Title: A simple derivation of the stationary action formulation of classical mechanics using differential forms

Authors: Andrew Grieco

Address: Department of Electrical and Computer Engineering, University of California, San Diego, 9500 Gilman Drive, San Diego, CA 92093, USA

E-mail: agrieco@eng.ucsd.edu

Abstract
This manuscript describes the stationary action formulation of classical mechanics as a constrained extremization problem using differential forms. The general method entails treating the parameterization relationship as a constraint applied to the action integral, and the conditions that arise from writing the equivalent unconstrained action are then associated with the force law to derive the Lagrangian and Hamiltonian terms.

Keywords
differential forms, classical mechanics, stationary action

1. Introduction

This manuscript describes the stationary action formulation of classical mechanics as a constrained extremization problem using differential forms. The essence of the method involves treating the relationship between parameters of the action integral as a constraint. Since the stationary action formulation constitutes an extremization problem, it is straightforward to incorporate the constraint using the method of Lagrange multipliers. The conditions that arise from writing the equivalent unconstrained action are then associated with the Newton's second law, from which the Lagrangian and Hamiltonian terms naturally emerge. The advantage of this approach is the elegant conceptual framework, which leads to concise derivations.

The remainder of the manuscript is arranged as follows. Section 2 contains an overview of the Euler-Lagrange equation. Section 3 describes the conventional formulation of classical mechanics, as well as an alternative formulation. Section 4 contains a short overview and conclusion. The Appendix demonstrates how the method may be applied in the context of relativity.

2. The Euler-Lagrange Equation

It is most appropriate to begin by solving the Euler-Lagrange equation (see Ch. 6 in [1]) in terms of differential forms. The objective is to determine under what condition the following integral is extremized:

$$S = \int L(x(t), v(t), t) \wedge dt \,. \tag{1}$$

The quantity S is known as the action. The quantity L is known as the Lagrangian, and is taken to be a function of parameters x, v, and t that bear the relationship:

$$dx - v \wedge dt = 0 \,. \tag{2}$$

Here the d operator represents the exterior derivative, and the $\wedge$ operator is the exterior product. In the context of mechanics it is convenient to think of x and v as the position and velocity of the mass, and t as time. However, in the broadest sense they represent generalized coordinates (see Ch. 1 in [2]).

The extremization condition may be formulated by observing that small deviations from the extremal path integral should not produce changes in the value of the action. Consequently a closed path integral ∂P within the vicinity of the extremal path should vanish (see p. 125 in [3]). This condition may be simplified using Stokes' theorem (see Ch. 3.4 in [4]):

$$0 = \int_{\partial P} L \wedge dt = \int_{P} d(L \wedge dt) = \int_{P} dL \wedge dt = dS\,. \tag{3}$$

Since the closed surface P is arbitrary this condition reduces to:

$$d(L \wedge dt) = dL \wedge dt = 0\,. \tag{4}$$

Note that this is equivalent to requiring that the exterior derivative of the action vanish along an open path integral, provided the endpoints of the integral are fixed. Consequently the action is a conserved quantity along such a path.

The extremization condition of (4) is subject to the parameterization relationship (2). Together these constitute a constrained extremization problem that may be solved using the method of Lagrange multipliers (see Ch. 5.4 in [4]). In this case the Lagrange multiplier λ obeys the expression:

$$d(L \wedge dt) = \lambda \wedge d(dx - v \wedge dt)\,. \tag{5}$$

By inspection it is obvious that:

$$\lambda = -\frac{\partial L}{\partial v}\,. \tag{6}$$

With this information it is possible to combine the Lagrangian and the parameterization relationship to obtain a single unconstrained expression for the action. When the unconstrained Lagrangian equation is substituted into (1), the unconstrained action S_u becomes:

$$S_u = \int \left[L \wedge dt - \lambda \wedge (dx - v \wedge dt) \right] = \int \left[\left(L - v \frac{\partial L}{\partial v} \right) \wedge dt + \frac{\partial L}{\partial v} \wedge dx \right]. \tag{7}$$

The desired solution may be recovered by performing the extremization procedure on the unconstrained action. In this case the analogue of the condition in (4) becomes:

$$
\begin{aligned}
0 &= d\left[\left(L - v\frac{\partial L}{\partial v}\right)\wedge dt + \frac{\partial L}{\partial v}\wedge dx\right] \\
&= \frac{\partial L}{\partial x}\wedge dx \wedge dt + d\left(\frac{\partial L}{\partial v}\right)\wedge\left(dx - v\wedge dt\right). \\
&= \left[-\frac{\partial L}{\partial x}\wedge dt + d\left(\frac{\partial L}{\partial v}\right)\right]\wedge\left(dx - v\wedge dt\right)
\end{aligned} \tag{8}
$$

