Neutrosophic linguistic variables and aggregation operators for multiple attribute group decision making

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Abstract

The paper proposes the new concepts of a neutrosophic linguistic variable (NLV) and a neutrosophic linguistic set (NLS) based on the combination of a linguistic variable and a neutrosophic number. Then, we introduce basic operational laws of NLVs and the expected value of a NLV to rank NLVs. Furthermore, we propose a neutrosophic linguistic weighted arithmetic average (NLWAA) operator and a neutrosophic linguistic weighted geometric average (NLWGA) operator and investigate their properties, and then establish a decision-making method based on the NLWAA and NLWGA operators to handle multiple attribute group decision-making problems with neutrosophic linguistic information. Finally, an illustrative example is given to demonstrate the application and effectiveness of the developed approach.

Keywords: Neutrosophic linguistic variable; Neutrosophic linguistic set; Expected value; Neutrosophic linguistic weighted arithmetic average (NLWAA) operator; Neutrosophic linguistic weighted geometric average (NLWGA) operator; Decision making

1. Introduction

Decision-making method is an important research topic in decision theory. Then various decision-making methods has been proposed and applied widely to engineering, economics, and management fields. However, there is a lot of qualitative information in complex decision-making problems, where the evaluation results of decision makers may be expressed easily by linguistic variables due to the uncertainty of decision environment and difference of decision makers' cultural and knowledge background. Hence, Zadeh [1] firstly presented the concept of the linguistic variable and applied it to fuzzy reasoning. Later, Herrera et al. [2] and Herrera and Herrera-Viedma [3] proposed linguistic decision analyses to solve decision-making problems with linguistic information. Then, Xu [4] introduced a linguistic hybrid arithmetic averaging operator for multiple attribute group decision-making problems with linguistic information. Xu [5] further put forward goal programming models to handle multiple attribute decision-making problems under linguistic environment. Also, Xu [6] presented the uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator for dealing with multiple attribute group decision-making problems with uncertain linguistic information. Xu [7] further developed some induced uncertain linguistic ordered weighted averaging (IULOWA) operators for multiple attribute group decision-making problems with uncertain linguistic information.

However, there is often incomplete, indeterminate, and inconsistent information in real life. To express

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this kind of information, Smarandache [8-10] proposed the concept of a neutrosophic number, denoted by $A = a + bI$, which consists of its determinate part a and its indeterminate part bI. Therefore, it can be easier and better to express incomplete, indeterminate and inconsistent information, which exists commonly in real life, while existing linguistic information cannot express indeterminate and inconsistent information. Since the neutrosophic number can effectively express incomplete and indeterminate information, Ye [11] proposed a group decision-making method with neutrosophic numbers, including a de-neutrosophication process and a possibility degree ranking method for neutrosophic numbers. Then, Kong et al. [12] presented a cosine similarity measure between neutrosophic numbers and applied it to the misfire fault diagnosis of gasoline engines.

Due to the ambiguity of people's thinking and the complexity of objective things in the real world, linguistic evaluation in decision-making problems can easily express and better handle the incomplete and indeterminate information than numerical evaluation. Because existing linguistic information cannot express indeterminacy, one needs to introduce neutrosophic linguistic information to overcome the difficulty of existing linguistic expression for indeterminacy. To do so, the purposes of this paper are: (1) to propose the concepts of a neutrosophic linguistic variable (NLV) and a neutrosophic linguistic set (NLS), (2) to introduce basic operational laws of NLVs and the expected value of a NLV for ranking NLVs, (3) to develop a neutrosophic linguistic weighted arithmetic average (NLWAA) operator and a neutrosophic linguistic weighted geometric average (NLWGA) operator and to investigate their properties, and (4) to establish a decision-making method based on the NLWAA and NLWGA operators for solving multiple attribute group decision-making problems with neutrosophic linguistic information.

To achieve the above purposes, the remainder of this paper is organized as follows. Section 2 briefly describes some concepts of linguistic variables, linguistic sets, and neutrosophic numbers. In Section 3, we propose NLVs and NLSs and define the operational laws of NLVs and the expected value of a NLV. Section 4 develops NLWAA and NLWGA operators of NLVs and investigates their properties. In Section 5, a multiple attribute group decision-making method based on the NLWAA and NLWGA operators is established under a neutrosophic linguistic environment. In Section 6, an illustrative example is given to demonstrate the application and effectiveness of the proposed method. Section 7 gives a conclusion and future work.

2. Preliminaries of linguistic sets and neutrosophic numbers

In this section, some basic concepts related to linguistic variables, linguistic sets and neutrosophic numbers are briefly introduced to utilize the subsequent analysis.

2.1 The concepts of a linguistic variable and a linguistic set

Zadeh [1] firstly proposed the concept of a linguistic variable in 1975.

Let $L = \{l_0, l_2, \ldots, l_{c-1}\}$ be a finite and totally ordered discrete linguistic term set with old cardinality c, where l_i in the linguistic term set L represents a linguistic variable and c is an odd value. For example, taking $c = 7$, one can specify a linguistic term set $L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6\}$ = {extremely low, very low, low, medium, high, very high, extremely high}.

