Multiple attribute decision-making method under hesitant interval neutrosophic linguistic environment

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Abstract: Motivated by the ideas of interval neutrosophic linguistic sets (INLSs) and hesitant fuzzy sets (HFSs), this paper proposes the concept of hesitant interval neutrosophic linguistic sets (HINLSs) and defines the operational laws of hesitant interval neutrosophic linguistic elements (HINLEs) and the score, accuracy and certainty functions for HINLEs. Then, a hesitant interval neutrosophic linguistic weighted average (HINLWA) operator and a hesitant interval neutrosophic linguistic weighted geometric (HINLWG) operator are developed to aggregate the hesitant interval neutrosophic linguistic information. Moreover, some desirable properties of the two operators are investigated. A decision-making method based on the HINLWA and HINLWG operators is developed to handle multiple attribute decision making problems, in which attribute values with respect to alternatives take the form of HINLEs, under a hesitant interval neutrosophic linguistic environment. Finally, an illustrative example about investment alternatives is given to demonstrate the application of the developed method.

Keywords: Hesitant interval neutrosophic linguistic set; Hesitant interval neutrosophic linguistic

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element; Score function; Hesitant interval neutrosophic linguistic weighted average (HINLWA) operator; Hesitant interval neutrosophic linguistic weighted geometric (HINLWG) operator; Decision making

1. Introduction

The neutrosophic set proposed firstly by Smarandache [1] generalizes an intuitionistic fuzzy set and an interval-valued intuitionistic fuzzy set from philosophical point of view and its functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e., $T_A(x)$: $X \rightarrow]^-0, 1^+[$, $I_A(x)$: $X \rightarrow]^-0, 1^+[$, and $F_A(x)$: $X \rightarrow]^-0, 1^+[$. The neutrosophic set can be better to express incomplete, indeterminate and inconsistent information. However, the nonstandard interval $]^-0, 1^+[$ is difficult to apply in real scientific and engineering areas. Hence, some researchers have introduced some subclasses of the neutrosophic set to easily apply in real scientific and engineering areas by constraining the nonstandard interval $]^-0, 1^+[$ into the real standard interval [0, 1] for its functions $T_A(x)$, $I_A(x)$ and $F_A(x)$. Firstly, Wang et al. [2, 3] introduced the concepts of an interval neutrosophic set (INS) and a single-valued neutrosophic set (SVNS), which are the subclasses of a neutrosophic set, and provided the set-theoretic operators and various properties of SVNSs and INSs. Then, Ye [4] proposed a correlation coefficient of SVNSs and applied it to multiple attribute decision-making problems with single-valued neutrosophic information. Ye [5] presented a single-valued neutrosophic cross-entropy measure for single-valued neutrosophic multiple attribute decision-making problems. Liu and Wang [6] presented single-valued neutrosophic normalized weighted Bonferroni mean operators and applied them to decision making problems with single-valued neutrosophic information. Further, Liu et al. [7] developed some generalized single-valued neutrosophic number Hamacher aggregation operators and their application to group decision making problems with single-valued neutrosophic information. On the other hand, Chi and Liu [8] extended a TOPSIS method to interval neutrosophic multiple attribute decision-making problems. Ye [9] introduced the distances-based similarity measures of INSs and their applications in multiple attribute decision-making under interval neutrosophic environment. Zhang et al. [10] put forward the score, accuracy, and certainty functions of an interval neutrosophic number (INN) and introduced the interval neutrosophic number weighted average (INNWA) operator and interval neutrosophic number weighted geometric (INNWG) operator for interval neutrosophic multiple attribute decision-making problems.

Currently, based on the combination of interval neutrosophic sets and linguistic variables [11], Ye [12] defined the concept of interval neutrosophic linguistic sets (INLSs) and the score, accuracy and certainty functions of an interval neutrosophic linguistic number (INLN), and then developed an interval neutrosophic linguistic weighted average (INLWA) operator and an interval neutrosophic linguistic weighted geometric (INLWG) operator to handle multiple attribute decision-making problems with interval neutrosophic linguistic information. Since a single-valued neutrosophic linguistic set is a special case of an interval neutrosophic linguistic set, Ye [13] proposed an extended TOPSIS method for multiple attribute group decision-making problems with single-valued neutrosophic linguistic numbers. Furthermore, by the combination of SVNSs and hesitant fuzzy sets (HFSs) [14, 15], Ye [16] presented a single-valued neutrosophic hesitant fuzzy set (SVNHFS), a single-valued neutrosophic hesitant fuzzy weighted average (SVNHFWA) operator and a single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator, then applied them to multiple attribute decision-making problems under a single-valued neutrosophic hesitant fuzzy environment. Then, Liu and Shi [17] further developed the generalized hybrid weighted average operator of interval neutrosophic hesitant set and its multiple attribute decision making method.

An INLN introduced in [12] contains the linguistic variable represented by decision maker's judgment to an evaluated object and the subjective evaluation value represented by an INN as the reliability of the given linguistic variable. However, in complex decision making problems, when decision makers give their assessments on attributes by the form of INLNs, they may also be hesitant among several possible INLNs. For example, for a predefined linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ = {extremely low, very low, low, medium, high, very high, extremely high}, evaluating the "growth" of a company, we can utilize a hesitant interval neutrosophic linguistic element (HINLE) $\{\langle s_2, ([0.7,0.8], [0.0,0.1], [0.1, 0.2])\rangle, \langle s_3, ([0.6,0.7], [0.1, 0.2], [0.1, 0.2])\rangle, \langle s_4, ([0.5,0.6], [0.2, 0.3], [0.2, 0.3])\rangle\}$ as its evaluation, where s_2, s_3 and s_4 indicate that the "growth" of a company may be "low", "medium" and "high", and the INNs "([0.7,0.8], [0.0,0.1], [0.1, 0.2])", "([0.6,0.7], [0.1, 0.2], [0.1, 0.2])" and "([0.5,0.6], [0.2, 0.3], [0.2, 0.3])" indicate that the "growth" of a company may contain truth degrees, indeterminacy degrees and falsity degrees belonging to s_2, s_3 and s_4, respectively. In this case, the existing methods are not suitable for dealing with the decision making problems with hesitant interval neutrosophic linguistic information. Motivated by the concepts of INLSs and HFSs, the purposes of this paper are: (1) to propose the concepts of a hesitant interval neutrosophic linguistic set (HINLS) and the HINLE which is composed of a set of INLNs, (2) to define the operational laws of HINLEs and the score, accuracy and certainty functions for HINLEs, (3) to propose a hesitant interval neutrosophic linguistic weighted average (HINLWA) operator and a hesitant interval neutrosophic linguistic weighted geometric (HINLWG) operator and to investigate their properties, and (4) to establish a decision-making method based on the HINLWA and HINLWG operators to handle multiple attribute decision-making problems with hesitant interval neutrosophic linguistic information. To do so, the rest of the paper is organized as follows. Section 2 briefly describes some concepts of INLSs and HFSs. In Section 3, we propose the concepts of HINLSs and HINLEs and define the operational laws of HINLEs and the score, accuracy and certainty functions for HINLEs. Section 4 develops the HINLWA and HINLWG operators and investigates their some properties. Section 5 establishes a multiple attribute decision-making approach based on the HINLWA and HINLWG operators and the score, accuracy and certainty functions. An illustrative example about investment alternatives is provided in Section 6. Section 7 gives conclusions and future research.

2. Preliminaries of INLSs and HFSs

In this section, some basic concepts related to INLSs and HFSs are briefly introduced to utilize the following analysis.

Let $S = \{s_0, s_1, \ldots, s_l\}$ be a finite ordered discrete linguistic term set with old cardinality, where s_i represents a possible value for a linguistic variable and l+1 is the cardinality of S. For example, when $l = 6$, we can give a linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ = {extremely low, very low, low, medium, high, very high, extremely high}.

In a linguistic term set S, any two linguistic variables s_i and s_j must satisfy the following properties [18, 19]:

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$,

(2) Negation operator is neg(s_i) = s_{l-i},

(3) Maximum operator is max(s_i, s_j) = s_i if $i > j$,

(4) Minimum operator is min(s_i, s_j) = s_j if $i > j$.

