Euler Equations Solutions for Incompressible Fluid Flow

Abstract

The simplest solution is usually the best solution---Albert Einstein

This paper covers the solutions of the Euler equations in 3-D and 4-D for incompressible fluid flow.The solutions are the spin-offs of the author's previous analytic solutions of the Navier-Stokes equations (vixra:1405.0251 of 2014). However, some of the solutions contained implicit terms. In this paper, the implicit terms have been expressed explicitly in terms of x, y, z and t.
The author applied a new law, the law of definite ratio for fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. This law evolved from the author's earlier solutions of the Navier-Stokes equations. In addition to the usual approach of solving these equations, the Euler equations have also been solved by a second method in which the three equations in the system are added to produce a single equation which is then integrated. The solutions by the two approaches are identical, except for the constants involved. From the experience gained in solving the linearized Navier-Stokes equations, only the equation with the gravity term as the subject of the equation was integrated. The experience was that when each of the terms of the Navier-Stokes equation was used as the subject of the equation, only the equation with the gravity term as the subject of the equation produced a solution. Ratios were used to split-up the x–direction Euler equation with the gravity term as the subject of the equation. The resulting five sub-equations were readily integrable, and even, the non-linear sub-equations were readily integrated. The integration results were combined. The combined results satisfied the corresponding equation. This equation which satisfied its corresponding equation would be defined as the driver equation; and each of the other equations which would not satisfy its corresponding equation would be called a supporter equation. A supporter equation does not satisfy its corresponding equation completely, but provides useful information which is not apparent in the solution of the driver equation. The solutions and relations revealed the role of each term of the Euler equations in fluid flow. The gravity term is the indispensable term in fluid flow, and it is involved in the forward motion of fluids. The pressure gradient term is also involved in the forward motion. The variable acceleration term is also involved in the forward motion. The fluid flow behavior in the Euler solution may be characterized as follows. The x–direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the x–nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The y–nonlinear term is similar parabolically to the x–nonlinear term. The characteristic curve for the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.

Solutions of the Euler Equations of Fluid flow
(Method 1)

In the Navier-Stokes equation, if $\mu = 0$, one obtains the Euler equation. From

$$\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) - \frac{\partial p}{\partial x} + \rho g_x = \rho(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z})$$, one obtains

Euler equation : $(\mu = 0)$ $-\frac{\partial p}{\partial x} + \rho g_x = \rho(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z})$ or

$$\boxed{\rho(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}) + \frac{\partial p_x}{\partial x} = \rho g_x}$$ <---driver equation.

Euler equation $(\mu = 0)$: $\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_x$ <---driver equation

Split the equation using the ratio terms f, h, n, q, d,, and solve. $(f + h + n + q + d = 1)$

1. $\frac{\partial V_x}{\partial t} = fg_x$ $V_{x4} = fg_x t$ $V_{x4} = fg_x t$	**2.** $V_x\frac{\partial V_x}{\partial x} = hg_x$ $V_x\frac{dV_x}{dx} = hg_x$ $V_x dV_x = hg_x dx$ $\frac{V_x^{\ 2}}{2} = hg_x x$ or $V_x^{\ 2} = 2hg_x x$ $V_x = \pm\sqrt{2hg_x x}$	**3.** $V_y\frac{\partial V_x}{\partial y} = ng_x$ $V_y\frac{dV_x}{dy} = ng_x$ $V_y dV_x = ng_x \mathrm{dy}$ $V_y V_x = ng_x \mathrm{y} + \psi_y(V_y)$ $V_{x6} = \frac{ng_x \mathrm{y}}{V_y} + \frac{\psi_y(V_y)}{V_y}$ $V_y \neq 0$	**4.** $V_z\frac{\partial V_x}{\partial z} = qg_x$ $V_z\frac{dV_x}{dz} = qg_x$ $V_z dV_x = qg_x \mathrm{dz}$; $V_z V_x = qg_x \mathrm{z} + \psi_z(V_z)$ $V_{x7} = \frac{qg_x \mathrm{z}}{V_z} + \frac{\psi_z(V_z)}{V_z}$ $V_z \neq 0$	**5.** $\frac{1}{\rho}\frac{\partial p}{\partial x} = dg_x$ $\frac{1}{\rho}\frac{\partial p}{\partial x} = dg_x$ $\frac{\partial p}{\partial x} = d\rho g_x$ $p = d\rho g_x x + C_7$

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x \mathrm{y}}{V_y} + \frac{qg_x \mathrm{z}}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C \\ P(x) = d\rho g_x x \qquad (f + h + n + q + d = 1)\ V_y \neq 0,\ V_z \neq 0 \end{array}}$$

Find the test derivatives to check in the original equation.