For a linguistic set L, any two linguistic variables l_i and l_j should satisfy the following properties [2, 3]:

(1) Ordering: $l_i \geq l_j$ if $i \geq j$;

(2) Negation operator: $\text{neg}(l_i) = l_{c-1-i}$;

(3) Maximum operator: $\max(l_i, l_j) = l_i$ if $i > j$;

(4) Minimum operator: $\min(l_i, l_j) = l_j$ if $i > j$.

To minimize the linguistic information loss in the operational process, the discrete linguistic set $L = \{l_0,$

l_1, l_2, l_3, l_4, l_5, l_6} can be extended to a continuous linguistic set $\overline{L} = \{l_a \mid a \in R\}$, which satisfied the aforementioned characteristics.

For any two linguistic variables l_i and l_j for $l_i, l_j \in \overline{L}$, the operational rules are defined as follows [4, 5]:

(1) $\rho l_i = l_{\rho \times i}, \quad \rho \geq 0$;

(2) $l_i + l_j = l_{i+j}$;

(3) $l_i \times l_j = l_{i \times j}$;

(4) $l_i / l_j = l_{i/j}$;

(5) $(l_i)^\rho = l_{i^\rho}, \quad \rho \geq 0$.

2.2 Some concepts of neutrosophic numbers

The neutrosophic number proposed by Smarandache [8-10] consists of the determinate part a and the indeterminate part bI, which is denoted by $A = a + bI$, where a and b are real numbers, and I is indeterminacy, such that $I^n = I$ for $n > 0$, $0 \times I = 0$, and bI/nI = undefinition for any real number n.

For example, assume that there is a neutrosophic number $A = 3 + 2I$. If $I \in [0, 0.3]$, it is equivalent to $A \in [3, 3.6]$ for sure $A \geq 3$, this means that its determinate part is 3 and its indeterminate part is $2I$ for the indeterminacy $I \in [0, 0.3]$ and the possibility for the number "A" is within the interval [3, 3.6].

Let $A_1 = a_1 + b_1I$ and $A_2 = a_2 + b_2I$ be two neutrosophic numbers for $a_1, b_1, a_2, b_2 \in R$ (all real numbers). the operational relationship for A_1 and A_2 is as follows [8-10]:

(1) $A_1 + A_2 = a_1 + a_2 + (b_1 + b_2)I$;

(2) $A_1 - A_2 = a_1 - a_2 + (b_1 - b_2)I$;

(3) $A_1 \times A_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I$;

(4) $A_1^2 = (a_1 + b_1I)^2 = a_1^2 + ((a_1+b_1)^2 - a_1^2)I$;

(5) $\dfrac{A_1}{A_2} = \dfrac{a_1 + b_1 I}{a_2 + b_2 I} = \dfrac{a_1}{a_2} + \dfrac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} \cdot I$ for $a_2 \neq 0$ and $a_2 \neq -b_2$;

(6) $$\sqrt{A_1} = \sqrt{a_1 + b_1 I} = \begin{cases} \sqrt{a_1} - (\sqrt{a_1} + \sqrt{a_1 + b_1})I \\ \sqrt{a_1} - (\sqrt{a_1} - \sqrt{a_1 + b_1})I \\ -\sqrt{a_1} + (\sqrt{a_1} + \sqrt{a_1 + b_1})I \\ -\sqrt{a_1} + (\sqrt{a_1} - \sqrt{a_1 + b_1})I \end{cases}.$$

Definition 1. Let $A = a + bI$ be a neutrosophic number. If $a, b \geq 0$, then A is called the positive neutrosophic number.

In the following, all neutrosophic numbers are considered as positive and are called neutrosophic numbers for short, unless they are stated.

3. Neutrosophic linguistic variable and Neutrosophic linguistic set

In this section, we propose NLVs and NLSs and give the operations of NLVs and the expected value of a NLV for ranking NLVs.

Definition 2. Let L be the pre-established finite and totally ordered linguistic term set with odd cardinality. Suppose $\bar{l} = l_{a+bI}$ and a and b are real numbers and I is indeterminacy, then $\bar{l}$ is called a NLV, where $\bar{l}$ is composed of the determinate linguistic part $l_a \in L$ and the indeterminate linguistic part l_{bI} (i.e. $\bar{l} = l_a + l_{bI}$).

Definition 3. Let $\bar{L}$ be the set of all NLVs. Assume that two NLVs are $\bar{l}_1 = l_{a_1+b_1I}$ and $\bar{l}_2 = l_{a_2+b_2I}$ for $\bar{l}_1, \bar{l}_2 \in \bar{L}$, then the operational laws are defined as follows:

(1) $\bar{l}_1 + \bar{l}_2 = l_{a_1+a_2+(b_1+b_2)I}$;

(2) $\bar{l}_1 - \bar{l}_2 = l_{a_1-a_2+(b_1-b_2)I}$;

(3) $\bar{l}_1 \times \bar{l}_2 = l_{a_1a_2+(a_1b_2+a_2b_1+b_1b_2)I}$;

(4) $\dfrac{\bar{l}_1}{\bar{l}_2} = \bar{l}_{\frac{a_1}{a_2}+\frac{a_2b_1-a_1b_2}{a_2(a_2+b_2)}\cdot I}$ for $a_2 \neq 0$ and $a_2 \neq -b_2$;

(5) $\rho\bar{l}_1 = \bar{l}_{\rho a_1+\rho b_1 I}$ for $\rho \geq 0$;

(6) $\bar{l}_1^{\rho} = l_{a_1^{\rho}+[(a_1+b_1)^{\rho}-a_1^{\rho}]I}$ for $\rho \geq 0$.