2.1. Interval neutrosophic linguistic sets

Ye [12] presented the concept of an INLS and gave its definition.

Definition 1 [12]. Let X be a finite universal set. An INLS in X is defined by

$$A = \left\{ \left\langle x, \left[s_{\theta(x)}, \left(T_A(x), I_A(x), F_A(x) \right) \right] \right\rangle \mid x \in X \right\},$$

where $s_{\theta(x)} \in S$, $T_A(x)$ = [inf $T_A(x)$, sup $T_A(x)$] $\subseteq$ [0, 1], $I_A(x)$ = [inf $I_A(x)$, sup $I_A(x)$] $\subseteq$ [0, 1] and $F_A(x)$ = [inf $F_A(x)$, sup $F_A(x)$] $\subseteq$ [0, 1] represent the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element x in X to the linguistic variable $s_{\theta(x)}$, respectively, with the condition 0 ≤ sup $T_A(x)$ + sup $I_A(x)$ + sup $F_A(x)$ ≤ 3 for any $x \in X$.

Then, the seven tuple $\langle s_{\theta(x)}$, ([inf $T_A(x)$, sup $T_A(x)$], [inf $I_A(x)$, sup $I_A(x)$,], [inf $F_A(x)$, sup $F_A(x)$])$\rangle$ in A is called an INLN. For convenience, an INLN can be represented as $a = \langle s_{\theta(a)}$, ([$T^L(a)$, $T^U(a)$], [$I^L(a)$, sup $I^U(a)$,], [$F^L(a)$, $F^U(a)$])$\rangle$.

Definition 2 [12]. Let $a_1 = \left\langle s_{\theta(a_1)}, \left([T^L(a_1), T^U(a_1)], [I^L(a_1), I^U(a_1)], [F^L(a_1), F^U(a_1)] \right) \right\rangle$ and $a_2 = \left\langle s_{\theta(a_2)}, \left([T^L(a_2), T^U(a_2)], [I^L(a_2), I^U(a_2)], [F^L(a_2), F^U(a_2)] \right) \right\rangle$ be two INLNs and λ ≥ 0, then the operational laws of INLNs are defined as follows:

(1) $$a_1 + a_2 = \left\langle s_{\theta(a_1)+\theta(a_2)}, \begin{pmatrix} [T^L(a_1) + T^L(a_2) - T^L(a_1)T^L(a_2), \\ T^U(a_1) + T^U(a_2) - T^U(a_1)T^U(a_2)], \\ [I^L(a_1)I^L(a_2), I^U(a_1)I^U(a_2)], \\ [F^L(a_1)F^L(a_2), F^U(a_1)F^U(a_2)] \end{pmatrix} \right\rangle,$$

(2) $a_1 \times a_2 = \left\langle s_{\theta(a_1)\times\theta(a_2)}, \begin{pmatrix} [T^L(a_1)T^L(a_2), T^U(a_1)T^U(a_2)], \\ [I^L(a_1)+I^L(a_2)-I^L(a_1)I^L(a_2), \\ I^U(a_1)+I^U(a_2)-I^U(a_1)I^U(a_2)], \\ [F^L(a_1)+F^L(a_2)-F^L(a_1)F^L(a_2), \\ F^U(a_1)+F^U(a_2)-F^U(a_1)F^U(a_2)] \end{pmatrix} \right\rangle$,

(3) $\lambda a_1 = \left\langle s_{\lambda\theta(a_1)}, \begin{pmatrix} [1-\left(1-T^L(a_1)\right)^\lambda, 1-\left(1-T^U(a_1)\right)^\lambda], \\ [\left(I^L(a_1)\right)^\lambda, \left(I^U(a_1)\right)^\lambda], [\left(F^L(a_1)\right)^\lambda, \left(F^U(a_1)\right)^\lambda] \end{pmatrix} \right\rangle$,

(4) $a_1^\lambda = \left\langle s_{\theta^\lambda(a_1)}, \begin{pmatrix} [\left(T^L(a_1)\right)^\lambda, \left(T^U(a_1)\right)^\lambda], [1-\left(1-I^L(a_1)\right)^\lambda, 1-\left(1-I^U(a_1)\right)^\lambda], \\ [1-\left(1-F^L(a_1)\right)^\lambda, 1-\left(1-F^U(a_1)\right)^\lambda] \end{pmatrix} \right\rangle$.

To rank INLNs, Ye [12] defined the score, accuracy and certainty functions of an INLN.

Definition 3 [12]. Let $a = \langle s_{\theta(a)}, ([T^L(a), T^U(a)], [I^L(a), I^U(a),], [F^L(a), F^U(a)])\rangle$ be an INLN. Then, the score, accuracy and certainty functions for the INLN a are defined, respectively, as follows:

$$E(a) = \frac{(4+T^L(a)-I^L(a)-F^L(a)+T^U(a)-I^U(a)-F^U(a))\theta(a)}{6l}, \qquad (1)$$

$$H(a) = \frac{(T^L(a)-F^L(a)+T^U(a)-F^U(a))\theta(a)}{2l}, \qquad (2)$$

$$C(a) = \frac{(T^L(a)+T^U(a))\theta(a)}{2l}. \qquad (3)$$

Definition 4 [12]. Let a_1 and a_2 be two INLNs. Then, the ranking method can be defined as follows:

(1) If $E(a_1) > E(a_2)$, then $a_1 \succ a_2$;

(2) If $E(a_1) = E(a_2)$ and $H(a_1) > H(a_2)$, then $a_1 \succ a_2$;

(3) If $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) > C(a_2)$, then $a_1 \succ a_2$;

(4) If $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) = C(a_2)$, then $a_1 = a_2$.

2. 2. Hesitant fuzzy sets

Torra and Narukawa [14] and Torra [15] firstly proposed the concept of a HFS, which is defined as follows.

Definition 5 [14, 15]. Let X be a fixed set, a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\},$$

where $h_A(x) = \bigcup_{\gamma_A(x)\in h_A(x)} \{\gamma_A(x)\}$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to A. For convenience, we call $h_A(x)$ a hesitant fuzzy element [20] denoted simply by h, which reads $h = \bigcup_{\gamma\in h}\{\gamma\}$ for $\gamma \in [0, 1]$.

According to the relationship between a hesitant fuzzy element and an intuitionistic fuzzy value, Xia and Xu [20] defined some operations on three hesitant fuzzy elements h, h_1, h_2 and a scale $\lambda \geq 0$:

(1) $h^{\lambda} = \bigcup_{\gamma\in h}\{\gamma^{\lambda}\}$,

(2) $\lambda h = \bigcup_{\gamma\in h}\{1-(1-\gamma)^{\lambda}\}$,

(3) $h_1 \oplus h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$,

(4) $h_1 \otimes h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2}\{\gamma_1\gamma_2\}$.

Definition 6 [20]. For a hesitant element h, $G(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two hesitant elements h_1 and h_2, if $G(h_1) > G(h_2)$, then $h_1 > h_2$; if $G(h_1) = G(h_2)$, then $h_1 = h_2$.

3. Hesitant interval neutrosophic linguistic set

Based on the combination of an INLS and a HFS, this section proposes the concepts of a HINLS, a HINLE, which is a basic element in a HINLS, and the operational laws of HINLEs, as well as the appropriate score, accuracy and certainty functions to be suitable for a HINLE.

Definition 7. Let X be a nonempty set of the universe and $S = \{s_0, s_1, \ldots, s_l\}$ be a finite ordered discrete linguistic set. Then, a HINLS in X can be expressed as the following mathematical symbol:

$$N = \{\langle x, \tilde{n}_N(x)\rangle \mid x \in X\},$$

where $\tilde{n}_N(x) = \bigcup_{a_N(x)\in\tilde{n}_N(x)}\{a_N(x)\}$ is a set of INLNs, denoting the possible INLNs of the element $x \in X$ to the set N, and $a_N(x) = \langle s_{\theta(x)}$, ([inf $T_N(x)$, sup $T_N(x)$], [inf $I_N(x)$, sup $I_N(x)$,], [inf $F_N(x)$, sup $F_N(x)$])$\rangle$ is an INLN. For convenience, $\tilde{n}_N(x) = \bigcup_{a_N(x)\in\tilde{n}_N(x)}\{a_N(x)\}$ in N is simply denoted by $\tilde{n} = \bigcup_{a\in\tilde{n}}\{a\}$, where $\tilde{n}$ is called a HINLE and $a = \langle s_{\theta(a)}$, ([$T^L(a)$, $T^U(a)$], [$I^L(a)$, $I^U(a)$,], [$F^L(a)$, $F^U(a)$])$\rangle$ is called an INLN. Then, N is the set of all HINLEs.