1. $\frac{\partial V_x}{\partial t} = fg_x$	**2.** $V_x^2 = 2hg_x x$; $2V_x\frac{\partial V_x}{\partial x} = 2hg_x$; $\frac{\partial V_x}{\partial x} = \frac{hg_x}{V_x}$, $V_x \neq 0$	3. $\frac{\partial V_x}{\partial y} = \frac{ng_x}{V_y}$ $V_y \neq 0$	4. $\frac{\partial V_x}{\partial z} = \frac{qg_x}{V_z}$ $V_z \neq 0$	5. $\frac{\partial p}{\partial x} = d\rho g_x$

$\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_x$ (Above, $\psi_y(V_y)$ and $\psi_z(V_z)$ are arbitrary functions)

$$fg_x + V_x\frac{hg_x}{V_x} + V_y\frac{ng_x}{V_y} + V_z\frac{qg_x}{V_z} + \frac{1}{\rho}d\rho g_x \overset{?}{=} g_x$$

$$fg_x + hg_x + ng_x + qg_x + dg_x \overset{?}{=} g_x$$

$$g_x(f + h + n + q + d) \overset{?}{=} g_x$$

$$g_x(1) \overset{?}{=} g_x \qquad (f + h + n + q + d = 1)$$

$$g_x \overset{?}{=} g_x$$ Yes

The relation obtained satisfies the Euler equation. Therefore the solution to the Euler equation

$$: \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}) = \rho g_x \text{ is}$$

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \dfrac{ng_x y}{V_y} + \dfrac{qg_x z}{V_z} + \underbrace{\dfrac{\psi_y(V_y)}{V_y} + \dfrac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};\ P(x) = d\rho g_x x \\ V_y \neq 0,\ V_z \neq 0;\ \ (d+f+h+n+q=1) \end{array}} \quad x\text{-direction}$$

Similarly, the equations and solutions for the other two directions are respectively

For V_y, $\dfrac{\partial p}{\partial y} + \rho \dfrac{\partial v_y}{\partial t} + \rho V_x \dfrac{\partial v_y}{\partial x} + \rho V_y \dfrac{\partial v_y}{\partial y} + \rho V_z \dfrac{\partial V_y}{\partial z} = \rho g_y$

$$\boxed{\begin{array}{l} V_y(x,y,z,t) = \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y \mathrm{y}} + \dfrac{\lambda_6 g_y x}{V_x} + \dfrac{\lambda_8 g_y \mathrm{z}}{V_z} + \dfrac{\psi_x(V_x)}{V_x} + \dfrac{\psi_z(V_z)}{V_z};\ P(y) = \lambda_4 \rho g_y y \\ V_x \neq 0,\ V_z \neq 0;\ \ (\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1) \end{array}} \quad y\text{-direction}$$

For V_z: $\dfrac{\partial p}{\partial z} + \rho \dfrac{\partial v_z}{\partial t} + \rho V_x \dfrac{\partial v_z}{\partial x} + \rho V_y \dfrac{\partial v_z}{\partial y} + \rho V_z \dfrac{\partial v_z}{\partial z} = \rho g_z$

$$\boxed{\begin{array}{l} V_z(x,y,z,t) = \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \dfrac{\beta_6 g_z x}{V_x} + \dfrac{\beta_7 g_z y}{V_y} + \dfrac{\psi_x(V_x)}{V_x} + \dfrac{\psi_y(V_y)}{V_y}\ ;\ P(z) = \beta_4 \rho g_z z \\ V_x \neq 0,\ V_y \neq 0;\ \ (\beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1) \end{array}} \quad z\text{-direction}$$

One will next solve the above system of solutions for V_x , V_y, V_z in order to express $\dfrac{ng_x y}{V_y}$ and $\dfrac{q_e g_x z}{V_z}$ in terms of x, y, z, and t.

Solving for V_x , V_y, V_z $\dfrac{ng_x y}{V_y}$, and $\dfrac{q_e g_x z}{V_z}$

$$\begin{cases} V_x = fg_x t \pm \sqrt{2hg_x x} + \dfrac{ng_x y}{V_y} + \dfrac{qg_x z}{V_z} + \dfrac{\psi_y(V_y)}{V_y} + \dfrac{\psi_z(V_z)}{V_z}. & (A) \\ V_y = \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y \mathrm{y}} + \dfrac{\lambda_6 g_y x}{V_x} + \dfrac{\lambda_8 g_y \mathrm{z}}{V_z} + \dfrac{\psi_x(V_x)}{V_x} + \dfrac{\psi_z(V_z)}{V_z}. & (B) \\ V_z = \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \dfrac{\beta_6 g_z x}{V_x} + \dfrac{\beta_7 g_z y}{V_y} + \dfrac{\psi_x(V_x)}{V_x} + \dfrac{\psi_y(V_y)}{V_y}\ . & (C) \end{cases}$$

Let $V_x = x$, $V_y = y$ and $V_z = z$. (x, y and z are being used for simplicity. They will be changed back to V_x , V_y, and V_z later, and they do not represent the variables x, y and z in the system of solutions)

