A Neutrosophic Description Logic

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***Abstract*— Description Logics (DLs) are appropriate, widely used, logics for managing structured knowledge. They allow reasoning about individuals and concepts, *i.e.* set of individuals with common properties. Typically, DLs are limited to dealing with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership: we may say that an individual is an instance of a concept only to a certain degree, depending on the individual's properties. The DLs that deal with such fuzzy concepts are called fuzzy DLs. In order to deal with fuzzy, incomplete, indeterminate and inconsistent concepts, we need to extend the capabilities of fuzzy DLs further.**

In this paper we will present an extension of fuzzy $\mathcal{ALC}$, combining Smarandache's neutrosophic logic with a classical DL. In particular, concepts become neutrosophic (here neutrosophic means fuzzy, incomplete, indeterminate and inconsistent), thus, reasoning about such neutrosophic concepts is supported. We will define its syntax, its semantics, describe its properties.

***Index Terms*— neutrosophic logic, description logic, neutrosophic description logic, fuzzy description logic**

I. INTRODUCTION

The modelling and reasoning with uncertainty and imprecision is an important research topic in the Artificial Intelligence community. Almost all the real world knowledge is imperfect. A lot of works have been carried out to extend existing knowledge-based systems to deal with such imperfect information, resulting in a number of concepts being investigated, a number of problems being identified and a number of solutions being developed [1], [2], [3], [4].

Description Logics (DLs) have been utilized in building a large amount of knowledge-based systems. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. A main point is that DLs are considered as to be attractive logics in knowledge based applications as they are a good compromise between expressive power and computational complexity.

Nowadays, a whole family of knowledge representation systems has been build using DLs, which differ with respect to their expressiveness, their complexity and the completeness of their algorithms, and they have been used for building a variety of applications [5], [6], [7], [8].

The classical DLs can only deal with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership. There are many works attempted to extend the DLs using fuzzy set theory [9], [10], [11], [12], [13], [14]. These fuzzy DLs can only deal with fuzzy concepts but not incomplete, indeterminate, and inconsistent concepts (neutrosophic concepts). For example, "Good Person" is a neutrosophic concepts, in the sense that by different subjective opinions, the truth-membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.6, which is inconsistent, or the truth- membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.3, which is incomplete.

The set and logic that can model and reason with fuzzy, incomplete, indeterminate, and inconsistent information are called neutrosophic set and neutrosophic logic, respectively [15], [16]. In Smarandache's neutrosophic set theory,a neutrosophic set A defined on universe of discourse X, associates each element x in X with three membership functions: truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$, and $T_A(x), I_A(x), F_A(x)$ are independent. For simplicity, in this paper, we will extend Straccia's fuzzy DLs [9], [11] with neutrosophic logic, called neutrosophic DLs, where we only use two components $T_A(x)$ and $F_A(x)$, with $T_A(x) \in [0,1], F_A(x) \in [0,1], 0 \leq T_A(x) + F_A(x) \leq 2$. The neutrosophic DLs is based on the DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs. This allows us to adapt it easily to the different DLs presented in the literature. Another important point is that we will show that the additional expressive power has no impact from a computational complexity point of view. The neutrosophic $\mathcal{ALC}$ is a strict generalization of fuzzy $\mathcal{ALC}$, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a corresponding neutrosophic concept and neutrosophic terminological axiom, but not *vice versa*.

II. A QUICK LOOK TO FUZZY $\mathcal{ALC}$

We assume three alphabets of symbols, called *atomic concepts* (denoted by A), *atomic roles* (denoted by R) and *individuals* (denoted by a and b). [1]

[1]Through this work we assume that every metavariable has an optional subscript or superscript.