Definition 8. Let $\tilde{n}$, $\tilde{n}_1$ and $\tilde{n}_2$ be any three HINLEs and $\lambda \geq 0$, then the operational laws of HINLEs are defined as follows:

$$\text{(1)}\quad \tilde{n}_1 + \tilde{n}_2 = \bigcup_{a_1\in\tilde{n}_1, a_2\in\tilde{n}_2}\left\{\left\langle s_{\theta(a_1)+\theta(a_2)}, \begin{pmatrix} [T^L(a_1)+T^L(a_2)-T^L(a_1)T^L(a_2), \\ T^U(a_1)+T^U(a_2)-T^U(a_1)T^U(a_2)], \\ [I^L(a_1)I^L(a_2), I^U(a_1)I^U(a_2)], \\ [F^L(a_1)F^L(a_2), F^U(a_1)F^U(a_2)] \end{pmatrix}\right\rangle\right\},$$

$$\text{(2)}\quad \tilde{n}_1 \times \tilde{n}_2 = \bigcup_{a_1\in\tilde{n}_1, a_2\in\tilde{n}_2}\left\{\left\langle s_{\theta(a_1)\times\theta(a_2)}, \begin{pmatrix} [T^L(a_1)T^L(a_2), T^U(a_1)T^U(a_2)], \\ [I^L(a_1)+I^L(a_2)-I^L(a_1)I^L(a_2), \\ I^U(a_1)+I^U(a_2)-I^U(a_1)I^U(a_2)], \\ [F^L(a_1)+F^L(a_2)-F^L(a_1)F^L(a_2), \\ F^U(a_1)+F^U(a_2)-F^U(a_1)F^U(a_2)] \end{pmatrix}\right\rangle\right\},$$

$$\text{(3)}\quad \lambda\tilde{n} = \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\lambda\theta(a)}, \begin{pmatrix} [1-(1-T^L(a))^\lambda, 1-(1-T^U(a))^\lambda], \\ [(I^L(a))^\lambda, (I^U(a))^\lambda], [(F^L(a))^\lambda, (F^U(a))^\lambda] \end{pmatrix}\right\rangle\right\},$$

$$\text{(4)}\quad \tilde{n}^\lambda = \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta^\lambda(a)}, \begin{pmatrix} [(T^L(a))^\lambda, (T^U(a))^\lambda], [1-(1-I^L(a))^\lambda, 1-(1-I^U(a))^\lambda], \\ [1-(1-F^L(a))^\lambda, 1-(1-F^U(a))^\lambda] \end{pmatrix}\right\rangle\right\}.$$

Definition 9. Let $\tilde{n}$ be an HINLE. Then, the score, accuracy and certainty functions of the HINLE $\tilde{n}$ are defined, respectively, as follows:

$$E_H(\tilde{n}) = \frac{1}{\#\tilde{n}} \sum_{a \in \tilde{n}} \frac{(4 + T^L(a) - I^L(a) - F^L(a) + T^U(a) - I^U(a) - F^U(a))\theta(a)}{6l}, \tag{4}$$

$$H_H(\tilde{n}) = \frac{1}{\#\tilde{n}} \sum_{a \in \tilde{n}} \frac{(T^L(a) - F^L(a) + T^U(a) - F^U(a))\theta(a)}{2l}, \tag{5}$$

$$C_H(\tilde{n}) = \frac{1}{\#\tilde{n}} \sum_{a \in \tilde{n}} \frac{(T^L(a) + T^U(a))\theta(a)}{2l}. \tag{6}$$

where $\#\tilde{n}$ is the number of INLNs in $\tilde{n}$ and l+1 is the cardinality of the linguistic term set S.

Definition 10. Let $\tilde{n}_1$ and $\tilde{n}_2$ be any two HINLEs. Then, the ranking method can be defined as follows:

(1) If $E_H(\tilde{n}_1) > E_H(\tilde{n}_2)$, then $\tilde{n}_1 \succ \tilde{n}_2$;

(2) If $E_H(\tilde{n}_1) = E_H(\tilde{n}_2)$ and $H_H(\tilde{n}_1) > H_H(\tilde{n}_2)$, then $\tilde{n}_1 \succ \tilde{n}_2$;

(3) If $E_H(\tilde{n}_1) = E_H(\tilde{n}_2)$, $H_H(\tilde{n}_1) = H_H(\tilde{n}_2)$, and $C_H(\tilde{n}_1) > C_H(\tilde{n}_2)$, then $\tilde{n}_1 \succ \tilde{n}_2$;

(4) If $E_H(\tilde{n}_1) = E_H(\tilde{n}_2)$, $H_H(\tilde{n}_1) = H_H(\tilde{n}_2)$, and $C_H(\tilde{n}_1) = C_H(\tilde{n}_2)$, then $\tilde{n}_1 = \tilde{n}_2$.

4. Hesitant interval neutrosophic linguistic weighted aggregation operators

Based on the operational laws of HINLEs, we can propose two hesitant interval neutrosophic weighted aggregation operators to aggregate hesitant interval neutrosophic linguistic information, which are usually utilized in multiple attribute decision making.

4.1 Hesitant interval neutrosophic linguistic weighted average operator

Definition 11. Let $\tilde{n}_j$ (j = 1, 2, …, n) be a collection of HINLEs. The HINLWA operator is defined as

$$HINLWA(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \sum_{j=1}^{n} w_j \tilde{n}_j \tag{7}$$

where $\boldsymbol{W}$ = $(w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{n}_j$ (j =1, 2, …, n), $w_j \in [0, 1]$ and

$\sum_{j=1}^{n} w_j = 1$.

Theorem 1. Let $\tilde{n}_j$ (j = 1, 2, …, n) be a collection of HINLEs. Then by Eq. (7) and the operational laws of HINLEs, we can obtain the following result:

$$HINLWA(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \bigcup_{a_1 \in \tilde{n}_1, a_2 \in \tilde{n}_2, \ldots, a_n \in \tilde{n}_n} \left\{ \left\langle s_{\sum_{j=1}^{n} w_j \theta(a_j)}, \left(\left[1 - \prod_{j=1}^{n} (1 - T^L(a_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - T^U(a_j))^{w_j} \right], \right. \right. \right.$$
$$\left. \left. \left. \left[\prod_{j=1}^{n} \left(I^L(a_j) \right)^{w_j}, \prod_{j=1}^{n} \left(I^U(a_j) \right)^{w_j} \right], \left[\prod_{j=1}^{n} \left(F^L(a_j) \right)^{w_j}, \prod_{j=1}^{n} \left(F^U(a_j) \right)^{w_j} \right] \right) \right\rangle \right\}, \tag{8}$$

where $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{n}_j$ (j = 1, 2, …, n), $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof. The proof of Eq. (8) can be done by means of mathematical induction.