Step 1 From the above system of solutions, let

$A = (fgt + \sqrt{2hg_x x})$; $D = (qg_x z)$; $E = (ng_x y)$

$B = (\lambda_5 g_y t + \sqrt{2\lambda_7 g_y y})$; $F = (\lambda_6 g_y x)$; $G = (\lambda_8 g_y z)$

$C = (\beta_5 g_z t + \sqrt{2\beta_8 g_z z})$; $J = (\beta_6 g_z x)$; $L = (\beta_7 g_z y)$

Step 2: Then the solutions to the Euler system of equations become (ignoring the arbitrary functions)

$$\left.\begin{aligned} x &= A + \frac{D}{z} + \frac{E}{y} \\ y &= B + \frac{F}{x} + \frac{G}{z} \\ z &= C + \frac{J}{x} + \frac{L}{y} \end{aligned}\right\} \text{M}$$

Step 3

$$\left.\begin{aligned} xyz &= Ayz + Dy + Ez && (1) \\ xyz &= Bxz + Fz + Gx && (2) \\ xyz &= Cxy + Jy + Lx && (3)) \end{aligned}\right\} \text{N}$$

Step 4

$$\left.\begin{aligned} 0 &= Ayz + Dy + Ez - Bxz - Fz - Gx && (4) \\ 0 &= Ayz + Dy + Ez - Cxy - Jy - Lx && (5) \\ 0 &= Bxz + Fz + Gx - Cxy - Jy + -Lx && (6) \end{aligned}\right\} \text{P}$$

Maples software was used to solve system P to obtain

Step 5

$$x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$$

$$V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \quad \text{(back to } V_x\text{)}$$

$$y = -\frac{L}{C};$$

$$V_y = -\frac{L}{C} \quad \text{(changing back to } V_y \text{ as agreed to)}$$

$$z = -\frac{L(D-J)}{LA - CE};$$

$$V_z = -\frac{L(D-J)}{LA - CE} \quad \text{(changing back to } V_z \text{ as agrred to)}$$

Note:
None of the popular academic programs could solve the system in M.
Maples solved system P (step 4 above) for x, y, and z in terms of A, B, C, D. E. F, G. J. and L.
Note also that x, y and z are not the same as the x, y and z in the system of equations.. They were used for convenience and simplicity .

Step 5: Apply and substitute from in steps 6-8 below

$A = (fgt \pm \sqrt{2hg_x x})$; $B = (\lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y})$; $C = (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})$; $D = (qg_x z)$;
$E = (ng_x y)$; $F = (\lambda_6 g_y x)$; $G = (\lambda_8 g_y z) J = (\beta_6 g_z x)$; $L = (\beta_7 g_z y)$,

Step 6

$$V_y = -\frac{L}{C} = -\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})}$$

$$\frac{ng_x y}{V_y} = ng_x y \div -\left(\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})}\right)$$

$$\frac{ng_x y}{V_y} = -\frac{(ng_x y)[\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}]}{\beta_7 g_z y}; \quad y \neq 0$$

Step 7

$$\frac{ng_x y}{V_y} = -\frac{[\beta_5 g_z t](ng_x y) \pm [\sqrt{2\beta_8 g_z z}](ng_x y)}{\beta_7 g_z y}$$

$$\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}$$

$$\boxed{\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z}(ng_x)}{\beta_7 g_z}}$$

Step 8:

$$V_z = -\frac{L(D-J)}{LA-CE}$$

$$V_z = -\frac{(\beta_7 g_z y)[q g_x z \ - \beta_6 g_z x]}{(\beta_7 g_z y)(f g_x t \pm \sqrt{2h g_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\ )(n g_x y)}$$

$$\frac{q g_x z}{V_z} = (q g_x z) \div -(\frac{(\beta_7 g_z y)[q g_x z \ - \beta_6 g_z x]}{(\beta_7 g_z y)(f g_x t \pm \sqrt{2h g_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\ )(n g_x y)})$$

$$= -\frac{(q g_x z)\bullet(\beta_7 g_z y)(f g_x t \pm \sqrt{2h g_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\ )(n g_x y)}{(\beta_7 g_z y)[q g_x z \ - \beta_6 g_z x]}$$

$$= -\frac{(q g_x z)\bullet(\beta_7 g_z y f g_x t \pm \sqrt{2h g_x x}\beta_7 g_z y - \beta_5 g_z t n g_x y \pm \sqrt{2\beta_8 g_z z}\ n g_x y}{(\beta_7 g_z y)[q g_x z \ - \beta_6 g_z x]}$$

$$= -\frac{q g_x g_x z \beta_7 f g_z y t \pm \sqrt{2h g_x x}\beta_7 g_x g_z q y z - \beta_5 g_x g_z n q t y z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q y z}{(\beta_7 g_z y)[q g_x z \ - \beta_6 g_z x]}$$

$$\frac{q g_x z}{V_z} = -\frac{q g_x g_x z \beta_7 f g_z t \pm \sqrt{2h g_x x}\beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q z}{(\beta_7 g_z)[q g_x z \ - \beta_6 g_z x]}$$