Obviously, the above operational results are still NLVs.

Then, we define an expected value of a NLV, which is an important index to rank NLVs in the following decision making problems.

Definition 4. Let $\bar{l}$ be a NLV for $I \in [\inf I, \sup I]$. Then, an expected value of the NLV $\bar{l}$ can be represented as follows

$$E(\bar{l}) = \frac{(a + b\inf I) + (a + b\sup I)}{2(c-1)} . \tag{1}$$

Obviously, the bigger the value of $E(\bar{l})$ is, the greater the corresponding NLV $\bar{l}$ is.

Based on Definition 4, a ranking method for NLVs can be given below.

Definition 5. Let $\bar{l}_1$ and $\bar{l}_2$ be two NLVs. Then, the ranking method can be defined as follows:

(1) If $E(\bar{l}_1) > E(\bar{l}_2)$, then $\bar{l}_1 \succ \bar{l}_2$;

(2) If $E(\bar{l}_1) = E(\bar{l}_2)$, then $\bar{l}_1 = \bar{l}_2$.

Example 1. Let $\bar{l}_1 = \bar{l}_{3+2I}$ and $\bar{l}_2 = \bar{l}_{2+3I}$ be two neutrosophic linguistic numbers (NLNs) (values of NLVs) for $I \in [0.1, 0.3]$ and the cardinality of a NLS is $c = 7$. Then, the ranking order between $\bar{l}_1$ and $\bar{l}_2$ is given in this case.

According to Eq. (1) we have $E(\bar{l}_1) = 0.5667 > E(\bar{l}_2) = 0.4333$, Hence, $\bar{l}_1 \succ \bar{l}_2$.

4. Weighted aggregation operators for NLVs

Weighted aggregation operators are important tools for information aggregation, which can capture the expressed interrelationship of the individual arguments. Based on the operational laws in Definition 3, this section proposes the following weighted arithmetic aggregation operator and weighted geometric aggregation operator for NLVs, which are usually utilized in decision making problems.

4.1 Neutrosophic linguistic weighted arithmetic average operator

Definition 6. Let $\bar{l}_j$ (j = 1, 2, …, n) be a collection of NLVs. The NLWAA operator is defined by

$$NLWAA(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n) = \sum_{j=1}^{n} w_j \bar{l}_j \qquad (2)$$

where $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)$ is the weight vector of $\bar{l}_j$ (j =1, 2, …, n), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 1. Let $\bar{l}_j$ (j = 1, 2, …, n) be a collection of NLVs. Then by Eq. (2) and the operational laws in Definition 3, we have the following aggregation formula:

$$NLWAA(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n) = \bar{l}_{\sum_{j=1}^{n} w_j a_j + I \sum_{j=1}^{n} w_j b_j}, \qquad (3)$$

where $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)$ is the weight vector of $\bar{l}_j$ (j = 1, 2, …, n) , $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Obviously, the proof of Eq. (3) can be easily obtained according to the operational laws in Definition 3. Hence, it is omitted here.

Especially, if $\boldsymbol{W}$ = (1/n, 1/n, …, 1/n), then the NLWAA operator reduces to a neutrosophic linguistic arithmetic average operator for NLVs.

It is obvious that the NLWAA operator contains the following properties:

(1) Idempotency: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs. If $\bar{l}_j$ (j = 1, 2, . . . , n) is equal, i.e., $\bar{l}_j = \bar{l}$ for j = 1, 2, . . . , n, then $NLWAA(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n) = \bar{l}$.

(2) Monotonicity: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs. If $\bar{l}_j \le \bar{l}_j^*$ for j = 1, 2, . . . , n, then $NLWAA(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n) \le NLWAA(\bar{l}_1^*, \bar{l}_2^*, \cdots, \bar{l}_n^*)$.

(3) Boundedness: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs and let $\bar{l}_{\min} = \min(\bar{l}_1, \bar{l}_2, ..., \bar{l}_n)$ and

$\bar{l}_{\max} = \max(\bar{l}_1, \bar{l}_2, ..., \bar{l}_n)$ for j = 1, 2, . . . , n, then $\bar{l}_{\min} \le NLWAA\left(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n\right) \le \bar{l}_{\max}$.

Since the above properties are obvious, their proofs are omitted here.

4.2 Neutrosophic linguistic weighted geometric average operator

Definition 7. Let $\bar{l}_j$ (j = 1, 2, …, n) be a collection of NLVs. Then the NLWGA operator is defined as

$$NLWGA\left(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n\right) = \prod_{j=1}^{n} \bar{l}_j^{\,w_j}, \tag{4}$$

where $\boldsymbol{W}$ = (w_1, w_2, …, w_n) is the weight vector of $\bar{l}_j$ (j = 1, 2, …, n) , $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 2. Let $\bar{l}_j$ (j = 1, 2, …, n) be a collection of NLVs. by Eq. (4) and the operational laws in Definition 3, we have the following aggregation formula:

$$NLWGA(\bar{l}_1, \bar{l}_{2,} \cdots, \bar{l}_n) = \bar{l}_{\prod_{j=1}^{n} a_j^{w_j} + (\prod_{j=1}^{n} (a_j + b_j)^{w_j} - \prod_{j=1}^{n} a_j^{w_j}) I}, \tag{5}$$

where $\boldsymbol{W}$ = (w_1, w_2, …, w_n) is the weight vector of $\bar{l}_j$ (j = 1, 2, …, n), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof: The proof of Eq. (5) can be done by means of mathematical induction.