(1) When n = 2, then,

$$w_1 \tilde{n}_1 = \bigcup_{a_1 \in \tilde{n}_1} \left\{ \left\langle s_{w_1 \theta(a_1)} ([1 - (1 - T^L(a_1))^{w_1}, 1 - (1 - T^U(a_1))^{w_1}], [\left(I^L(a_1)\right)^{w_1}, \left(I^U(a_1)\right)^{w_1}], [\left(F^L(a_1)\right)^{w_1}, \left(F^U(a_1)\right)^{w_1}]) \right\rangle \right\},$$

$$w_2 \tilde{n}_2 = \bigcup_{a_2 \in \tilde{n}_2} \left\{ \left\langle s_{w_2 \theta(a_2)} ([1 - (1 - T^L(a_2))^{w_2}, 1 - (1 - T^U(a_2))^{w_2}], [\left(I^L(a_2)\right)^{w_2}, \left(I^U(a_2)\right)^{w_2}], [\left(F^L(a_2)\right)^{w_2}, \left(F^U(a_2)\right)^{w_2}]) \right\rangle \right\}.$$

Thus,

$$HINLWA(\tilde{n}_1, \tilde{n}_2) = w_1 \tilde{n}_1 + w_2 \tilde{n}_2$$
$$= \bigcup_{a_1 \in \tilde{n}_1, a_2 \in \tilde{n}_2} \left\{ \left\langle s_{\sum_{j=1}^{2} w_j \theta(a_j)}, ([1 - (1 - T^L(a_1))^{w_1} + 1 - (1 - T^L(a_2))^{w_2} - (1 - (1 - T^L(a_1))^{w_1})(1 - (1 - T^L(a_2))^{w_2}), \right. \right.$$
$$1 - (1 - T^U(a_1))^{w_1} + 1 - (1 - T^U(a_2))^{w_2} - (1 - (1 - T^U(a_1))^{w_1})(1 - (1 - T^U(a_2))^{w_2})],$$
$$\left. \left. [(I^L(a_1))^{w_1} (I^L(a_2))^{w_2}, (I^U(a_1))^{w_1} (I^U(a_2))^{w_2}], [(F^L(a_1))^{w_1} (F^L(a_2))^{w_2}, (F^U(a_1))^{w_1} (F^U(a_2))^{w_2}]) \right\rangle \right\}$$
$$= \bigcup_{a_1 \in \tilde{n}_1, a_2 \in \tilde{n}_2} \left\{ \left\langle s_{\sum_{j=1}^{2} w_j \theta(a_j)}, ([1 - (1 - T^L(a_1))^{w_1} (1 - T^L(a_2))^{w_2}, 1 - (1 - T^U(a_1))^{w_1} (1 - T^U(a_2))^{w_2}], \right. \right.$$
$$\left. \left. [\prod_{j=1}^{2} (I^L(a_j))^{w_j}, \prod_{j=1}^{2} (I^U(a_j))^{w_j}], [\prod_{j=1}^{2} (F^L(a_j))^{w_j}, \prod_{j=1}^{2} (F^U(a_j))^{w_j}]) \right\rangle \right\}.$$

(9)

(2) When $n = k$, by applying Eq. (8), we obtain

$$HINLWA(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_k) = \bigcup_{a_1\in\tilde{n}_1,a_2\in\tilde{n}_2,\ldots,a_k\in\tilde{n}_k}\left\{\left\langle s_{\sum_{j=1}^{k} w_j\theta(a_j)},\left([1-\prod_{j=1}^{k}(1-T^L(a_j))^{w_j},1-\prod_{j=1}^{k}(1-T^U(a_j))^{w_j}],\right.\right.\right. \tag{10}$$
$$\left.\left.\left.[\prod_{j=1}^{k}(I^L(a_j))^{w_j},\prod_{j=1}^{k}(I^U(a_j))^{w_j}],[\prod_{j=1}^{k}(F^L(a_j))^{w_j},\prod_{j=1}^{k}(F^U(a_j))^{w_j}]\right)\right\rangle\right\}$$

(3) When $n = k + 1$, by applying Eqs. (9) and (10), we get

$$HINLWA(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_{k+1}) = \bigcup_{a_1\in\tilde{n}_1,a_2\in\tilde{n}_2,\ldots,a_{k+1}\in\tilde{n}_{k+1}}\left\{\left\langle s_{\sum_{j=1}^{k+1} w_j\theta(a_j)},\left([1-\prod_{j=1}^{k}(1-T^L(a_j))^{w_j}+1-(1-T^L(a_{k+1}))^{w_{k+1}}\right.\right.\right.$$
$$-(1-\prod_{j=1}^{k}(1-T^L(a_j))^{w_j})(1-(1-T^L(a_{k+1}))^{w_{k+1}}),$$
$$1-\prod_{j=1}^{k}(1-T^U(a_j))^{w_j}+1-(1-T^U(a_{k+1}))^{w_{k+1}}$$
$$-(1-\prod_{j=1}^{k}(1-T^U(a_j))^{w_j})(1-(1-T^U(a_{k+1}))^{w_{k+1}})],$$
$$\left.\left.\left.[\prod_{j=1}^{k+1}(I^L(a_j))^{w_j},\prod_{j=1}^{k+1}(I^U(a_j))^{w_j}],[\prod_{j=1}^{k+1}(F^L(a_j))^{w_j},\prod_{j=1}^{k+1}(F^U(a_j))^{w_j}]\right)\right\rangle\right\}$$
$$= \bigcup_{a_1\in\tilde{n}_1,a_2\in\tilde{n}_2,\ldots,a_{k+1}\in\tilde{n}_{k+1}}\left\{\left\langle s_{\sum_{j=1}^{k+1} w_j\theta(a_j)},\left([1-\prod_{j=1}^{k+1}(1-T^L(a_j))^{w_j},1-\prod_{j=1}^{k+1}(1-T^U(a_j))^{w_j}],\right.\right.\right.$$
$$\left.\left.\left.[\prod_{j=1}^{k+1}(I^L(a_j))^{w_j},\prod_{j=1}^{k+1}(I^U(a_j))^{w_j}],[\prod_{j=1}^{k+1}(F^L(a_j))^{w_j},\prod_{j=1}^{k+1}(F^U(a_j))^{w_j}]\right)\right\rangle\right\}.$$

Therefore, from the above results, we have Eq. (8) for any *n*. This completes the proof. □

Especially, if $\boldsymbol{W} = (1/n, 1/n, \ldots, 1/n)^T$, then the *HINLWA* operator reduces to a hesitant interval neutrosophic linguistic average operator for HINLEs.

It is obvious that some desired properties of the *HINLWA* operator are given as follows:

(1) Idempotency: Let $\tilde{n}_j$ ($j = 1, 2, \ldots, n$) be a collection of HINLEs. If $\tilde{n}_j$ ($j = 1, 2, \ldots, n$) is

equal and $\tilde{n}_j = \tilde{n} = \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta(a)}, \left(T(a), I(a), F(a)\right)\right\rangle\right\}$ for j = 1, 2, . . . , n and T(a), T(a), T(a) ⊆ [0, 1], then there is $HINLWA\left(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n\right) = \tilde{n}$.

(2) Boundedness: Let $\tilde{n}_j$ (j = 1, 2, . . . , n) be a collection of HINLEs. If $s_{\theta^-} = \min_{1\le j\le n}\left\{s_{\theta(a_j)} \mid a_j \in \tilde{n}_j\right\}$, $s_{\theta^+} = \max_{1\le j\le n}\left\{s_{\theta(a_j)} \mid a_j \in \tilde{n}_j\right\}$, $T^- = \min_{1\le j\le n}\left\{T(a_j) \mid a_j \in \tilde{n}_j\right\}$, $T^+ = \max_{1\le j\le n}\left\{T(a_j) \mid a_j \in \tilde{n}_j\right\}$, $I^- = \min_{1\le j\le n}\left\{I(a_j) \mid a_j \in \tilde{n}_j\right\}$, $I^+ = \max_{1\le j\le n}\left\{I(a_j) \mid a_j \in \tilde{n}_j\right\}$, $F^- = \min_{1\le j\le n}\left\{F(a_j) \mid a_j \in \tilde{n}_j\right\}$, $F^+ = \max_{1\le j\le n}\left\{F(a_j) \mid a_j \in \tilde{n}_j\right\}$ for j = 1, 2, . . . , n, then there is $\left\langle s_{\theta^-}, \left(T^-, I^+, F^+\right)\right\rangle \le HINLWA\left(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n\right) \le \left\langle s_{\theta^+}, \left(T^+, I^-, F^-\right)\right\rangle$.

Proof.

