_{4l} &= i\epsilon_{ljk}\bar{\Phi}_j p_4 p_4 \otimes \gamma_5\, \Phi_k - 2i\bar{\Phi}_k p_l p_4 \Phi_k + i\bar{\Phi}_k p_k p_4 \Phi_l + i\bar{\Phi}_l p_4 p_k \Phi_k, \\
T_{lm} &= \mathcal{L}\delta_{lm} + i\epsilon_{mjk}\bar{\Phi}_j p_l p_4 \otimes \gamma_5\, \Phi_k - 2i\bar{\Phi}_k p_l p_m \Phi_k + i\bar{\Phi}_m p_l p_k \Phi_k + i\bar{\Phi}_k p_k p_l \Phi_m. \qquad (47)
\end{aligned}$$

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