A Lower Bound for the Smarandache Function Value $S(n!\pm 1)$

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***Abstract*—For any integer $a>0$, the famous Smarandache function $S(a)$ is defined as the smallest positive integer k such that $a|k!$. That is $S(a)=\min\{k|k\in \mathrm{N}, a|k!\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the lower bound of the Smarandache function $S(a)$, and get that if $n>10^3$, then $S(n!\pm 1)/n\ge[\log n/\log\log n]$, where $[\log n/\log\log n]$ is the integral part of $\log n/\log\log n$.**

Keywords-Smarandache function;shifted factorial; lower bound

I. INTRODUCTION

Let Z be the set of all integers,N be the set of all positive integers. For any positive integer a, the Smarandache function $S(a)$ is defined as the smallest integer $k>0$ such that $a|k!$ .That is

$$S(a)=\min\{k|k\in N, a|k!\}. \tag{1}$$

For any positive integer n ,the shifted factorial is defined as the positive integer $n!\pm 1$. About these two sequences,some authors had studied them,and obtained many interesting results.For example,recently,in [1] , J.Sándor and F.Luca had studied the properties of $S(n!+1)/n$ by the prime divisors results involving the shifted factorial in [2]and prove that

$$\limsup_{n\to\infty}\frac{S(n!+1)}{n}\ge 5.5. \tag{2}$$

By abc-conjecture supposing results first proposed by M.Murthy and S.Wong in[3] , they also proved that if abc-conjecture is correct,then

$$\liminf_{n\to\infty}\frac{S(n!+1)}{n}=\infty, \tag{3}$$

where the famous abc-conjecture was proposed by J.Oesterlé[4] and D.W.Masser [5]: Let a , b and c are coprime positive integers satisfying $a+b=c$,for any positive number ε , the $rad(abc)$ of distinct prime divisors of abc satisfy $c<C(\varepsilon)(rad(abc))^{1+\varepsilon}$, where $C(\varepsilon)$ is a computable constant involving ε .It is a difficult and unsolved problem for ours today(see problem B19 in [6]) .

The main purpose of this paper is using the elementary methods to study the distribution properties of $S(n!\pm 1)/n$, and give a general result for it under no conditions. That is ,we shall prove the following conclusion:

Theorem Let n be any positive integer satisfying $n>10^3$,then we have the inequality

$$\frac{S(n!\pm 1)}{n}\ge\left[\frac{\log n}{\log\log n}\right], \tag{4}$$

where $[x]$ denotes the greatest integer less than or equal to x .

From this Theorem we may immediately deduce the following corollary:

Corollary 1

$$\lim_{n\to\infty}\frac{S(n!\pm 1)}{n}=\infty. \tag{5}$$

It is clear that corollary1 not only improves the result (2),but also proves the result (3) under no conditions.

In addition,using the above Theorem,we can give the lower bound for prime divisors of the shifted factorial. Let p_n is the largest prime divisors of $n!+1$,P.Erdös and C.L.Stewart [7] proved :There exist infinite positive integers such that $p_n>2n$ .Lately, [2]obtained the following deeply result:For any positive number ε ,the positive integer n satisfying the inequality $p_n>(5.5-\varepsilon)n$ have the positive density rate.On the other hand, [3] proved: Suppose that abc-conjecture is true,we can get

$$\liminf_{n\to\infty}\frac{p_n}{n}=\infty.$$

From the Theorem, we may also derive:

Corollary 2 Let n be any positive integer satisfying

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$n > 10^3$,then there must exist a prime divisor p of the shifted shifted factorial $n!\pm 1$ such that

$$\frac{pr}{n} \ge \left[\frac{\log n}{\log\log n} \right], \quad (6)$$

where r denotes the frequency of p in the prime power factorization for $n!\pm 1$.

II. Four Lemmas

From the properties of the Smarandache function $S(a)$ in [8],we can give easyly the following two lemmas:

Lemma 1 Let $a = p_1^{r_1} \cdots p_k^{r_k}$ be the factorization of a into prime powers,then

$$S(a) = \max\left\{S\left(p_1^{r_1}\right), \cdots, S\left(p_k^{r_k}\right)\right\}.$$

Lemma 2 For any prime p and positive integer r, we have the inequality

$$p \le S\left(p^r\right) \le pr.$$

Lemma 3 Let $x_1, x_2, \cdots, x_k$ $(k > 1)$ be arbitrary real numbers and $m, n_1, n_2, \cdots, n_k$ be positive integers, then

$$\left(x_1 + x_2 + \cdots + x_k\right)^m = \sum \binom{m}{n_1, n_2, \cdots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $\sum$ denotes the summation over all the sollutions $(n_1, n_2, \cdots, n_k)$ for the equation

$$n_1 + n_2 + \cdots + n_k = m$$

and $\binom{m}{n_1, n_2, \cdots, n_k} = \frac{m!}{n_1! n_2! \cdots n_k!}$ are the positive integers.

Proof see [9].