(1) If $\tilde{n}_j = \tilde{n} = \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta(a)}, \left(T(a), I(a), F(a)\right)\right\rangle\right\}$ for j = 1, 2, . . . , n and T(a), T(a), T(a) ⊆ [0, 1], we have

$$
\begin{aligned}
HINLWA\left(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n\right) &= \bigcup_{a_1\in\tilde{n}_1, a_2\in\tilde{n}_2, \ldots, a_n\in\tilde{n}_n}\left\{\left\langle s_{\sum_{j=1}^{n} w_j\theta(a_j)}, \left([1-\prod_{j=1}^{n}(1-T^L(a_j))^{w_j}, 1-\prod_{j=1}^{n}(1-T^U(a_j))^{w_j}],\right.\right.\right. \\
&\qquad \left.\left.\left.[\prod_{j=1}^{n}(I^L(a_j))^{w_j}, \prod_{j=1}^{n}(I^U(a_j))^{w_j}], [\prod_{j=1}^{n}(F^L(a_j))^{w_j}, \prod_{j=1}^{n}(F^U(a_j))^{w_j}]\right)\right\rangle\right\} \\
&= \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta(a)\sum_{j=1}^{n} w_j}, \left(\begin{array}{l}[1-(1-T^L(a))^{\sum_{j=1}^{n} w_j}, 1-(1-T^U(a))^{\sum_{j=1}^{n} w_j}], \\ [(I^L(a))^{\sum_{j=1}^{n} w_j}, (I^U(a))^{\sum_{j=1}^{n} w_j}], [(F^L(a))^{\sum_{j=1}^{n} w_j}, (F^U(a))^{\sum_{j=1}^{n} w_j}]\end{array}\right)\right\rangle\right\}. \\
&= \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta(a)}, \left([T^L(a)), T^U(a))], [I^L(a), I^U(a)], [F^L(a), F^U(a)]\right)\right\rangle\right\} \\
&= \bigcup_{a\in\tilde{n}}\left\{\left\langle s_{\theta(a)}, \left(T(a), I(a), F(a)\right)\right\rangle\right\} = \tilde{n}
\end{aligned}
$$

(2) Since $s_{\theta^-} = \min_{1\le j\le n}\left\{s_{\theta(a_j)} \mid a_j \in \tilde{n}_j\right\}$, $s_{\theta^+} = \max_{1\le j\le n}\left\{s_{\theta(a_j)} \mid a_j \in \tilde{n}_j\right\}$, $T^- = \min_{1\le j\le n}\left\{T(a_j) \mid a_j \in \tilde{n}_j\right\}$, $T^+ = \max_{1\le j\le n}\left\{T(a_j) \mid a_j \in \tilde{n}_j\right\}$, $I^- = \min_{1\le j\le n}\left\{I(a_j) \mid a_j \in \tilde{n}_j\right\}$, $I^+ = \max_{1\le j\le n}\left\{I(a_j) \mid a_j \in \tilde{n}_j\right\}$, $F^- = \min_{1\le j\le n}\left\{F(a_j) \mid a_j \in \tilde{n}_j\right\}$, $F^+ = \max_{1\le j\le n}\left\{F(a_j) \mid a_j \in \tilde{n}_j\right\}$ for j = 1, 2, . . . , n, then there are $\theta^- \le \theta(a_j) \le \theta^+$, $T^- \le T(a_j) \le T^+$, $I^- \le I(a_j) \le I^+$, $F^- \le F(a_j) \le F^+$ for j = 1, 2, . . . , n. We have

$$\theta(a)=\sum_{j=1}^{n} w_j\theta(a_j)\geq\sum_{j=1}^{n} w_j\theta^{-}=\theta^{-},$$

$$T(a)=\left[1-\prod_{j=1}^{n}(1-T^{L}(a_j))^{w_j},1-\prod_{j=1}^{n}(1-T^{U}(a_j))^{w_j}\right]\geq\left[1-\prod_{j=1}^{n}(1-T^{L-})^{w_j},1-\prod_{j=1}^{n}(1-T^{U-})^{w_j}\right]=T^{-},$$

$$I(a)=\left[\prod_{j=1}^{n}(I^{L}(a_j))^{w_j},\prod_{j=1}^{n}(I^{U}(a_j))^{w_j}\right]\leq\left[\prod_{j=1}^{n}\left(I^{L+}\right)^{w_j},\prod_{j=1}^{n}\left(I^{U+}\right)^{w_j}\right]=I^{+},$$

$$F(a)=\left[\prod_{j=1}^{n}(F^{L}(a_j))^{w_j},\prod_{j=1}^{n}(F^{U}(a_j))^{w_j}\right]\leq\left[\prod_{j=1}^{n}\left(F^{L+}\right)^{w_j},\prod_{j=1}^{n}\left(F^{U+}\right)^{w_j}\right]=F^{+}.$$

Then, there are the following scores:

$$\frac{1}{\#\tilde{n}}\sum_{a\in\tilde{n}}\frac{\theta(a)\left(4+T^{L}(a)+T^{U}(a)-I^{L}(a)-I^{U}(a)-F^{L}(a)-F^{U}(a)\right)}{6l}=$$

$$\frac{1}{\#\tilde{n}}\sum_{a\in\tilde{n}}\left\{\frac{1}{6l}\left(\sum_{j=1}^{n}w_j\theta(a_j)\right)\left(\begin{array}{l}4+1-\prod_{j=1}^{n}(1-T^{L}(a_j))^{w_j}+1-\prod_{j=1}^{n}(1-T^{U}(a_j))^{w_j}\\-\prod_{j=1}^{n}(I^{L}(a_j))^{w_j}-\prod_{j=1}^{n}(I^{U}(a_j))^{w_j}-\prod_{j=1}^{n}(F^{L}(a_j))^{w_j}-\prod_{j=1}^{n}(F^{U}(a_j))^{w_j}\end{array}\right)\right\}$$

$$\geq\frac{\theta^{-}\left(4+T^{L-}+T^{U-}-I^{L+}-I^{U+}-F^{L+}-F^{U+}\right)}{6l},$$

where $\#\tilde{n}$ is the number of INLNs in $\tilde{n}=HINLWA(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n)$ and l+1 is the cardinality of the linguistic term set S. Therefore, according to Definition 10, there is $\left\langle s_{\theta^{-}},\left(T^{-},I^{+},F^{+}\right)\right\rangle\leq HINLWA(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n)$.

Similarly, there is $HINLWA(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n)\leq\left\langle s_{\theta^{+}},\left(T^{+},I^{-},F^{-}\right)\right\rangle$.

Thus, there is $\left\langle s_{\theta^{-}},\left(T^{-},I^{+},F^{+}\right)\right\rangle\leq HINLWA(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n)\leq\left\langle s_{\theta^{+}},\left(T^{+},I^{-},F^{-}\right)\right\rangle$.

Hence, we complete the proofs of these properties. □

4.2 Hesitant interval neutrosophic linguistic weighted geometric operator

Definition 12. Let $\tilde{n}_j$ (j = 1, 2, …, n) be a collection of HINLEs. Then the HINLWG operator is defined as

$$HINLWG(\tilde{n}_1,n_2,\cdots,\tilde{n}_n)=\prod_{j=1}^{n}\tilde{n}_j^{w_j}\qquad(11)$$

where $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{n}_j$ (j = 1, 2, …, n), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 2. Let $\tilde{n}_j$ (j = 1, 2, …, n) be a collection of HINLEs. by Eq. (11) and the operational laws of HINLEs, we have the following result:

$$HINLWG(\tilde{n}_1, \tilde{n}_{2,} \cdots, \tilde{n}_n) = \bigcup_{a_1 \in \tilde{n}_1, a_2 \in \tilde{n}_2, \ldots, a_n \in \tilde{n}_n} \left\{ \left\langle s_{\prod_{j=1}^{n} \theta^{w_j}(a_j)}, ([\prod_{j=1}^{n} (T^L(a_j))^{w_j}, \prod_{j=1}^{n} (T^U(a_j))^{w_j}], \right.\right.$$
$$\left.\left. [1 - \prod_{j=1}^{n} (1 - I^L(a_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - I^U(a_j))^{w_j}], [1 - \prod_{j=1}^{n} (1 - F^L(a_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - F^U(a_j))^{w_j}]) \right\rangle \right\}, \quad (12)$$

where $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{n}_j$ (j = 1, 2, …, n), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

The proof of Theorem 2 can be also given by the similar proof of Theorem 1 (omitted).