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A *concept* (denoted by C or D) of the language $\mathcal{ALC}$ is built out of atomic concepts according to the following syntax rules:

$$
\begin{array}{lllr l}
C, D & \longrightarrow & & \top| & \text{(top concept)} \\
& & & \bot| & \text{(bottom concept)} \\
& & & A| & \text{(atomic concept)} \\
& & & C \sqcap D| & \text{(concept conjunction)} \\
& & & C \sqcup D| & \text{(concept disjunction)} \\
& & & \neg C| & \text{(concept negation)} \\
& & & \forall R.C| & \text{(universal quantification)} \\
& & & \exists R.C & \text{(existential quantification)}
\end{array}
$$

Fuzzy DL extends classical DL under the framework of Zadeh's fuzzy sets [17].A *fuzzy set* S with respect to an universe U is characterized by a *membership function* $\mu_S : U \to [0, 1]$, assigning an S-membership degree, $\mu_S(u)$, to each element u in U. In fuzzy DL, (*i*) a concept C, rather than being interpreted as a classical set, will be interpreted as a fuzzy set and, thus, concepts become *fuzzy*; and, consequently, (*ii*) the statement "a is C", *i.e.* $C(a)$, will have a truth-value in $[0, 1]$ given by the degree of membership of being the individual a a member of the fuzzy set C.

A. Fuzzy Interpretation

A *fuzzy interpretation* is now a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is, as for the crisp case, the *domain*, whereas $.^{\mathcal{I}}$ is an *interpretation function* mapping

1) individual as for the crisp case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
2) a concept C into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$;
3) a role R into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$.

If C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as the *membership degree function* of the fuzzy concept (set) C w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the fuzzy concept C under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $.^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$
\begin{array}{lcl}
\top^{\mathcal{I}}(d) & = & 1 \\
\bot^{\mathcal{I}}(d) & = & 0 \\
(C \sqcap D)^{\mathcal{I}}(d) & = & \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) & = & \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) & = & 1 - C^{\mathcal{I}}(d) \\
(\forall R.C)^{\mathcal{I}}(d) & = & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) & = & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}.
\end{array}
$$

We will say that two concepts C and D are said to be *equivalent* (denoted by $C \cong D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the crisp non fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong \neg\bot$, $(C \sqcap D) \cong \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong \neg(\exists R.\neg C)$.

B. Fuzzy Assertion

A *fuzzy assertion* (denoted by ψ) is an expression having one of the following forms $\langle \alpha \geq n \rangle$ or $\langle \alpha \leq m \rangle$, where α is an $\mathcal{ALC}$ assertion, $n \in (0, 1]$ and $m \in [0, 1)$. From a semantics point of view, a fuzzy assertion $\langle \alpha \leq n \rangle$ constrains the truth-value of α to be less or equal to n (similarly for $\geq$). Consequently, *e.g.* $\langle$ (**Video** $\sqcap$ $\exists$**About.Basket**)(**v1**) $\geq 0.8 \rangle$ states that video **v1** is likely about basket. Formally, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \geq n \rangle$ (resp. $\langle R(a, b) \geq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \leq n \rangle$ (resp. $\langle R(a, b) \leq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$). Two fuzzy assertion ψ_1 and ψ_2 are said to be *equivalent* (denoted by $\psi_1 \cong \psi_2$) iff they are satisfied by the same set of interpretations. An *atomic fuzzy assertion* is a fuzzy assertion involving an atomic assertion (assertion of the form $A(a)$ or $R(a, b)$).

C. Fuzzy Terminological Axiom

From a syntax point of view, a *fuzzy terminological axiom* (denoted by $\tilde{\tau}$ is either a fuzzy concept specialization or a fuzzy concept definition. A *fuzzy concept specialization* is an expression of the form $A \prec C$, where A is an atomic concept and C is a concept. On the other hand, a *fuzzy concept definition* is an expression of the form $A :\approx C$, where A is an atomic concept and C is a concept. From a semantics point of view, a fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy concept specialization $A \prec C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d), \tag{1}$$

whereas $\mathcal{I}$ *satisfies* a fuzzy concept definition $A :\approx C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d). \tag{2}$$

D. Fuzzy Knowlege Base, Fuzzy Entailment and Fuzzy Subsumption

A *fuzzy knowledge base* is a finite set of fuzzy assertions and fuzzy terminological axioms. Σ_A denotes the set of fuzzy assertions in Σ, Σ_T denotes the set of fuzzy terminological axioms in Σ (the terminology), if $\Sigma_T = \emptyset$ then Σ is *purely assertional*, and we will assume that a terminology Σ_T is such that no concept A appears more than once on the left hand side of a fuzzy terminological axiom $\tilde{\tau} \in \Sigma_T$ and that no cyclic definitions are present in Σ_T.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a set of fuzzy Σ iff $\mathcal{I}$ satisfies each element of Σ. A fuzzy KB Σ *fuzzy entails* a fuzzy assertion ψ (denoted by $\Sigma \models_f \psi$) iff every model of Σ also satisfies ψ.