This expression is only satisfied when the leading term in the last line vanishes, which results in the solution of the Euler-Lagrange equation:

$$
d\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x}\wedge dt\,. \tag{9}
$$

Although the extremization condition is formulated differently, the remaining algebra of this section is essentially the same as that found in Ch. 35 of [5]. Note that since each spatial dimension is represented by a constraint it is trivial to extend this approach to additional dimensions simply by including additional Lagrange multipliers.

3. Classical Mechanics

3.1 Conventional Formulation

The connection between the Euler-Lagrange equation and classical mechanics is made by comparing (9) with Newton's second law that relates the total force F and momentum $p = mv$ of an object with mass m (see Ch. 7 in [1]):

$$
d(mv) = F\wedge dt\,. \tag{10}
$$

The force is therefore the product of mass and acceleration $F = ma$. This expression is very general and can accommodate non-inertial reference frames by including pseudo force contributions to the total force (see Ch. 9 in [1]). Given the similarity of (9) and (10), it is clear that the equations of motion are equivalent to the solution of the Euler-Lagrange equation when the following relationships hold:

$$
\begin{aligned}
\frac{\partial L}{\partial v} &= mv \\
\frac{\partial L}{\partial x} &= F
\end{aligned}\,. \tag{11}
$$

Consequently the equations of motion may be derived from a principle of stationary action.

It is easy to determine the action that results in the proper equations of motion. First derive the expression for the Lagrangian by integrating (11):

$$
L = \frac{mv^2}{2} - \int -F\wedge dx + g(t)\,. \tag{12}
$$

The first term in the Lagrangian is the kinetic energy and the second term is the potential energy. The quantity g is an arbitrary time-dependent function. The expression for both the constrained (1) and unconstrained (7) forms of the action follows from the substitution of the Lagrangian in (12):

$$\begin{aligned} S &= \int \left(\frac{mv^2}{2} - \int -F \wedge dx + g \right) \wedge dt \\ S_u &= \int \left[\left(-\frac{mv^2}{2} - \int -F \wedge dx + g \right) \wedge dt + mv \wedge dx \right] \end{aligned} . \tag{13}$$

By inspection of the integrals (7) and (13) the partial derivatives of the unconstrained action may be identified:

$$\begin{aligned} \frac{\partial S_u}{\partial t} &= L - v\frac{\partial L}{\partial v} = -\left(\frac{mv^2}{2} + \int -F \wedge dx \right) + g = -H + g \\ \frac{\partial S_u}{\partial x} &= \frac{\partial L}{\partial v} = mv = p \end{aligned} . \tag{14}$$

The first expression in (14) entails that the time derivative of the unconstrained action may be related to the total energy which is also known as the Hamiltonian *H*. This may be recognized as the Hamilton-Jacobi equation (see Ch. 10 in [2]). The second expression in (14) entails that the spatial derivative of the unconstrained action is the momentum.

3.2 Alternative Formulation

From a philosophical standpoint a few aspects of the conventional formulation are uncomfortable. For example, the force law (10) takes a form similar to that of parameterization relationship (2). However, only the parameterization relationship is treated as a constraint for the method of determining Lagrange multipliers. It is also disappointing that energy conservation is only recovered when additional conditions are placed on the action.

It is possible to formulate the principle of stationary action in an alternative manner that accommodates these objections. In particular this leads to a less ambiguous formulation of energy conservation. Intuitively the new method arises from a motivation to eliminate the inconsistent treatment of constraints. Unfortunately, treating the two constraints on equal footing is intractable. Since it is unfeasible to treat the two different constraints simultaneously, in the alternative method one is abandoned. Specifically the force law is retained in favor of the parameterization relationship. This can be justified by noting that the force law determines the acceleration and velocity, which is sufficient to infer the position. Consequently, the explicit relationship between the position and velocity posed by the parameterization constraint is tautological. Equivalently, this approach might be interpreted as reparameterizing the Lagrangian such that the parameterization constraint and the force law constraint become identical.