((Dividing out the "y" in the numerator and the denominator)

$$\frac{q g_x z}{V_z} = -\frac{q g_x g_x z \beta_7 f g_z t \pm \sqrt{2h g_x x}\beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q z}{(\beta_7 g_z)[q g_x z \ - \beta_6 g_z x]}$$ or

$$\frac{q g_x z}{V_z} = \frac{q g_x g_x z \beta_7 f g_z t \pm \sqrt{2h g_x x}\beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z}$$

$$\boxed{\frac{q g_x z}{V_z} = \frac{(\beta_7 f g_z g_x g_x q - \beta_5 g_x g_z n q) t z \pm \sqrt{2h g_x x}\beta_7 g_x g_z q z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z}}$$

Summary for the fractional terms of the *x*-direction solution

$\frac{n g_x y}{V_y}$ and $\frac{q g_x z}{V_z}$ in terms of x, y, z and t

$\frac{n g_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(n g_x)}{\beta_7 g_z}$ } B	$\frac{n g_x y}{V_y} = -k_1 g_z t \pm \frac{\sqrt{2k_2 g_z z}\bullet g_x k_3}{g_z}$ $k_1 = \frac{n\beta_5}{\beta_7}$; $k_2 = \beta_8$; $k_3 = \frac{n}{\beta_7}$;

$$\frac{q g_x z}{V_z} = \frac{(\beta_7 f_e g_z g_x g_x q - \beta_5 g_x g_z n_e q) t z \pm \sqrt{2h g_x x}\beta_7 g_x g_z q z \pm \sqrt{2\beta_8 g_z z}\ g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z} \Bigg\} \text{C}$$

$$\frac{q g_x z}{V_z} = \frac{(g_x^2 g_z k_4 - g_x g_z k_5) t z \pm \sqrt{2 g_x k_6 x}\bullet g_x g_z k_7 z \pm \sqrt{2 g_z k_8 z}\bullet g_x^2 k_9 z}{g_z^2 k_{10} x - g_x g_z k_{11} z}$$

$k_4 = \beta_7 f q$; $k_5 = \beta_5 n q$;; $k_6 = h$; $k_8 = \beta_8$; $k_9 = nq$ $k_{10} = \beta_7\beta_6$ $k_{11} = \beta_7 q$

Analysis of Solutions

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}; \; P(x) = d\rho g_x x \quad x\text{-direction}$$

$$V_y \neq 0,\; V_z \neq 0; \;\; (d+f+h+n+q=1)$$

$$\left.\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}\right\} \text{B}$$

$$\left.\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_z g_x^2 q - \beta_5 g_x g_z nq)tz \pm \sqrt{2hg_x x}\,\beta_7 g_x g_z qz \pm \sqrt{2\beta_8 g_z z}\; g_x^2 nqz}{\beta_7\beta_6 g_z^2 x - \beta_7 q g_z g_x z}\right\} \text{C}$$

$$d+f+h+n+q=1; \quad \lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1; \quad \beta_4+\beta_5+\beta_6+\beta_7+\beta_8=1$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g_x were zero, the first four terms of the velocity solution and $P(x)$ would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

More Observations: Comparison of the Euler solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion under gravity equations: (B): $V = gt$; (C): $V = \sqrt{2gx}$;

Liquid Pressure, P at the bottom of a liquid of depth h units is given by $P = \rho gh$

Observe the following similarities above:

1. Observe the "gt" in $V = gt$ of (B) of the motion equations and the $fg_x t$ in the Euler solution.

2. Observe the "$\sqrt{2gx}$" in $V = \sqrt{2gx}$ of (C) and the $\sqrt{2hg_x x}$ in the Euler solution.

3. Observe the $P = \rho gh$ of the liquid pressure and the $P(x) = d\rho g_x x$ of the Euler solution.

There are five main terms (ignoring the arbitrary functions) in the Euler solution. Of these five terms, three terms, namely, $fg_x t$, $\sqrt{2hg_x x}$, $d\rho g_x x$ are the same (except for the constants involved) as the terms in the equations of motion under gravity. This similarity means that the approach used in solving the Euler equation is sound. One should also note that to obtain these three terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these three terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Euler equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

Velocity profile

The x–direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the x–nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The y–nonlinear term is similar parabolically to the x–nonlinear term. The characteristic curve for the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent

Standard form of the x-direction Euler equation for incompressible fluid flow
One will call the Euler equation with the gravity term as the subject of equation in (A) , the standard form of the Euler equation for the ratio method of solving these equations, since this form produces a solution on integration. None of the other forms in (B), (C), (D), (E), or (F), produces a solution. That is, the integration results of each of the other five equations do not satisfy the corresponding equation.