Furthermore, let Σ_T be a terminology and let C, D be two concepts. We will say that D *fuzzy subsumes* C w.r.t. Σ_T (denoted by $C \preceq_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of Σ_T, $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$ holds.

III. A NEUTROSOPHIC DL

Our neutrosophic extension directly relates to Smarandache's work on neutrosophic sets [15], [16]. A neutrosophic set S defined on universe of discourse U, associates each element u in U with three membership functions: truth-membership function $T_S(u)$, indeterminacy-membership function $I_S(u)$, and falsity-membership function $F_S(u)$, where $T_S(u), I_S(u), F_S(u)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$, and $T_S(u), I_S(u), F_S(u)$ are independent. For simplicity, here we only use two components $T_S(u)$ and $F_S(u)$, with $T_S(u) \in [0,1], F_S(u) \in [0,1], 0 \leq T_S(u) + F_S(u) \leq 2$. It is easy to extend our method to include indeterminacy-membership function. $T_S(u)$ gives us an estimation of degree of u belonging to U and $F_S(u)$ gives us an estimation of degree of u not belonging to U. $T_S(u) + F_S(u)$ can be 1 (just as in classical fuzzy sets theory). But it is not necessary. If $T_S(u)+F_S(u) < 1$, for all u in U, we say the set S is incomplete, if $T_S(u)+F_S(u) > 1$, for all u in U, we say the set S is inconsistent. According to Wang [16], the truth-membership function and falsity-membership function has to satisfy three restrictions: for all $u \in U$ and for all neutrosophic sets S_1, S_2 with respect to U

$$T_{S_1 \cap S_2}(u) = \min\{T_{S_1}(u), T_{S_2}(u)\},$$

$$F_{S_1 \cap S_2}(u) = \max\{F_{S_1}(u), F_{S_2}(u)\},$$

$$T_{S_1 \cup S_2}(u) = \max\{T_{S_1}(u), T_{S_2}(u)\},$$

$$F_{S_1 \cup S_2}(u) = \min\{F_{S_1}(u), F_{S_2}(u)\}$$

$$T_{\overline{S_1}}(u) = F_{S_1}(u), F_{\overline{S_1}}(u) = T_{S_1}(u),$$

where $\overline{S_1}$ is the complement of S_1 in U. Wang [16] gives the definition of N-norm and N-conorm of neutrosophic sets, $\min$ and $\max$ is only one of the choices. In general case, they may be the simplest and the best.

When we switch to neutrosophic logic, the notion of degree of truth-membership $T_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set S over U is regarded as the *truth-value* of the statement "u is S", and the notion of degree of falsity-membership $F_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set S over U is regarded as the *falsity-value* of the statement "u is S". Accordingly, in our neutrosophic DL, (*i*) a concept C, rather than being interpreted as a fuzzy set, will be interpreted as a neutrosophic set and, thus, concepts become *imprecise* (fuzzy, incomplete, and inconsistent); and, consequently, (*ii*) the statement "a is C", *i.e.* $C(a)$ will have a truth-value in $[0, 1]$ given by the degree of truth-membership of being the individual a a member of the neutrosophic set C and a falsity-value in $[0, 1]$ given by the degree of falsity-membership of being the individual a not a member of the neutrosophic set C.