For this approach the most appropriate parameters are momentum, force, and time. From Newton's second law (10) the force law constraint may be written:

$$dp - F \wedge dt = 0 . \tag{15}$$

Consequently the objective is to extremize the following action:

$$S=\int L(p(t),F(t),t)\wedge dt\,. \tag{16}$$

The same extremization procedure is followed as before. The solution may be obtained by inspection through the comparison of the expression in (16) with the expressions in (1) and (9):

$$d\left(\frac{\partial L}{\partial F}\right)-\frac{\partial L}{\partial p}\wedge dt=0\,. \tag{17}$$

The solution in (17) is the relationship that must exist for the action to be extremal when subject to the force law constraint. Notably, it has the same form as that of the constraint. The next step in this approach is simply to demand a self-consistency whereby the solution of the unconstrained action reproduces the constraint applied to the constrained action. Applying this self-consistency principle, it is clear that expression (17) and the force law constraint (15) are equivalent when the following relationships hold:

$$\begin{aligned}\frac{\partial L}{\partial F}&=p\\ \frac{\partial L}{\partial p}&=F\end{aligned}\,. \tag{18}$$

It is easy to determine the action that results in the proper equations of motion. First derive the expression for the Lagrangian by integrating (18):

$$L=pF+g(t). \tag{19}$$

The quantity g is an arbitrary time-dependent function.

It is simple to determine the constrained and unconstrained forms of the alternative extremal action (16) by substituting the Lagrangian in (19) into expressions in (1) and (7):

$$\begin{aligned}S&=\int(pF+g)\wedge dt=\int pF\wedge dt+\int g\wedge dt\\ S_u&=\int(g\wedge dt+p\wedge dp)=\int g\wedge dt+\frac{p^2}{2}-C\end{aligned}\,. \tag{20}$$

Here the quantity C is a constant of integration that appears when the evaluated integral is left indefinite. Energy conservation arises from comparing the distinct but equivalent forms of the action. To see this subtract the integrals in (20) and rearrange the terms:

$$C+S_u-S=\frac{p^2}{2}+\int -pF\wedge dt=\frac{m^2v^2}{2}+\int -m^2va\wedge dt=mH\,. \tag{21}$$

Along the extremal path the actions are constant and equal to one another. The nonconstant terms in (21) may be recognized as the product of kinetic energy and mass, and the product of potential energy and mass. Consequently the expression may be identified as the product of total energy and mass. Energy conservation arises by evaluating the definite form of the integral in (21) for two points of the extremal path given by times t_1 and t_2:

$$mH(t_2) = mH(t_1). \tag{22}$$

Since the starting and ending times are arbitrary the quantities on either side of (22) must be conserved along the entire extremal path. The mass may be divided out, leaving only the total energy terms.

4. Conclusion

The previous sections detail the formulation of classical mechanics in terms of a constrained extremal action principle through the use of differential forms. The constrained action approach provides a general conceptual framework that is readily extended to relativistic mechanics (see the Appendix). Specifically the method entails casting the relationship between parameters of the action integral as constraint. The conditions that arise from writing the equivalent unconstrained action are then associated with the force law, which leads to the Lagrangian and Hamiltonian terms. An alternative approach is also presented that may be interpreted in terms of a self-consistency principle. The alternative method results in a less ambiguous formulation of energy conservation.

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Appendix: Relativity

A.1 Conventional Formulation

The same approach may be applied in the context of special relativity. Mathematically the only modification necessary is to replace the momentum and force with their relativistic equivalents that arise as a consequence of Lorentz invariance. The derivation begins identically to the classical case, since the relativistic corrections appear only in the force law (see Ch. 15 in [1]). The relativistic version of Newton's second law assumes the form:

$$\begin{aligned} d(\gamma m v) &= F_r \wedge dt \\ \gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned} \quad . \tag{A1}$$

Here c is the vacuum speed of light, m is the rest mass, the relativistic momentum is $p_r = \gamma m v$, and the relativistic force is $F_r = \gamma^3 m a$. Given the similarity between the relativistic force law and the formulation of the problem leading to the solution of the extremized action in (9) it is clear from inspection that the relativistic Lagrangian may be determined by the following relationships:

$$\begin{aligned}\frac{\partial L}{\partial v} &= \gamma m v \\ \frac{\partial L}{\partial x} &= F_r\end{aligned}.$$

(A2)

The action that results in the proper equations of motion may be determined as before, although the integrals are more difficult. First derive the expression for the Lagrangian by integrating (A2):

$$L = -\frac{mc^2}{\gamma} - \int -F_r \wedge dx + g(t).$$

(A3)