(1) When n = 2, then

$$\begin{aligned} l_1^{w_1} \times l_2^{w_2} &= \bar{l}_{(a_1^{w1} + ((a_1+b_1)^{w1} - a_1^{w1}) I) \times (a_2^{w2} + ((a_2+b_2)^{w2} - a_2^{w2}) I)} \\ &= \bar{l}_{a_1^{w1} a_2^{w2} + a_1^{w1} ((a_2+b_2)^{w2} - a_2^{w2}) I + a_2^{w2} ((a_1+b_1)^{w1} - a_1^{w1}) I + ((a_2+b_2)^{w2} - a_2^{w2})((a_1+b_1)^{w1} - a_1^{w1}) I} \\ &= \bar{l}_{a_1^{w1} a_2^{w2} + (a_1^{w1} (a_2+b_2)^{w2} - a_1^{w1} a_2^{w2}) I + (a_2^{w2} (a_1+b_1)^{w1} - a_2^{w2} a_1^{w1}) I + ((a_2+b_2)^{w2} (a_1+b_1)^{w1} - a_2^{w2} (a_1+b_1)^{w1} - a_1^{w1} (a_2+b_2)^{w2} + a_1^{w1} a_2^{w2}) I} \\ &= \bar{l}_{a_1^{w1} a_2^{w2} + ((a_2+b_2)^{w2} (a_1+b_1)^{w1} - a_1^{w1} a_2^{w2}) I}. \end{aligned} \tag{6}$$

(2) When n = k, by using Eq. (5), we obtain

$$NLWGA(\bar{l}_1, \bar{l}_{2,} \cdots, \bar{l}_k) = \bar{l}_{\prod_{j=1}^{k} a_j^{w_j} + (\prod_{j=1}^{k} (a_j + b_j)^{w_j} - \prod_{j=1}^{k} a_j^{w_j}) I}. \tag{7}$$

(3) When n = k+1, by using Eqs. (6) and (7), we obtain

$$\begin{aligned} NLWGA(\bar{l}_1, \bar{l}_{2,} \cdots, \bar{l}_{k+1}) &= \bar{l}_{\left(\prod_{j=1}^{k} a_j^{w_j} + (\prod_{j=1}^{k} (a_j+b_j)^{w_j} - \prod_{j=1}^{k} a_j^{w_j}) I\right)(a_{k+1}^{w_{k+1}} + ((a_{k+1}+b_{k+1})^{w_{k+1}} - a_{k+1}^{w_{k+1}}) I} \\ &= l_{\prod_{j=1}^{k} a_j^{w_j} a_{k+1}^{w_{j+1}} + \prod_{j=1}^{k} a_j^{w_j} ((a_{k+1}+b_{k+1})^{w_{k+1}} - a_{k+1}^{w_{k+1}}) I + a_{k+1}^{w_{k+1}} ((\prod_{j=1}^{k} (a_j+b_j)^{w_j} - \prod_{j=1}^{k} a_j^{w_j}) I + ((a_{k+1}+b_{k+1})^{w_{k+1}} - a_{k+1}^{w_{k+1}})((\prod_{j=1}^{k} (a_j+b_j)^{w_j} - \prod_{j=1}^{k} a_j^{w_j}) I} \\ &= \bar{l}_{\prod_{j=1}^{k+1} a_j^{w_j} + (\prod_{j=1}^{k+1} (a_j+b_j)^{w_j} - \prod_{j=1}^{k+1} a_j^{w_j}) I}. \end{aligned}$$

Therefore, according to the above results, we have Eq. (5) for any n. This completes the proof. □

Especially when $\boldsymbol{W}$ = (1/n, 1/n, …, 1/n), the NLWGA operator reduces to a neutrosophic linguistic geometric average operator.

It is obvious that the NLWGA operator contains the following properties:

(1) Idempotency: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs. If $\bar{l}_j$ (j = 1, 2, . . . , n) is equal, i.e., $\bar{l}_j = \bar{l}$ for j = 1, 2, . . . , n, then $NLWGA\left(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n\right) = \bar{l}$.

(2) Monotonicity: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs. If $\bar{l}_j \le \bar{l}_j^*$ for j = 1, 2, . . . , n, then

$$NLWGA\left(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n\right) \le NLWGA\left(\bar{l}_1^*, \bar{l}_2^*, \cdots, \bar{l}_n^*\right).$$

(3) Boundedness: Let $\bar{l}_j$ (j = 1, 2, . . . , n) be a collection of NLVs and let $\bar{l}_{\min} = \min(\bar{l}_1, \bar{l}_2, ..., \bar{l}_n)$ and $\bar{l}_{\max} = \max(\bar{l}_1, \bar{l}_2, ..., \bar{l}_n)$ for j = 1, 2, . . . , n, then $\bar{l}_{\min} \le NLWGA\left(\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_n\right) \le \bar{l}_{\max}$.

Since the above properties are obvious, their proofs are omitted here.

5. **Group decision-making method with NLVs**

In this section, we present a handling method for multiple attribute group decision-making problems with NLVs.