A. Neutrosophic Interpretation

A em neutrosophic interpretation is now a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, (\cdot)^{\mathcal{I}}, |\cdot|^t, |\cdot|^f)$, where $\Delta^{\mathcal{I}}$ is, as for the fuzzy case, the *domain*, and

1) $(\cdot)^{\mathcal{I}}$ is an *interpretation function* mapping
   a) individuals as for the fuzzy case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
   b) a concept C into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1] \times [0,1]$;
   c) a role R into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1] \times [0,1]$.
2) $|\cdot|^t$ and $|\cdot|^f$ are *neutrosophic valuation*, *i.e.* $|\cdot|^t$ and $|\cdot|^f$ map
   a) every atomic concept into a function from $\Delta^{\mathcal{I}}$ to $[0, 1]$;
   b) every atomic role into a function from $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to $[0, 1]$.

If C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as a pair of membership functions $\langle |C|^t, |C|^f \rangle$ of the neutrosophic concept (set) C w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the neutrosophic concept C and the degree of being the object d not an element of the neutrosophic concept C under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $(\cdot)^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$\begin{array}{lcl}
\top^{\mathcal{I}}(d) & = & \langle 1, 0 \rangle \\
\bot^{\mathcal{I}}(d) & = & \langle 0, 1 \rangle \\
(C \sqcap D)^{\mathcal{I}}(d) & = & \langle \min\{|C|^t(d), |D|^t(d)\}, \\
 & & \max\{|C|^f(d), |D|^f(d)\}\rangle \\
(C \sqcup D)^{\mathcal{I}}(d) & = & \langle \max\{|C|^t(d), |D|^t(d)\}, \\
 & & \min\{|C|^f(d), |D|^f(d)\}\rangle \\
(\neg C)^{\mathcal{I}}(d) & = & \langle |C|^f(d), |C|^t \rangle \\
(\forall R.C)^{\mathcal{I}}(d) & = & \langle \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d,d'), |C|^t(d')\}\}, \\
 & & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d,d'), |C|^f(d')\}\}\rangle \\
(\exists R.C)^{\mathcal{I}}(d) & = & \langle \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d,d'), |C|^t(d')\}\}, \\
 & & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d,d'), |C|^f(d')\}\}\rangle
\end{array}$$

Note that the semantics of $\forall R.C$
$(\forall R.C)^{\mathcal{I}}(d) = \langle \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d,d'), |C|^t(d')\}\}, \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d,d'), |C|^f(d')\}\}\rangle$
is the result of viewing $\forall R.C$ as the open first order formula $\forall y.\neg F_R(x,y) \vee F_C(y)$, where the universal quantifier $\forall$ is viewed as a conjunction over the elements of the domain. Similarly, the semantics of $\exists R.C$
$(\exists R.C)^{\mathcal{I}}(d) = \langle \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d,d'), |C|^t(d')\}\}, \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d,d'), |C|^f(d')\}\}\rangle$
is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x,y) \wedge F_C(y)$ and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Moreover, $|\cdot|^t$ and $|\cdot|^f$ are extended to complex concepts as follows: for all $d \in \Delta^{\mathcal{I}}$

$$\begin{array}{lcl}
|C \sqcap D|^t(d) & = & \min\{|C|^t(d), |D|^t(d)\} \\
|C \sqcap D|^f(d) & = & \max\{|C|^f(d), |D|^f(d)\} \\
 & & \\
|C \sqcup D|^t(d) & = & \max\{|C|^t(d), |D|^t(d)\} \\
|C \sqcup D|^f(d) & = & \min\{|C|^f(d), |D|^f(d)\} \\
 & & \\
|\neg C|^t(d) & = & |C|^f(d) \\
|\neg C|^f(d) & = & |C|^t(d) \\
 & & \\
|\forall R.C|^t(d) & = & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R(d,d')|^f, |C|^t(d)\}\} \\
|\forall R.C|^f(d) & = & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R(d,d')|^t, |C|^f(d)\}\} \\
 & & \\
|\exists R.C|^t(d) & = & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R(d,d')|^t, |C|^t(d)\}\} \\
|\exists R.C|^f(d) & = & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R(d,d')|^f, |C|^f(d)\}\}
\end{array}$$

We will say that two concepts C and D are said to be *equivalent* (denoted by $C \cong^n D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong^n \neg\bot, (C \sqcap D) \cong^n \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong^n \neg(\exists R.\neg C)$.