$$\boxed{\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\frac{\partial p_x}{\partial x}=\rho g_x \quad \text{(A)}} \quad < \text{standard form}$$

$$-\rho\frac{\partial V_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial V_x}{\partial y}-\rho V_z\frac{\partial V_x}{\partial z}+\rho g_x=\frac{\partial p_x}{\partial x} \quad \text{(B)}$$

$$-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial V_x}{\partial y}-\rho V_z\frac{\partial V_x}{\partial z}+\frac{\partial p_x}{\partial x}+\rho g_x=\rho\frac{\partial V_x}{\partial t} \quad \text{(C)}$$

$$-\rho\frac{\partial V_x}{\partial t}-\rho V_y\frac{\partial V_x}{\partial y}-\rho V_z\frac{\partial V_x}{\partial z}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_x\frac{\partial V_x}{\partial x} \quad \text{(D)}$$

$$-\rho\frac{\partial V_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_z\frac{\partial V_x}{\partial z}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_y\frac{\partial V_x}{\partial y} \quad \text{(E)}$$

$$-\rho\frac{\partial V_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial V_x}{\partial y}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_z\frac{\partial V_x}{\partial z} \quad \text{(F)}$$

Uniqueness of the solution of the Euler equation
When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. (vixra:1405.0251 of 2014). Similarly. the solution of the Euler solution is unique.

Extra:

Linearized Euler Equation: If one linearizes the Euler equation as was done in the linearization of the Navier-Stokes equation, one obtains $4\frac{\partial V_x}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=g_x$; whose solution is

$V_x=\frac{fg_x}{4}t+C;\quad P(x)=d\rho g_x x$.

Note: By comparison with Navier-Stokes equation and its relation, a relation to Euler equation can be found by deleting the Navier-Stokes relation resulting from the μ-terms.

Also note: The liquid pressure, P at the bottom of a liquid of depth h units is given by $\boxed{P=\rho gh}$. From the Euler solution in this paper, $\boxed{P(x)=d\rho gx}$ from integrating $\frac{dp}{dx}=d\rho g$ where d is ratio term. Each of the other terms in the Euler equation must also be set equal to the product of a ratio term and ρg. This result implies that the approach used in solving the Euler equations is valid.

Conclusion

The x–direction Euler equation with the gravity term as the subject of the equation was split-up into sub-equations using ratios. The sub-equations were solved and combined. The relation obtained from the integration satisfied the corresponding Euler equation. Similarly the y–direction and the z–direction equations were solved. By solving algebraically and simultaneously for V_x, V_y and V_z, the $(ng_x y/V_y)$ and $(qg_x z/V_z)$ terms have been expressed explicitly in terms of x, y, z and t. One may note that in checking the relations obtained for integrating the equations for possible solutions, one needs not have explicit expressions for V_x, V_y, and V_z, since these behave as constants in the identity checking process. The above solution is the solution everyone has been waiting for, for nearly 150 years. It was obtained in two simple steps, namely, splitting the equation using ratios, integrating and successfully checking for identity in the original equation.

The fluid flow behavior in the Euler solution may be characterized as follows. The x–direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the x–nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The y–nonlinear term is similar parabolically to the x–nonlinear term. The characteristic curve for the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.

The impediment to solving the Euler equations has been due to how to obtain sub-equations from the six-term equation. The above solution was made possible after pairing the terms of the equation using ratios (ratio terms). The author was encouraged by Lagrange's use of ratios and proportion in solving differential equations. One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

Finally, to solve a partial differential equation, one integrates properly, and check the integration relations for identity in the original equation. Since the above procedure has been followed and completed successfully, the Euler equations for incompressible fluid flow have been solved.

P.S. The liquid pressure, P at the bottom of a liquid of depth h units is given by $\boxed{P = \rho gh}$.
From the Euler solution in this paper, $\boxed{P(x) = d\rho gx}$ from integrating $\frac{dp}{dx} = d\rho g$ where d is ratio term. Each of the other terms in the Euler equation must also be set equal to the product of a ratio term and ρg. This result implies that the approach used in solving the Euler equations is valid. and perhaps may be the only approach to produce such a result by integrating the Euler equation.