For a multiple attribute group decision-making problem with NLVs, let $U = \{u_1, u_2, . . . , u_m\}$ be a discrete set of alternatives, $G = \{g_1, g_2, . . . , g_n\}$ be a set of attributes, and $E = \{e_1, e_2, . . . , e_v\}$ be a set of decision makers. If the kth (k = 1, 2,…, v) decision maker provides the evaluation of the alternative u_i (i = 1, 2, . . . , m) on the attribute g_j (j = 1, 2,…, n) under some linguistic term set, such as $L = \{l_0$: extremely poor, l_1: very poor, l_2: poor, l_3: medium, l_4: good, l_5: very good, l_6: extremely good$\}$, the evaluation value with indeterminacy I can be represented by the form of a NLV $\bar{l}_{ij}^k = l_{a_{ij}^k + b_{ij}^k I}$ for $a_{ij}^k, b_{ij}^k \in R$ (k = 1, 2,…, v; j = 1, 2,…, n; i = 1, 2,…, m). Therefore, we can obtain the kth neutrosophic linguistic decision matrix D^k:

$$D^k = \begin{bmatrix} \bar{l}_{11}^k & \bar{l}_{12}^k & \cdots & \bar{l}_{1n}^k \\ \bar{l}_{21}^k & \bar{l}_{22}^k & \cdots & \bar{l}_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{m1}^k & \bar{l}_{m2}^k & \cdots & \bar{l}_{mn}^k \end{bmatrix}.$$

If the weight vector of attributes is $W = (w_1, w_2,…, w_n)$ with $w_j \ge 0$ and $\sum_{j=1}^{n} w_j = 1$ and the weight vector of decision makers is $Q = (q_1, q_2,…, q_v)$ with $q_k \ge 0$ and $\sum_{k=1}^{v} q_k = 1$. Then, the steps of the decision-making problem are described as follows:

Step 1: According to the decision matrix D^k (k = 1, 2, …, v) provided by decision makers, by the following formula:

$$\bar{l}_{ij} = NLWAA\left(\bar{l}_{ij}^1, \bar{l}_{ij}^2, \cdots, \bar{l}_{ij}^v\right) = l_{\sum_{k=1}^{v} q_k a_{ij}^k + I \sum_{k=1}^{v} q_k b_{ij}^k}, \quad (8)$$

we can get a collective neutrosophic linguistic decision matrix:

$$D = \begin{bmatrix} \bar{l}_{11} & \bar{l}_{12} & \cdots & \bar{l}_{1n} \\ \bar{l}_{21} & \bar{l}_{22} & \cdots & \bar{l}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{m1} & \bar{l}_{m2} & \cdots & \bar{l}_{mn} \end{bmatrix}.$$

Step 2: The individual overall NLV $\bar{l}_i$ for u_i (i = 1, 2, …, m) is calculated by the following aggregation formula:

$$\bar{l}_i = NLWAA(\bar{l}_{i1}, \bar{l}_{i2}, \cdots, \bar{l}_{in}) = \bar{l}_{\sum_{j=1}^{n} w_j a_{ij} + I \sum_{j=1}^{n} w_j b_{ij}}, \tag{9}$$

or $$\bar{l}_i = NLWGA(\bar{l}_{i1}, \bar{l}_{i2,} \cdots, \bar{l}_{in}) = \bar{l}_{\prod_{j=1}^{n} a_{ij}^{w_j} + (\prod_{j=1}^{n} (a_{ij}+b_{ij})^{w_j} - \prod_{j=1}^{n} a_{ij}^{w_j}) I}. \tag{10}$$

Step 3: We introduce a de-neutrosophication process in the decision-making problem based on $I \in$ [*inf I, sup I*] $\subset$ [-1, 1]. A NLV $\bar{l}_i$ (i = 1, 2, …, m) can be transformed to an interval NLV, which is equivalent to $\bar{l}_i = l_{a_i + b_i I} \in l_{[a_i + b_i \inf I, a_i + b_i \sup I]}$. Then, the expected value of $E(\bar{l}_i)$ (i = 1, 2, …, m) is calculated by applying Eq. (1).

Step 4: The alternatives are ranked according to the values of $E(\bar{l}_i)$ (i = 1, 2, …, m) by the ranking method in Definition 5, and then the best one(s) can be selected according to the largest expected values of $E(\bar{l}_i)$.

Step 5: End.

6. Illustrative example

In this section, an illustrative example for a multiple attribute group decision-making problem with NLVs is provided to demonstrate the applications of the proposed decision-making method in realistic scenarios.

There is a decision-making problem of manufacturing alternatives in the flexible manufacturing system. Suppose a set of four alternatives for the flexible manufacturing system is U= {u_1, u_2, u_3, u_4}. Then, a decision is made according to the three attributes: (1) g_1 is the improvement of manufacturing quality; (2) g_2 is the market response; (3) g_3 is the manufacturing cost. The four possible alternatives on the three attributes are to be evaluated by a group of three decision makers corresponding to the linguistic term set L = {l_0: extremely poor, l_1: very poor, l_2: poor, l_3: medium, l_4: good, l_5: very good, l_6: extremely good}. Assume that the weight vector of the three attributes is $\boldsymbol{W}$ = (0.2, 0.5, 0.3) and the weight vector of the three decision makers is $\boldsymbol{Q}$ = (0.3, 0.36, 0.34).