Especially, if $\boldsymbol{W} = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$, then the HINLWG operator reduces to a hesitant interval neutrosophic linguistic geometric operator.

It is obvious that some desired properties of the HINLWG operator are also given as follows:

(1) Idempotency: Let $\tilde{n}_j$ (j = 1, 2, . . . , n) be a collection of HINLEs. If $\tilde{n}_j$ (j = 1, 2, . . . , n) is equal and $\tilde{n}_j = \tilde{n} = \bigcup_{a \in \tilde{n}} \left\{ \left\langle s_{\theta(a)}, \left(T(a), I(a), F(a)\right) \right\rangle \right\}$ for j = 1, 2, . . . , n and $T(a)$, $T(a)$, $T(a) \subseteq$ [0, 1], then there is $HINLWG(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \tilde{n}$.

(2) Boundedness: Let $\tilde{n}_j$ (j = 1, 2, . . . , n) be a collection of HINLEs. If $s_{\theta^-} = \min_{1 \le j \le n} \{ s_{\theta(a_j)} \mid a_j \in \tilde{n}_j \}$, $s_{\theta^+} = \max_{1 \le j \le n} \{ s_{\theta(a_j)} \mid a_j \in \tilde{n}_j \}$, $T^- = \min_{1 \le j \le n} \{ T(a_j) \mid a_j \in \tilde{n}_j \}$, $T^+ = \max_{1 \le j \le n} \{ T(a_j) \mid a_j \in \tilde{n}_j \}$, $I^- = \min_{1 \le j \le n} \{ I(a_j) \mid a_j \in \tilde{n}_j \}$, $I^+ = \max_{1 \le j \le n} \{ I(a_j) \mid a_j \in \tilde{n}_j \}$, $F^- = \min_{1 \le j \le n} \{ F(a_j) \mid a_j \in \tilde{n}_j \}$, $F^+ = \max_{1 \le j \le n} \{ F(a_j) \mid a_j \in \tilde{n}_j \}$ for j = 1, 2, . . . , n, then there is $\left\langle s_{\theta^-}, \left(T^-, I^+, F^+\right) \right\rangle \le HINLWG(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) \le \left\langle s_{\theta^+}, \left(T^+, I^-, F^-\right) \right\rangle$.

Since the process to prove these properties is similar to the above proofs, it is not repeated here.

5. Decision-making method based on the HINLWA and HINLWG operators

This section proposes a multiple attribute decision-making method based on the HINLWA and HINLWG operators and the score, accuracy and certainty functions of HINLEs under a hesitant interval neutrosophic linguistic environment.

For a multiple attribute decision-making problem, let A = {A_1, A_2, …, A_m} be a set of alternatives and let C = {C_1, C_2, …, C_n} be a set of attributes. Assume that the weight of the attribute C_j (j = 1, 2, …, n), entered by the decision-maker, is w_j, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In the decision process, the evaluation information of the alternative A_i (i = 1, 2, …, m) on the attribute C_j (j = 1, 2, …, n) is represented by a hesitant interval neutrosophic linguistic decision matrix denoted by $D = \left(\tilde{n}_{ij}\right)_{n\times m}$, where $\tilde{n}_{ij}$ (i = 1, 2, …, m; j = 1, 2, …, n) is a HINLE $\tilde{n}_{ij} = \bigcup_{a_{ij}\in\tilde{n}_{ij}} \{a_{ij}\}$ and $a_{ij} = \left\langle s_{\theta(a_{ij})}, \left([T^L(a_{ij}), T^U(a_{ij})], [I^L(a_{ij}), I^U(a_{ij})], [F^L(a_{ij}), F^U(a_{ij})]\right)\right\rangle$ is an INLN.

Then, the HINLWA operator or the HINLWG operator is utilized to establish a multiple attribute decision making method under a hesitant interval neutrosophic linguistic environment, which includes the following steps:

Step 1: By applying Eq. (8) or Eq. (12), the individual overall HINLE $\tilde{n}_i$ for A_i (i = 1, 2, …, m) is calculated by

$$\tilde{n}_i = \bigcup_{a_i\in\tilde{n}_i} \{a_i\} = HINLWA(\tilde{n}_{i1}, \tilde{n}_{i2}, ..., \tilde{n}_{in})$$
$$= \bigcup_{a_{i1}\in\tilde{n}_{i1}, a_{i2}\in\tilde{n}_{i2}, ..., a_{in}\in\tilde{n}_{in}} \left\{\left\langle s_{\sum_{j=1}^{n} w_j\theta(a_{ij})}, \left(\begin{array}{l}\left[1-\prod_{j=1}^{n}(1-T^L(a_{ij}))^{w_j}, 1-\prod_{j=1}^{n}(1-T^U(a_{ij}))^{w_j}\right],\\ \left[\prod_{j=1}^{n}\left(I^L(a_{ij})\right)^{w_j}, \prod_{j=1}^{n}\left(I^U(a_{ij})\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(F^L(a_{ij})\right)^{w_j}, \prod_{j=1}^{n}\left(F^U(a_{ij})\right)^{w_j}\right]\end{array}\right)\right\rangle\right\},$$

c(13)

or

$$\tilde{n}_i = \bigcup_{a_i \in \tilde{n}_i} \{a_i\} = HINLWG(\tilde{n}_{i1}, \tilde{n}_{i2}, \ldots, \tilde{n}_{in})$$

$$= \bigcup_{a_{i1} \in \tilde{n}_{i1}, a_{i2} \in \tilde{n}_{i2}, \ldots, a_{in} \in \tilde{n}_{in}} \left\{ \left\langle s_{\prod_{j=1}^{n} \theta^{w_j}(a_{ij})}, \left(\begin{array}{l} \left[\prod_{j=1}^{n} \left(T^L(a_{ij})\right)^{w_j}, \prod_{j=1}^{n} \left(T^U(a_{ij})\right)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} (1 - I^L(a_{ij}))^{w_j}, 1 - \prod_{j=1}^{n} (1 - I^U(a_{ij}))^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} (1 - F^L(a_{ij}))^{w_j}, 1 - \prod_{j=1}^{n} (1 - F^U(a_{ij}))^{w_j} \right] \end{array} \right) \right\rangle \right\}. \quad (14)$$

Step 2: Calculate the values of the score function $E_H(\tilde{n}_i)$ (i = 1, 2, …, m) (accuracy function $H_H(\tilde{n}_i)$, certainty function $C_H(\tilde{n}_i)$) by using Eq. (4) (Eqs. (5) and (6)).

Step 3: Rank the alternatives according to the values of $E_H(\tilde{n}_i)$ ($H_H(\tilde{n}_i)$ and $C_H(\tilde{n}_i)$) (i = 1, 2, …, m) and then the largest score value is the best one.

Step 4: End.

6. Illustrative example

An illustrative example about investment alternatives adapted from Ye [12, 16] is used as the application of the proposed decision-making method under a hesitant interval neutrosophic linguistic environment. An investment company wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: A_1 (a car company), A_2 (a food company), A_3 (a computer company) and A_4 (an arms company). The investment company must take a decision according to the three attributes: C_1 (the risk), C_2 (the growth) and C_3 (the environmental impact). The vector of the attribute weights is given as $\boldsymbol{W}$ = (0.35, 0.25, 0.4)^T. Three decision makers are invited to evaluate the four possible alternatives of A_i (i = 1, 2, 3, 4) with respect to the three attributes of C_j (j = 1, 2, 3) by the form of HINLEs under the linguistic term set S = {s_1 = extremely poor, s_2 = very poor, s_3 = poor, s_4 = medium, s_5 = good, s_6 = very good, s_7 = extremely good}.