Lemma 4 Let x *and* y be the positive numbers satisfying

$$(x+1)^{x+1} > \frac{y}{e}, \quad (7)$$

then for $y > 10^3$,we have the inequality

$$x + 1 > \frac{\log y}{\log\log y}. \quad (8)$$

Proof If $x + 1 \le \frac{\log y}{\log\log y}$,then from (7),we can deduce that

$$\frac{\log y}{\log\log y}\left(\log\log y - \log\log\log y\right) > \log y - 1. \quad (9)$$

Moreover,since $\log\log y > 0$ for $y > 10^3$, hence from (9) we know that

$$\log\log y > (\log y)(\log\log\log y). \quad (10)$$

Let $z = \log\log y$,then from(10) we find that

$$\frac{\log\log y}{\log y} = \frac{z}{e^z} < \frac{z}{z + z^2/2} = \frac{1}{1 + z/2}. \quad (11)$$

It follows from (10) and (11) that

$$1 > \left(1 + \frac{z}{2}\right)\log z. \quad (12)$$

On the other hand,it is clear that

$$z > 1.93 \ , \ \log z > 0.65$$

beause of $y > 10^3$,then we obtain $1 > 1.27$ from (12).This is a contradiction.Since Lemma 4 is correct.

This completes the proof of Lemma 4.

III. Proof of the Theorem

In this section, we use above Lemmas to complete the proof of Theorem. Let $m = S(n!\pm 1)$. We know that

$$m! = (n!\pm 1)a, \quad a \in N \quad (13)$$

according to (1). We immediately get $m > n$ from (13). Suppose that $q = [m/n]$,then q is a positive integer and we can write

$$m = nq + s, s \in Z, 0 \le s < n. \quad (14)$$

Let

$$b = s!(n!)^q, \quad (15)$$

then from Lemma 3, (14) and (15),we deduce that

$$\frac{m!}{b} = \frac{m!}{s! n! \cdots n!} = \binom{m}{s, n, \cdots, n} \quad (16)$$

and $m!/b$ is a positive integer. Meanwhile, suppose that

$$x_1 = x_2 = \cdots = x_k \text{ and } k = q + 1,$$

then according to (14), (16) and Lemma 3, we obtain

$$\frac{m!}{b} < (q+1)^m. \quad (17)$$

Moreover,since $s < n$ by (14), hence $\gcd(b, n!\pm 1) = 1$ by (15).We further deduce from (13) that

$$\frac{m!}{b} = (n!\pm 1)\frac{a}{b} \ge n!\pm 1. \quad (18)$$

Combining (17) and (18),we may immediately obtain the inequality

$$(q+1)^m \ge n!. \quad (19)$$

According to $n! > (n/e)^n$ from Stirling formula,we have

$$(q+1)^m > \left(\frac{n}{e}\right)^n \qquad (20)$$

from (19).Because $m < n(q+1)$ from (14),so we get

$$(q+1)^{q+1} > \frac{n}{e} \qquad (21)$$

by (20). When $n > 10^3$, using Lemma4,we infer by(21) that

$$q+1 > \frac{\log n}{\log\log n}. \qquad (22)$$

Since $q \in N$, hence from (22)

$$q+1 > \left[\frac{\log n}{\log\log n}\right]. \qquad (23)$$

Finally, according to $m \geq nq$ by (14),we obtain (4) by (23).

This completes the proof of Theorem.

Now we prove Corollary 2. Let

$$n!\pm 1 = p_1^{r_1} \cdots p_k^{r_k} \qquad (24)$$

be the factorization of $n!\pm 1$ into prime powers and

$$S(p^r) = \max\left\{S(p_1^{r_1}), \cdots, S(p_k^{r_k})\right\},$$

then from Lemma 1,we may give equation

$$S(n!\pm 1) = S(p^r). \qquad (25)$$

Meanwhile $S(p^r) < pr$ by Lemma 2, thus , by (25) we know that

$$S(n!\pm 1) < pr. \qquad (26)$$

Combining (4) and(26),we may immediately obtain (6).

Thus,the Corollary 2 is proved.

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REFERENCES

[1] J.Sándor and F.Luca, On $S(n!+1)$, where S is the Smarandache function , Octogon Math.Mag., 2008,16(2) , pp.1024-1026.

[2] C.L.Stewart, On the greatest and least prime factors of $n!+1$, II , Publ.Math. Debrecen,2004,65(3-4) , pp.461-480.

[3] M.Murthy and S.wong, The abc-conjecture and prime divisors of the Lucas and Lehmer sequences , K.Peters, Number theory for the millenium III, Natick, MA, 2002, pp. 43-54.

[4] J.Oesterlé, Nouvelles approaches du Théoréme Fermat , Sem. Bourbaki, 1988,No. 694, pp. 1-21.

[5] D.W. Masser, Open problems, Analytic number theory, London: Imperial College,1995, pp.27-35.

[6] R.K.Guy, Unsolved problems in number theory, third edition,New York:Springer verlag, 2004.

[7] P.Erdös and C.L.Stewart, On the greatest and least prime factors of $n!+1$, J.London Math. Soc., 1976, 13(4), 513-519.

[8] J.Sándor, On certain new inequalities and limits for the Smarandache function, Smarandache Notions Journal,1998,9(1-2), pp.63-69.

[9] Fan Xuhui, On the divisibility of the Smarandache combinatorial sequence, Scientia Magna, vol. 3,No.3, 2007, pp.70-73.