Then, the three decision makers are invited to make judgments and to give the evaluation with indeterminacy I according to the linguistic term set. Thus, the evaluation results of an alternative u_i (i = 1, 2, 3, 4) on an attribute g_j (j = 1, 2, 3) are given as the following three neutrosophic linguistic decision matrices:

$$D^1 = \begin{bmatrix} l_5 & l_{4+I} & l_{3+I} \\ l_4 & l_5 & l_{4+I} \\ l_{4+I} & l_{4+I} & l_4 \\ l_5 & l_{4+I} & l_4 \end{bmatrix}, \quad D^2 = \begin{bmatrix} l_{4+I} & l_5 & l_3 \\ l_5 & l_4 & l_{3+I} \\ l_5 & l_{4+I} & l_4 \\ l_{4+I} & l_5 & l_{5+I} \end{bmatrix}, \quad D^3 = \begin{bmatrix} l_{5+I} & l_4 & l_{3+I} \\ l_{4+I} & l_4 & l_3 \\ l_5 & l_5 & l_{4+I} \\ l_4 & l_{4+I} & l_4 \end{bmatrix}.$$

Whereas, we use the developed approach to rank the alternatives and to select the most desirable one(s), which can be described as the following steps:

Step 1: According to the above three decision matrices of D^k (k = 1, 2, 3), the collective neutrosophic linguistic decision matrix is obtained by applying Eq. (8) as follows:

$$D=\begin{bmatrix} l_{4.64+0.7I} & l_{4.36+0.3I} & l_{3+0.64I} \\ l_{4.36+0.34I} & l_{4.3} & l_{3.3+0.66I} \\ l_{4.7+0.3I} & l_{4.34+0.66I} & l_{4+0.34I} \\ l_{4.3+0.36I} & l_{4.36+0.64I} & l_{4.36+0.36I} \end{bmatrix}.$$

Step 2: By applying Eq. (9), we can obtain the individual overall NLNs of $\bar{l}_i$ for u_i (i =1, 2, 3, 4):

$$\bar{l}_1 = l_{4.008+0.482I},\ \bar{l}_2 = l_{4.012+0.266I},\ \bar{l}_3 = l_{4.31+0.492I}, \text{ and } \bar{l}_4 = l_{4.348+0.5I}$$

Step 3: For the de-neutrosophication in the decision making problem, assume that the infimum of I is taken as $\inf I = 0$ and the supremum of I is taken as $\sup I = 0.1$ to consider the maximum and minimum values for indeterminacy I, which are determined by decision makers' preference or requirements in real situations. Thus by applying Eq. (1), we can obtain the expected values of $E(\bar{l}_i)$ (i =1, 2, 3, 4):

$$E(\bar{l}_1) = 0.672,\ E(\bar{l}_2) = 0.6709,\ E(\bar{l}_3) = 0.7224, \text{ and } E(\bar{l}_4) = 0.7288.$$

Step 4: Since $E(\bar{l}_4) > E(\bar{l}_3) > E(\bar{l}_1) > E(\bar{l}_2)$, the ranking order of four alternatives is $u_4 \succ u_3 \succ u_1 \succ u_2$. Therefore, we can see that the alternative u_4 is the best choice among all the alternatives.

Or we can also utilize the NLWGA operator as the following computational steps:

Step 1': It is the same result as Step 1.

Step 2': By applying Eq. (9), we can obtain the individual overall NLNs of $\bar{l}_i$ for u_i (i =1, 2, 3, 4):

$$\bar{l}_1 = l_{3.9462+0.5004I},\ \bar{l}_2 = l_{3.9828+0.2876I},\ \bar{l}_3 = l_{4.3031+0.489I}, \text{ and } \bar{l}_4 = l_{4.3479+0.4976I}$$

Step 3': By applying Eq. (1) for $I \in [0, 0.1]$, we can obtain the expected values of $E(\bar{l}_i)$ (i =1, 2, 3, 4):

$$E(\bar{l}_1) = 0.6619,\ E(\bar{l}_2) = 0.6662,\ E(\bar{l}_3) = 0.7213, \text{ and } E(\bar{l}_4) = 0.7288.$$

Step 4': Since $E(\bar{l}_4) > E(\bar{l}_3) > E(\bar{l}_2) > E(\bar{l}_1)$, the ranking order of four alternatives is $u_4 \succ u_3 \succ u_2 \succ u_1$. Therefore, we can see that the alternative u_4 is the best choice among all the alternatives.

Similarly, if one considers different ranges of the indeterminate degree for I in NLNs, by Steps 3 and 4 or Steps' 3 and 4, one can obtain different results, as shown in Table 1 and Table 2.