Solutions of 3-D Euler Equations (Method 2)

Here, the three equations below, will be added together; and a single equation will be integrated

$$\begin{cases} \frac{\partial p}{\partial x} + \rho(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}) = \rho g_x \quad (1) \\ \frac{\partial p}{\partial y} + \rho(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z}) = \rho g_y \quad (2) \\ \frac{\partial p}{\partial z} + \rho(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z}) = \rho g_z \quad (3) \end{cases}$$

Step 1: Apply the axiom, if $a = b$ and $c = d$, then $a + c = b + d$; and therefore, add the left sides and add the right sides of the above equations . That is, $(1) + (2) + (3) = \rho g_x + \rho g_y + \rho g_z$

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} + \rho \frac{\partial V_x}{\partial t} + \rho \frac{\partial V_y}{\partial t} + \rho \frac{\partial V_z}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_x \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y}$$
$$+\rho V_z \frac{\partial V_y}{\partial z} + \rho V_x \frac{\partial V_z}{\partial x} + \rho V_y \frac{\partial V_z}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = (\rho g_x + \rho g_y + \rho g_z)$$ (Two lines per equation)

Let $\rho g_x + \rho g_y + \rho g_z = \rho G$ **,** where $G = \left| g_x + g_y + g_z \right|$ to obtain

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} + +\rho \frac{\partial V_x}{\partial t} + \rho \frac{\partial V_y}{\partial t} + \rho \frac{\partial V_z}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}$$
$$+ \rho V_x \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_y}{\partial z} + \rho V_x \frac{\partial V_z}{\partial x} + \rho V_y \frac{\partial V_z}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = \rho G$$

Step 2: Solve the above 16-term equation using the ratio method. (15 ratio terms)
The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)
$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9$

$\frac{\partial p}{\partial x} = \beta_1 \rho G$; $\quad \frac{\partial p}{\partial y} = \beta_2 \rho G$; $\quad \frac{\partial p}{\partial z} = \beta_3 \rho G$;

$\rho \frac{\partial V_x}{\partial t} = \beta_4 \rho G$; $\quad \rho \frac{\partial V_y}{\partial t} = \beta_5 \rho G$; $\quad \rho \frac{\partial V_z}{\partial t} = \beta_6 \rho G$; $\quad \rho V_x \frac{\partial V_x}{\partial x} = \lambda_1 \rho G$;

$\rho V_y \frac{\partial V_x}{\partial y} = \lambda_2 \rho G$; $\quad \rho V_z \frac{\partial V_x}{\partial z} = \lambda_3 \rho G$; $\quad \rho V_x \frac{\partial V_y}{\partial x} = \lambda_4 \rho G$; $\quad \rho V_y \frac{\partial V_y}{\partial y} = \lambda_5 \rho G$;

$\rho V_z \frac{\partial V_y}{\partial z} = \lambda_6 \rho G$; $\quad \rho V_x \frac{\partial V_z}{\partial x} = \lambda_7 \rho G$; $\quad \rho V_y \frac{\partial V_z}{\partial y} = \lambda_8 \rho G$; $\quad \rho V_z \frac{\partial V_z}{\partial z} = \lambda_9 \rho G$

1.	**2**	**3**
$\frac{\partial p}{\partial x} = \beta_1 \rho G$	$\frac{\partial p}{\partial y} = \beta_2 \rho G$	$\frac{\partial p}{\partial z} = \beta_3 \rho G$
$\frac{dp}{dx} = \beta_1 \rho G$	$\frac{dp}{dy} = \beta_2 \rho G$	$\frac{dp}{dz} = \beta_3 \rho G$
$P(x) = \beta_1 \rho G x + C_{19}$	$P(y) = \beta_2 \rho G y + C_{20}$	$P(z) = \beta_3 \rho G z + C_{21}$