For example, the HINLE of an alternative A_1 with respect to an attribute C_1 is given as $\{\langle s_4$, ([0.5, 0.6], [0.1, 0.2], [0.2,0.3])$\rangle$, $\langle s_5$, ([0.3, 0.4], [0.2, 0.3], [0.3, 0.4])$\rangle\}$ by the three decision makers, which indicates that the assessment of the alternative A_1 with respect to the attribute C_1 is about the linguistic value s_4 with the satisfaction degree [0.5, 0.6], dissatisfaction degree [0.2, 0.3], and indeterminacy degree [0.1, 0.2] given by two experts of them and about the linguistic value s_5 with the satisfaction degree [0.3, 0.4], dissatisfaction degree [0.3, 0.4] and indeterminacy degree [0.2, 0.3] given by one expert of them. Thus, when the four possible alternatives with respect to the above three attributes are evaluated by the three decision makers, the hesitant interval neutrosophic linguistic decision matrix is constructed as shown in Table 1.

Table 1 Hesitant interval neutrosophic linguistic decision matrix D

	C_1	C_2	C_3
A_1	$\{\langle s_4$, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])$\rangle$, $\langle s_5$, ([0.3, 0.4], [0.2, 0.3], [0.3, 0.4])$\rangle\}$	$\{\langle s_5$, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])$\rangle\}$	$\{\langle s_3$, ([0.4, 0.5], [0.1, 0.2], [0.3, 0.5])$\rangle$, $\langle s_4$, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])$\rangle\}$
A_2	$\{\langle s_4$, ([0.7, 0.8], [0.1, 0.2], [0.2, 0.3])$\rangle$, $\langle s_5$, ([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])$\rangle\}$	$\{\langle s_4$, ([0.6, 0.7], [0, 0.1], [0.2, 0.3])$\rangle\}$	$\{\langle s_3$, ([0.7, 0.8], [0, 0.1], [0.1, 0.2])$\rangle$, $\langle s_4$, ([0.6, 0.7], [0.1,0.2], [0.1, 0.2])$\rangle$, $\langle s_5$, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])$\rangle\}$
A_3	$\{\langle s_4$, ([0.7, 0.9], [0.2, 0.4], [0.1, 0.2])$\rangle$, $\langle s_5$, ([0.5, 0.6], [0.3, 0.4], [0.2, 0.3])$\rangle\}$	$\{\langle s_4$, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])$\rangle$, $\langle s_5$, ([0.3, 0.5}, [0.1, 0.2], [0.4, 0.5]]$\rangle\}$	$\{\langle s_3$, ([0.5, 0.6], [0.1, 0.3], [0.2, 0.3])$\rangle\}$

A_4	{⟨s_3, ([0.7, 0.8], [0, 0.1], [0.1, 0.2])⟩}	{⟨s_3, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])⟩, ⟨s_4, ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4])⟩, ⟨s_5, ([0.4, 0.5], [0.2, 0.3], [0.1, 0.2])⟩}	{⟨s_4, ([0.5, 0.6], [0.1, 0.2], [0.1, 0.2])⟩, ⟨s_5, ([0.3, 0.5], [0, 0.2], [0.1, 0.2])⟩}

Then, the developed approach is utilized to obtain the ranking order of the alternatives and the most desirable one(s), which can be described as the following steps:

Step 1: Aggregate all HINLEs of $\tilde{n}_{ij}$ (i = 1, 2, 3, 4; j = 1, 2, 3) by using the HINLWA operator to derive the collective HINLE $\tilde{n}_i$ (i = 1, 2, 3, 4) for an alternative A_i (i = 1, 2, 3, 4). Taking an alternative A_1 for an example, we have

$$\tilde{n}_1 = HINLWA(\tilde{n}_{11}, \tilde{n}_{12}, \tilde{n}_{13})$$
$$= \bigcup_{a_{11}\in\tilde{n}_{11}, a_{12}\in\tilde{n}_{12}, a_{13}\in\tilde{n}_{13}} \left\{ \left\langle s_{\sum_{j=1}^{3} w_j \theta(a_{1j})}, \left(\begin{array}{l} \left[1-\prod_{j=1}^{3}(1-T^L(a_{1j}))^{w_j}, 1-\prod_{j=1}^{3}(1-T^U(a_{1j}))^{w_j}\right], \\ \left[\prod_{j=1}^{3}\left(I^L(a_{1j})\right)^{w_j}, \prod_{j=1}^{3}\left(I^U(a_{1j})\right)^{w_j}\right], \\ \left[\prod_{j=1}^{3}\left(F^L(a_{1j})\right)^{w_j}, \prod_{j=1}^{3}\left(F^U(a_{1j})\right)^{w_j}\right] \end{array} \right) \right\rangle \right\}$$

= {⟨$s_{3.85}$, ([0.4622, 0.5627], [0.1189, 0.2213], [0.2603, 0.3955])⟩, ⟨$s_{4.2}$, ([0.395, 0.496], [0.1516, 0.2551], [0.3000, 0.4373])⟩, ⟨$s_{4.25}$, ([0.3966, 0.4996], [0.1189, 0.2213], [0.3193, 0.4254])⟩, ⟨$s_{4.6}$, ([0.3212, 0.4234], [0.1516, 0.2551], [0.3680, 0.4704])⟩}.

Similarly, we can derive the following collective HINLFEs of $\tilde{n}_i$ (i = 2, 3, 4):

$\tilde{n}_2$ = {⟨$s_{3.6}$, ([0.6776, 0.7787], [0, 0.1275], [0.1516, 0.2551])⟩, ⟨s_4, ([0.6383, 0.7397], [0, 0.1682], [0.1516, 0.2551])⟩, ⟨$s_{4.4}$, ([0.6045, 0.7079], [0, 0.1682], [0.2000, 0.3000])⟩, ⟨$s_{3.95}$, ([0.6435, 0.7449], [0, 0.1275], [0.1189, 0.2551])⟩, ⟨$s_{4.35}$, ([0.6000, 0.7000], [0, 0.1682], [0.1189, 0.2551])⟩, ⟨$s_{4.75}$, ([0.5627, 0.6634], [0, 0.1682], [0.1569, 0.3000])⟩};

$\tilde{n}_3$ = {⟨$s_{3.6}$, ([0.5819, 0.7538], [0.1516, 0.3318], [0.1737, 0.2797])⟩, ⟨$s_{3.85}$, ([0.5452, 0.7396], [0.1275, 0.2998], [0.1866, 0.2958])⟩, ⟨$s_{3.95}$, ([0.5, 0.6], [0.1747, 0.3318], [0.2213, 0.3224])⟩, ⟨$s_{4.2}$, ([0.4561, 0.5771], [0.1469, 0.2998], [0.2378, 0.3409])⟩};

$\tilde{n}_4$ = {⟨$s_{3.4}$, ([0.6045, 0.7079], [0, 0.1569], [0.1189, 0.2213])⟩, ⟨$s_{3.8}$, ([0.5476, 0.6807], [0, 0.1569], [0.1189, 0.2213])⟩, ⟨$s_{3.65}$, ([0.5819, 0.6862], [0, 0.1569], [0.1316, 0.2378])⟩, ⟨$s_{4.05}$, ([0.5216, 0.6569], [0, 0.1569], [0.1316, 0.2378])⟩, ⟨$s_{3.9}$, ([0.5624, 0.6682], [0, 0.1737], [0.1, 0.2])⟩, ⟨$s_{4.3}$, ([0.4993, 0.6372], [0, 0.1737], [0.1, 0.2])⟩}.

Step 2: Calculate the score values of the collective HINLFE $\tilde{n}_i$ (i = 1, 2, 3 4) by Eq. (4):

$E_H(\tilde{n}_1) = 0.4413$, $E_H(\tilde{n}_2) = 0.5519$, $E_H(\tilde{n}_3) = 0.4549$, and $E_H(\tilde{n}_4) = 0.5051$.

Step 3: Rank the alternatives in accordance with the score values: $A_2 \succ A_4 \succ A_3 \succ A_1$. Therefore, the alternative A_2 is the best choice according to the largest score value.