Table 1. Decision results based on the NLWAA operator by choosing different indeterminate ranges for I in NLNs

I	NLWAA	Ranking order
$I \in [-0.7, 0]$	$E(\bar{l}_1) = 0.6399$, $E(\bar{l}_2) = 0.6532$, $E(\bar{l}_3) = 0.6896$, $E(\bar{l}_4) = 0.6955$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.5, 0]$	$E(\bar{l}_1) = 0.6479$, $E(\bar{l}_2) = 0.6576$, $E(\bar{l}_3) = 0.6978$, $E(\bar{l}_4) = 0.7038$	$u_4 \succ u_3 \succ u_2 \succ u_1$

$I \in [-0.3, 0]$	$E(\bar{l}_1) = 0.6559$, $E(\bar{l}_2) = 0.6620$, $E(\bar{l}_3) = 0.7060$, $E(\bar{l}_4) = 0.7122$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.1, 0]$	$E(\bar{l}_1) = 0.6640$, $E(\bar{l}_2) = 0.6665$, $E(\bar{l}_3) = 0.7142$, $E(\bar{l}_4) = 0.7205$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I = 0$	$E(\bar{l}_1) = 0.6680$, $E(\bar{l}_2) = 0.6687$, $E(\bar{l}_3) = 0.7183$, $E(\bar{l}_4) = 0.7247$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.1]$	$E(\bar{l}_1) =0.6720$, $E(\bar{l}_2) = 0.6709$, $E(\bar{l}_3) = 0.7224$, $E(\bar{l}_4) = 0.7288$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.3]$	$E(\bar{l}_1) = 0.6801$, $E(\bar{l}_2) = 0.6753$, $E(\bar{l}_3) = 0.7306$, $E(\bar{l}_4) = 0.7372$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.5]$	$E(\bar{l}_1) = 0.6881$, $E(\bar{l}_2) = 0.6797$, $E(\bar{l}_3) = 0.7388$, $E(\bar{l}_4) = 0.7455$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.7]$	$E(\bar{l}_1) = 0.6961$ $E(\bar{l}_2) = 0.6842$ $E(\bar{l}_3) = 0.7470$ $E(\bar{l}_4) = 0.7538$	$u_4 \succ u_3 \succ u_1 \succ u_2$

Table 2. Decision results based on the NLWGA operator by choosing different indeterminate ranges for I in NLNs

I	NLWGA	Ranking order
$I \in [-0.7, 0]$	$E(\bar{l}_1) = 0.6285$, $E(\bar{l}_2) = 0.6470$, $E(\bar{l}_3) = 0.6887$, $E(\bar{l}_4) = 0.6956$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.5, 0]$	$E(\bar{l}_1) = 0.6369$, $E(\bar{l}_2) = 0.6518$, $E(\bar{l}_3) = 0.6968$, $E(\bar{l}_4) = 0.7039$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.3, 0]$	$E(\bar{l}_1) =0.6452$, $E(\bar{l}_2) = 0.6566$, $E(\bar{l}_3) = 0.7050$, $E(\bar{l}_4) = 0.7122$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.1, 0]$	$E(\bar{l}_1) =0.6535$, $E(\bar{l}_2) = 0.6614$, $E(\bar{l}_3) = 0.7131$, $E(\bar{l}_4) = 0.7205$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I = 0$	$E(\bar{l}_1) = 0.6577$, $E(\bar{l}_2) = 0.6638$, $E(\bar{l}_3) = 0.7172$, $E(\bar{l}_4) = 0.7247$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.1]$	$E(\bar{l}_1) = 0.6619$, $E(\bar{l}_2) = 0.6662$, $E(\bar{l}_3) = 0.7213$, $E(\bar{l}_4) = 0.7288$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.3]$	$E(\bar{l}_1) = 0.6702$, $E(\bar{l}_2) = 0.6710$, $E(\bar{l}_3) = 0.7294$, $E(\bar{l}_4) = 0.7371$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.5]$	$E(\bar{l}_1) = 0.6786$, $E(\bar{l}_2) = 0.6758$, $E(\bar{l}_3) = 0.7376$, $E(\bar{l}_4) = 0.7454$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.7]$	$E(\bar{l}_1) = 0.6869$, $E(\bar{l}_2) = 0.6806$, $E(\bar{l}_3) = 0.7457$, $E(\bar{l}_4) = 0.7537$	$u_4 \succ u_3 \succ u_1 \succ u_2$

For the decision results based on the NLWAA operator in Table 1, we can see that the ranking orders of the four alternatives are $u_4 \succ u_3 \succ u_2 \succ u_1$ from $I \in [-0.7, 0]$ to $I = 0$ and $u_4 \succ u_3 \succ u_1 \succ u_2$ from $I \in [0, 0.1]$ to $I \in [0, 0.7]$, and then the best alternative is u_4. For the decision results based on the NLWGA operator in Table 2, we can see that the ranking orders of the four alternatives are $u_4 \succ u_3 \succ u_2 \succ u_1$ from $I \in [-0.7, 0]$ to $I \in [0, 0.3]$ and $u_4 \succ u_3 \succ u_1 \succ u_2$ from $I \in [0, 0.5]$ to $I \in [0, 0.7]$, and then the best alternative is also u_4.

The illustrative example demonstrates that different ranges of indeterminate degrees for *I* in NLNs result in different ranking orders of alternatives. Then, the group decision-making method proposed in this paper can deal with the decision making problems with neutrosophic linguistic information (indeterminate linguistic information). If we do not consider the indeterminacy *I* in NLNs (i.e., *I* = 0), then this group decision-making method reduces to classical one with crisp linguistic values.

Furthermore, since the indeterminate linguistic part $l_{b_i I}$ in NLVs can affect the ranking order of alternatives in the group decision-making problem, the method proposed in this paper can provide more general and more flexible selecting way for decision makers when the indeterminate degree for *I* in NLNs is assigned different ranges in de-neutrosophication process. Therefore, the decision makers can select some ranges of indeterminate degrees for *I* in NLNs according to their preference or real requirements and have flexibility in real decision-making problems.