4. $\rho \frac{\partial V_x}{\partial t} = \beta_4 \rho G$ $\frac{dV_x}{dt} = \beta_4 G$ $V_x = \beta_4 G t + C_{22}$	**5** $\rho \frac{\partial V_y}{\partial t} = \beta_5 \rho G$ $\frac{dV_y}{dt} = \beta_5 G$ $V_y = \beta_5 G t + C_{23}$	**6** $\rho \frac{\partial V_z}{\partial t} = \beta_6 \rho G$ $\frac{dV_z}{dt} = \beta_6 G$ $V_z = \beta_6 G t + C_{24}$
7, $\rho V_x \frac{\partial V_x}{\partial x} = \lambda_1 \rho G$ $V_x \frac{\partial V_x}{\partial x} = \lambda_1 G$ $V_x \frac{dV_x}{dx} = \lambda_1 G$ $V_x dV_x = \lambda_1 G\, dx$ $\frac{V^2{}_x}{2} = \lambda_1 G\, x$ $V^2{}_x = 2\lambda_1 G\, x$ $V_x = \pm\sqrt{2\lambda_1 G\, x} + C_{25}$	**8** $\rho V_y \frac{\partial V_x}{\partial y} = \lambda_2 \rho G$ $V_y \frac{dV_x}{dy} = \lambda_2 G$ $V_y dV_x = \lambda_2 G\, dy$ $V_y V_x = \lambda_2 G\, y + \psi_y(V_y)$ $V_x = \frac{\lambda_2 G\, y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	**9** $\rho V_z \frac{\partial V_x}{\partial z} = \lambda_3 \rho G$ $V_z \frac{dV_x}{dz} = \lambda_3 G$ $V_z dV_x = \lambda_3 G\, dz$ $V_z V_x = \lambda_3 G\, z + \psi_z(V_z)$ $V_x = \frac{\lambda_3 G\, z}{V_z} + \frac{\psi_z(V_z)}{V_z}$
10 $\rho V_x \frac{\partial V_y}{\partial x} = \lambda_4 \rho G$ $V_x \frac{dV_y}{dx} = \lambda_4 G$ $V_x dV_y = \lambda_4 G\, dx$ $V_x V_y = \lambda_4 G\, x + \psi_x(V_x)$ $V_y = \frac{\lambda_4 G\, x}{V_x} + \frac{\psi_x(V_x)}{V_x}$	**11** $\rho V_y \frac{\partial V_y}{\partial y} = \lambda_5 \rho G$ $V_y \frac{dV_y}{dy} = \lambda_5 G$ $V_y dV_y = \lambda_5 G\, dy$ $\frac{V_y^{\,2}}{2} = \lambda_5 G\, y$ $V_y^{\,2} = 2\lambda_5 G\, y$ $V_y = \pm\sqrt{2\lambda_5 G\, y} + C_{26}$	**12.** $\rho V_z \frac{\partial V_y}{\partial z} = \lambda_6 \rho G$ $V_z \frac{dV_y}{dz} = \lambda_6 G$ $V_z dV_y = \lambda_6 G\, dz$ $V_z V_y = \lambda_6 G\, z + \psi_z(V_z)$ $V_y = \frac{\lambda_6 G\, z}{V_z} + \frac{\psi_z(V_z)}{V_z}$
13 $\rho V_x \frac{\partial V_z}{\partial x} = \lambda_7 \rho G$ $V_x \frac{dV_z}{dx} = \lambda_7 G$ $V_x dV_z = \lambda_7 G\, dx$ $V_x V_z = \lambda_7 G\, x + \psi_x(V_x)$ $V_z = \frac{\lambda_7 G\, x}{V_x} + \frac{\psi_x(V_x)}{V_x}$	**14** $\rho V_y \frac{\partial V_z}{\partial y} = \lambda_8 \rho G$ $V_y \frac{dV_z}{dy} = \lambda_8 G$ $V_y dV_z = \lambda_8 G\, dy$ $V_y V_z = \lambda_8 G\, y + \psi_y(V_y)$ $V_z = \frac{\lambda_8 G\, y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	**15** $\rho V_z \frac{\partial V_z}{\partial z} = \lambda_9 \rho G$ $V_z \frac{dV_z}{dz} = \lambda_9 G$ $V_z dV_z = \lambda_9 G\, dz$ $\frac{V_z^{\,2}}{2} = \lambda_9 G\, z$ $V_z^{\,2} = 2\lambda_9 G\, z$ $V_z = \pm\sqrt{2\lambda_9 G\, z} + C_{27}$

Step 3 : One Collects the integrals of the sub-equations, above, for $V_x, V_y, V_z, P(x), P(y), P(z)$

For V_x, $P(x)$	For V_y, $P(y)$	For V_z, $P(z)$
Sum of integrals from sub-equations #4, #7, #8, #9, #1,	**S**um of integrals from sub-equations #5, #10, #11, #12, #2.	**S**um of integrals from sub-equations #6, #13, #14, #15, #3,
$V_x = \beta_4 Gt + C_{22}$	$V_y = \beta_5 Gt + C_{21}$	$V_z = \beta_6 Gt + C_{24}$
$V_x = \pm\sqrt{2\lambda_1 G\, x} + C_{25}$	$V_y = \dfrac{\lambda_4 G\, x}{V_x} + \dfrac{\psi_x(V_x)}{V_x}$	$V_z = \dfrac{\lambda_7 G\, x}{V_x} + \dfrac{\psi_x(V_x)}{V_x}$
$V_x = \dfrac{\lambda_2 G\, y}{V_y} + \dfrac{\psi_y(V_y)}{V_y}$	$V_y = \pm\sqrt{2\lambda_5 G\, y} + C_{26}$	$V_z = \dfrac{\lambda_8 G\, y}{V_y} + \dfrac{\psi_y(V_y)}{V_y}$
$V_x = \dfrac{\lambda_3 G\, z}{V_z} + \dfrac{\psi_z(V_z)}{V_z}$	$V_y = \dfrac{\lambda_6 G\, z}{V_z} + \dfrac{\psi_z(V_z)}{V_z}$	$V_z = \pm\sqrt{2\lambda_9 G\, z} + C_{27}$
$P(x) = \beta_1 \rho G x + C_{19}$	$P(y) = \beta_2 \rho G y + C_{20}$	$P(z) = \beta_3 \rho G z + C_{21}$