On the other hand, if the HINLWG operator is utilized in the multiple attribute decision-making problem, the decision-making steps can be described as following:

Step 1': Aggregate all HINLEs of $\tilde{n}_{ij}$ (i = 1, 2, 3, 4; j = 1, 2, 3) by using the HINLWG operator to derive the collective HINLEs of $\tilde{n}_i$ (i = 1, 2, 3, 4) for the alternative A_i (i = 1, 2, 3, 4):

$\tilde{n}_1$ = {⟨$s_{3.7697}$, ([0.4573, 0.5578], [0.1261, 0.2263], [0.2665, 0.4113])⟩, ⟨$s_{4.2295}$, ([0.3466, 0.4547], [0.1261, 0.2263], [0.3589, 0.4615])⟩, ⟨$s_{4.076}$, ([0.3824, 0.484], [0.1614, 0.2616], [0.3, 0.4422])⟩, ⟨$s_{4.5731}$, ([0.2898, 0.3946], [0.1614, 0.2616], [0.3881, 0.4898])⟩};

$\tilde{n}_2$ = {⟨$s_{3.5652}$, ([0.6735, 0.7737], [0.0362, 0.1363], [0.1614, 0.2616])⟩, ⟨s_4, ([0.6333, 0.7335], [0.076, 0.1761], [0.1614, 0.2616])⟩, ⟨$s_{4.3734}$, ([0.5887, 0.6896], [0.076, 0.1761], [0.2, 0.3])⟩, ⟨$s_{3.8548}$, ([0.6382, 0.7384], [0.0362, 0.1363], [0.1261, 0.2616])⟩, ⟨$s_{4.3249}$, ([0.6, 0.7], [0.076, 0.1761], [0.1261, 0.2616])⟩, ⟨$s_{4.7287}$, ([0.5578, 0.6581], [0.076, 0.1761], [0.1663, 0.3])⟩};

$\tilde{n}_3$ = {⟨$s_{3.5652}$, ([0.5625, 0.6915], [0.1614, 0.3368], [0.1937, 0.2942])⟩, ⟨$s_{3.7697}$, ([0.4951, 0.6607], [0.1363, 0.3143], [0.2242, 0.3257])⟩, ⟨$s_{3.8548}$, ([0.5, 0.6], [0.1997, 0.3368], [0.2263, 0.3265])⟩, ⟨$s_{4.076}$, ([0.4401, 0.5733], [0.1758, 0.3143], [0.2555, 0.3565])⟩};

$\tilde{n}_4$ = {⟨$s_{3.3659}$, ([0.5887, 0.6896], [0.0662, 0.1663], [0.1261, 0.2263])⟩, ⟨$s_{3.6801}$, ([0.4799, 0.6411], [0.026, 0.1663], [0.1261, 0.2263])⟩, ⟨$s_{3.6169}$, ([0.5625, 0.6636], [0.0662, 0.1663], [0.1548, 0.2555])⟩, ⟨$s_{3.9545}$, ([0.4585, 0.6169], [0.026, 0.1663], [0.1548, 0.2555])⟩, ⟨$s_{3.8244}$, ([0.532, 0.634], [0.0933, 0.1937]. [0.1, 0.2])⟩, ⟨$s_{4.1814}$, ([0.4337, 0.5894], [0.0543, 0.1937], [0.1, 0.2])⟩}.

Step 2': Calculate the score values of the collective HINLE $\tilde{n}_i$ (i = 1, 2, 3, 4) for the alternative A_i (i = 1, 2, 3, 4) by Eq. (4) as follows:

$E_H(\tilde{n}_1) = 0.4231$, $E_H(\tilde{n}_2) = 0.5363$, $E_H(\tilde{n}_3) = 0.4325$, and $E_H(\tilde{n}_4) = 0.4774$.

Step 3': Rank the alternatives in accordance with the score values: $A_2 \succ A_4 \succ A_3 \succ A_1$. Therefore, the alternative A_2 is also the best choice according to the largest score value.

Obviously, above two kinds of ranking orders are identical and the same as the ones in Ye [12, 16]. Although two kinds of ranking orders based on the HINLWA and HINLWG operators are identical, there are different focal points [16] between the HINLWA operator and the HINLWG operator. The HINLWA operator emphasizes the group's major points, while the HINLWG operator emphasizes the individual major points. Then, decision makers may select one of them according to their preference or real requirements.

Compared with the relative decision making methods based on INLSs and SVNHFSs, the decision making method in this paper uses HINLS information, while the decision making methods in [12, 16] use INLS information and SVNHFS information, respectively. Since HINLS is a further generalization of INLS and SVNHFS, HINLS information includes INLS information and SVNHFS information, and also the INLWA and INLWG operators and the SVNHFWA and SVNHFWG operators are special cases of the HINLWA and HINLWG operators. Therefore, the decision-making method proposed in this paper can deal with not only hesitant interval neutrosophic linguistic decision-making problems but also single-valued neutrosophic hesitant fuzzy multiple attribute decision-making problems and interval neutrosophic linguistic multiple attribute decision-making problems. To some extent, the decision-making method in hesitant interval neutrosophic linguistic setting is more general and more feasible than existing decision-making methods in single-valued neutrosophic hesitant fuzzy setting and interval neutrosophic linguistic setting.

7. Conclusion

This paper introduced the concept of HINLSs based on the combination of both HFSs and INLSs as a further generalization of these fuzzy concepts and defined some operational laws of HINLEs and the score, accuracy and certainty functions of HINLEs. Then, we proposed the HINLWA and HINLWG operators to aggregate hesitant interval neutrosophic linguistic information and investigated their some properties. Furthermore, the HINLWA and HINLWG operators were applied to multiple attribute decision-making problems under a hesitant interval neutrosophic linguistic environment, in which attribute values with respect to alternatives are evaluated by the form of HINLEs and the attribute weights are known information. We utilized the score function (accuracy and certainty functions) to rank the alternatives and to determine the best one(s). Finally, an illustrative example was provided to demonstrate the application of the developed decision-making approach. The main advantage of the developed method is that it can describe the incomplete, indeterminate and inconsistent information by several INLNs in which linguistic variables indicate whether an attribute is good or bad in qualitative and INNs are adopted to demonstrate the satisfaction degrees, dissatisfaction degrees and indeterminacy degrees to a linguistic variable in quantitative. Therefore, the proposed multiple attribute decision-making method under a hesitant interval neutrosophic linguistic environment is more suitable for real scientific and engineering applications. In the future, we shall further develop more aggregation operators for HINLEs and apply them to these areas such as group decision making, expert system, information fusion system, fault diagnoses, and medical diagnoses.

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Table 1. Hesitant interval neutrosophic linguistic decision matrix D

	C_1	C_2	C_3
A_1	$\{\langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_5, ([0.3, 0.4], [0.2, 0.3], [0.3, 0.4])\rangle\}$	$\{\langle s_5, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])\rangle\}$	$\{\langle s_3, ([0.4, 0.5], [0.1, 0.2], [0.3, 0.5])\rangle, \langle s_4, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle\}$
A_2	$\{\langle s_4, ([0.7, 0.8], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])\rangle\}$	$\{\langle s_4, ([0.6, 0.7], [0, 0.1], [0.2, 0.3])\rangle\}$	$\{\langle s_3, ([0.7, 0.8], [0, 0.1], [0.1, 0.2])\rangle, \langle s_4, ([0.6, 0.7], [0.1,0.2], [0.1, 0.2])\rangle, \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle\}$
A_3	$\{\langle s_4, ([0.7, 0.9], [0.2, 0.4], [0.1, 0.2])\rangle, \langle s_5, ([0.5, 0.6], [0.3, 0.4], [0.2, 0.3])\rangle\}$	$\{\langle s_4, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])\rangle, \langle s_5, ([0.3, 0.5\}, [0.1, 0.2], [0.4, 0.5]]\rangle\}$	$\{\langle s_3, ([0.5, 0.6], [0.1, 0.3], [0.2, 0.3])\rangle\}$
A_4	$\{\langle s_3, ([0.7, 0.8], [0, 0.1], [0.1, 0.2])\rangle\}$	$\{\langle s_3, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4])\rangle, \langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.1, 0.2])\rangle\}$	$\{\langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.1, 0.2])\rangle, \langle s_5, ([0.3, 0.5], [0, 0.2], [0.1, 0.2])\rangle\}$