7. Conclusion

This paper proposed the concepts of NLVs and NLSs and introduced the operational laws and the expected value of a NLV for ranking NLVs. Then, we proposed the two aggregation operators of NLWAA and NLWGA to aggregate neutrosophic linguistic information and investigated their properties. Furthermore, a decision-making method based on the NLWAA and NLWGA operators was established to handle group decision-making problems with neutrosophic linguistic information. Finally, an illustrative example was given to demonstrate the application of the proposed method. The proposed neutrosophic linguistic multiple attribute group decision-making method is more suitable for real scientific and engineering applications because it easily express and handle the indeterminate linguistic information which exists commonly in real life. In future work, we should further extend the developed method to assignment and resource allocation problems where the indeterminate information of the problems is specified uncertainly.

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I	NLWAA	Ranking order
$I \in [-0.7, 0]$	$E(\bar{l}_1) = 0.6399$, $E(\bar{l}_2) = 0.6532$, $E(\bar{l}_3) = 0.6896$, $E(\bar{l}_4) = 0.6955$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.5, 0]$	$E(\bar{l}_1) = 0.6479$, $E(\bar{l}_2) = 0.6576$, $E(\bar{l}_3) = 0.6978$, $E(\bar{l}_4) = 0.7038$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.3, 0]$	$E(\bar{l}_1) = 0.6559$, $E(\bar{l}_2) = 0.6620$, $E(\bar{l}_3) = 0.7060$, $E(\bar{l}_4) = 0.7122$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.1, 0]$	$E(\bar{l}_1) = 0.6640$, $E(\bar{l}_2) = 0.6665$, $E(\bar{l}_3) = 0.7142$, $E(\bar{l}_4) = 0.7205$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I = 0$	$E(\bar{l}_1) = 0.6680$, $E(\bar{l}_2) = 0.6687$, $E(\bar{l}_3) = 0.7183$, $E(\bar{l}_4) = 0.7247$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.1]$	$E(\bar{l}_1) =0.6720$, $E(\bar{l}_2) = 0.6709$, $E(\bar{l}_3) = 0.7224$, $E(\bar{l}_4) = 0.7288$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.3]$	$E(\bar{l}_1) = 0.6801$, $E(\bar{l}_2) = 0.6753$, $E(\bar{l}_3) = 0.7306$, $E(\bar{l}_4) = 0.7372$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.5]$	$E(\bar{l}_1) = 0.6881$, $E(\bar{l}_2) = 0.6797$, $E(\bar{l}_3) = 0.7388$, $E(\bar{l}_4) = 0.7455$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.7]$	$E(\bar{l}_1) = 0.6961$ $E(\bar{l}_2) = 0.6842$ $E(\bar{l}_3) = 0.7470$ $E(\bar{l}_4) = 0.7538$	$u_4 \succ u_3 \succ u_1 \succ u_2$

Table 2. Decision results based on the NLWGA operator by choosing different indeterminate ranges for I in NLNs

I	NLWGA	Ranking order
$I \in [-0.7, 0]$	$E(\bar{l}_1) = 0.6285$, $E(\bar{l}_2) = 0.6470$, $E(\bar{l}_3) = 0.6887$, $E(\bar{l}_4) = 0.6956$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.5, 0]$	$E(\bar{l}_1) = 0.6369$, $E(\bar{l}_2) = 0.6518$, $E(\bar{l}_3) = 0.6968$, $E(\bar{l}_4) = 0.7039$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.3, 0]$	$E(\bar{l}_1) =0.6452$, $E(\bar{l}_2) = 0.6566$, $E(\bar{l}_3) = 0.7050$, $E(\bar{l}_4) = 0.7122$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [-0.1, 0]$	$E(\bar{l}_1) =0.6535$, $E(\bar{l}_2) = 0.6614$, $E(\bar{l}_3) = 0.7131$, $E(\bar{l}_4) = 0.7205$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I = 0$	$E(\bar{l}_1) = 0.6577$, $E(\bar{l}_2) = 0.6638$, $E(\bar{l}_3) = 0.7172$, $E(\bar{l}_4) = 0.7247$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.1]$	$E(\bar{l}_1) = 0.6619$, $E(\bar{l}_2) = 0.6662$, $E(\bar{l}_3) = 0.7213$, $E(\bar{l}_4) = 0.7288$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.3]$	$E(\bar{l}_1) = 0.6702$, $E(\bar{l}_2) = 0.6710$, $E(\bar{l}_3) = 0.7294$, $E(\bar{l}_4) = 0.7371$	$u_4 \succ u_3 \succ u_2 \succ u_1$
$I \in [0, 0.5]$	$E(\bar{l}_1) = 0.6786$, $E(\bar{l}_2) = 0.6758$, $E(\bar{l}_3) = 0.7376$, $E(\bar{l}_4) = 0.7454$	$u_4 \succ u_3 \succ u_1 \succ u_2$
$I \in [0, 0.7]$	$E(\bar{l}_1) = 0.6869$, $E(\bar{l}_2) = 0.6806$, $E(\bar{l}_3) = 0.7457$, $E(\bar{l}_4) = 0.7537$	$u_4 \succ u_3 \succ u_1 \succ u_2$