From above**,**

For V_x, Sum of integrals from sub-equations #4, #7, #8, #9, #1,

$$V_x(x,y,z,t) = \beta_4 G\mathrm{t} \pm \sqrt{2\lambda_1 G\, x} + \frac{\lambda_2 G\, y}{V_y} + \frac{\lambda_3 G\, z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \; ; \qquad P(x) = \beta_1 \rho G x + C_{19}$$

For V_y : Sum of integrals from sub-equations #5, #10, #11, #12, #2.

$$V_y(x,y,z,t) = \beta_5 Gt + \pm\sqrt{2\lambda_5 G\, y} + \frac{\lambda_4 G\, x}{V_x} + \frac{\lambda_6 G\, z}{V_z} + \underbrace{\frac{\psi_x(V_x)}{V_x} \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \; ; \qquad P(y) = \beta_2 \rho G y + C_{20}$$

For V_z: Sum of integrals from sub-equations #6, #13, #14, #15, #3,

$$V_z = \beta_6 Gt \pm \sqrt{2\lambda_9 G\, z} + \frac{\lambda_7 G\, x}{V_x} + \frac{\lambda_8 G\, y}{V_y} + \underbrace{\frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}}_{\text{arbitrary functions}} \qquad P(z) = \beta_3 \rho G z + C_{21}$$

Step 4: Simplify the sums of the integrals from above..(**Method 2 solutions of N-S equations**

$$V_x(x,y,z,t) = \beta_4 Gt \pm \sqrt{2\lambda_1 G\, x} + \frac{\lambda_2 G\, y}{V_y} + \frac{\lambda_3 G\, z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$P(x) = \beta_1 \rho G x + C_{19} \qquad (V_y \neq 0,\ V_z \neq 0)$$

$$V_y(x,y,z,t) = \beta_5 Gt \pm \sqrt{2\lambda_5 G\, y} + \frac{\lambda_4 G\, x}{V_x} + \frac{\lambda_6 G\, z}{V_z} + \underbrace{\frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$P(y) = \beta_2 \rho G y + C_{20} \qquad (V_x \neq 0,\ V_z \neq 0)$$

$$V_z(x,y,z,t) = \beta_6 Gt \pm \sqrt{2\lambda_9 G z} + \frac{\lambda_7 G x}{V_x} + \frac{\lambda_8 G y}{V_y} + \underbrace{\frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}}_{\text{arbitrary functions}}$$

$$P(z) = \beta_3 \rho G z + C_{21} \qquad (V_y \neq 0,\ V_y \neq 0)$$

The above are solutions for V_x V_y, V_z $P(x)$, $P(y)$, $P(z)$ .of the Euler Equations

Comparison of Method 1 and Method 2 of Solutions of Euler Equations

Method 1: x–direction solution of Euler equation

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C_9 \qquad \text{(A)}$$

$$P(x) = d\rho g_x x;\ (d + f + h + n + q = 1) \quad (V_y \neq 0,\ V_z \neq 0)$$

Method 2: x–direction solution of Eulerequation

$$V_x(x,y,z,t) = \beta_4 Gt \pm \sqrt{2\lambda_1 G x} + \frac{\lambda_2 G y}{V_y} + \frac{\lambda_3 G z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \qquad \textbf{(B)}$$

$$P(x) = \beta_1 \rho G x + C_{19} \qquad (V_y \neq 0,\ V_z \neq 0)$$

It is pleasantly surprising that the above solutions (A) and (B) are almost identical (except for the constants), even though they were obtained by different approaches as in Method 1 and Method 2. . For the system of Euler equations, there are three driver equations, since each equation contains the gravity term. In Method 1, the gravity term was ρg. In Method 2, the gravity term was ρG, where G is the magnitude of the vector sum of the gravity terms. Note that in Method 1, the sum of the ratio terms (7 ratio terms for each equation) equals unity, but in Method 2, the sum of the ratio terms (15 ratio terms) for the single driver equation solved equals unity. Note that in Method 2, only a single "driver" equation was solved, but in Method 1, three "driver" equations were solved. In Method 2, one could say that the system of Euler equations was "more simultaneously" solved than in Method 1.

About integrating only a single equation

If one asked for help in solving the Euler equations, and one was told to add the three equations together and then solve them, one would think that one was being given a nonsensical advice; but now, after studying Method 2 above, one would appreciate such a suggestion.

References:

For paper edition of the above paper, see the book entitled "Power of Ratios" by A. A. Frempong, published by Yellowtextbooks.com. Without using ratios or proportion, the author would never be able to split-up the Euler equations into sub-equations which were readily integrable. The impediment to solving the Euler equations for over 150 years has been due to finding a way to split-up the equations. Since ratios were the key to splitting the Euler equations and solving them, the solutions have also been published in the " Power of Ratios" book which covers definition of ratio and applications of ratio in mathematics, science, engineering, economics and business fields.