Similar to before the second term in the Lagrangian is the potential energy and the quantity g is an arbitrary time-dependent function. The first term is not properly identified as the kinetic energy in this instance, however. From the Lagrangian in (A3) it is easy to derive the expression for both the constrained and unconstrained forms of the action using the relativistic equivalents of (1) and (7):

$$\begin{aligned}S &= \int\left(-\frac{mc^2}{\gamma} - \int -F_r \wedge dx + g\right)\wedge dt \\ S_u &= \int\left[\left(-\gamma mc^2 - \int -F_r \wedge dx + g\right)\wedge dt + \gamma m v \wedge dx\right]\end{aligned}.$$

(A4)

By inspection of the integrals in (A4) the partial derivatives of the unconstrained action may be identified:

$$\begin{aligned}\frac{\partial S_u}{\partial t} &= L - v\frac{\partial L}{\partial v} = -\gamma mc^2 - \int -F_r \wedge dx + g = -H_r + g \\ \frac{\partial S_u}{\partial x} &= \frac{\partial L}{\partial v} = \gamma m v = p_r\end{aligned}.$$

(A5)

Similar to before the first expression in (A5) entails that the time derivative of the unconstrained action may be related to the total relativistic energy H_r and may be recognized as the relativistic Hamilton-Jacobi equation. Likewise the second expression in (A5) entails that the spatial derivative of the unconstrained action is the relativistic momentum.

A.2 Alternative Formulation

The alternative method is also amenable to relativistic corrections in the same vein. Given the similarity between the relativistic force law (A1), and the alternative formulation of the problem leading to the solution of the extremized action in (17) it is clear that they are equivalent when the following relationships hold:

$$\frac{\partial L}{\partial F_r} = p_r$$
$$\frac{\partial L}{\partial p_r} = F_r \; .$$
(A6)

The action that results in the proper equations of motion may be determined as before, although the integrals are more difficult. First derive the expression for the Lagrangian by integrating (A6):

$$L = p_r F_r + g(t).$$
(A7)

The quantity $g(t)$ is an arbitrary time-dependent function. From the Lagrangian in (A7) it is possible to write expressions for the relativistic equivalents of both the constrained and unconstrained forms of the extremal action analogous to the expressions in (20):

$$S = \int (p_r F_r + g) \wedge dt = \int p_r F_r \wedge dt + \int g \wedge dt$$
$$S_u = \int (g \wedge dt + p_r \wedge dp_r) = \int g \wedge dt + \frac{p_r^2}{2} - C \; .$$
(A8)

Here the quantity C is a constant of integration that appears when the evaluated integral is left indefinite.

Energy conservation arises as before, although some effort must be made to arrange the equation in the anticipated form. To see this subtract the distinct but equivalent forms of the action in (A8) and rearrange the terms:

$$C + S_u - S = \frac{p_r^2}{2} + \int -p_r F_r \wedge dt = \frac{\gamma^2 m^2 v^2}{2} + \int -\gamma^4 m^2 va \wedge dt \; .$$
(A9)

This may be rewritten as follows:

$$C + S_u - S = \frac{\gamma^2 m^2 v^2}{2} + \left(\gamma m^2 c^2 - \frac{\gamma^2 m^2 v^2}{2} \right) - \left(\gamma m^2 c^2 - \frac{\gamma^2 m^2 v^2}{2} \right) + \int -\gamma^4 m^2 va \wedge dt \, .$$
(A10)

Next gather terms and simplify the expression using the following integral:

$$d\left(\gamma m^2 c^2 - \frac{\gamma^2 m^2 v^2}{2} \right) = \gamma^3 m^2 va \wedge dt - \gamma^4 m^2 va \wedge dt \; .$$
(A11)

Combined with (A11) the expression in (A10) reduces to:

$$C + S_u - S = \gamma m^2 c^2 + \int -\gamma^3 m^2 va \wedge dt = mH_r \, .$$
(A12)

Along the extremal path the actions are constant and equal to one another. The nonconstant terms in (A12) may be recognized as the product of the total relativistic energy and rest mass. Energy conservation arises by evaluating the definite form of the integral in (A12) for two points of the extremal path given by times t_1 and t_2:

$$mH_r(t_2) = mH_r(t_1). \tag{A13}$$

Since the starting and ending times are arbitrary the quantities on either side of (A13) must be conserved along the entire extremal path. The rest mass may be divided out, leaving only the total relativistic energy terms.

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