B. Neutrosophic Assertion

A *neutrosophic assertion* (denoted by φ) is an expression having one of the following form $\langle \alpha :\geq n, \leq m \rangle$ or $\langle \alpha :\leq n, \geq m \rangle$, where α is an $\mathcal{ALC}$ assertion, $n \in [0,1]$ and $m \in [0,1]$. From a semantics point of view, a neutrosophic assertion $\langle \alpha :\geq n, \leq m \rangle$ constrains the truth-value of α to be greater or equal to n and falsity-value of α to be less or equal to m (similarly for $\langle \alpha :\leq n, \geq m \rangle$). Consequently, *e.g.* $\langle$(**Poll** $\sqcap$ $\exists$**Support.War_x**)(**p1**) $:\geq 0.8, \leq 0.1 \rangle$ states that poll **p1** is close to support War_x. Formally, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\geq n, \leq m \rangle$ (resp. $\langle R(a,b) :\geq n, \leq m \rangle$) iff $|C|^t(a^{\mathcal{I}}) \geq n$ and $|C|^f(a^{\mathcal{I}}) \leq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$ and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq m$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\leq n, \geq m \rangle$ (resp. $\langle R(a,b) :\leq n, \geq m \rangle$) iff $|C|^t(a^{\mathcal{I}}) \leq n$ and $|C|^f(a^{\mathcal{I}}) \geq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$ and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq m$). Two fuzzy assertion φ_1 and φ_2 are said to be *equivalent* (denoted by $\varphi_1 \cong^n \varphi_2$) iff they are satisfied by the same set of interpretations. Notice that $\langle \neg C(a) :\geq n, \leq m \rangle \cong^n \langle C(a) :\leq m, \geq n \rangle$ and $\langle \neg C(a) :\leq n, \geq m \rangle \cong^n \langle C(a) :\geq m, \leq n \rangle$. An *atomic neutrosophic assertion* is a neutrosophic assertion involving an atomic assertion.

C. Neutrosophic Terminological Axiom

Neutrosophic terminological axioms we will consider are a natural extension of fuzzy terminological axioms to the neutrosophic case. From a syntax point of view, a *neutrosophic terminological axiom* (denoted by $\hat{\tau}$) is either a neutrosophic concept specialization or a neutrosophic concept definition. A *neutrosophic concept specialization* is an expression of the form $A \prec^n C$, where A is an atomic concept and C is a concept. On the other hand, a *neutrosophic concept definition* is an expression of the form $A :\approx^n C$, where A is an atomic concept and C is a concept. From a semantics point of view, we consider the natural extension of fuzzy set to the neutrosophic case [15], [16]. A neutrosophic interpretation $\mathcal{I}$ *satisfies* a neutrosophic concept specialization $A \prec^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) \leq |C|^t(d), |A|^f(d) \geq |C|^f(d), \quad (3)$$

whereas $\mathcal{I}$ *satisfies* a neutrosophic concept definition $A :\approx^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) = |C|^t(d), |A|^f(d) = |C|^f(d). \quad (4)$$

D. Neutrosophic Knowledge Base, Neutrosophic Entailment and Neutrosophic Subsumption

A *neutrosophic knowledge base* is a finite set of neutrosophic assertions and neutrosophic terminological axioms. As for the fuzzy case, with Σ_A we will denote the set of neutrosophic assertions in Σ, with Σ_T we will denote the set of neutrosophic terminological axioms in Σ (the terminology), if $\Sigma_T = \emptyset$ then Σ is *purely assertional*, and we will assume that a terminology Σ_T is such that no concept A appears more than once on the left hand side of a neutrosophic terminological axiom $\hat{\tau} \in \Sigma_T$ and that no cyclic definitions are present in Σ_T.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a neutrosophic Σ iff $\mathcal{I}$ satisfies each element of Σ. A neutrosophic KB Σ *neutrosophically entails* a neutrosophic assertion φ (denoted by $\Sigma \models^n \varphi$) iff every model of Σ also satisfies φ.

Furthermore, let Σ_T be a terminology and let C, D be two concepts. We will say that D *neutrosophically subsumes* C w.r.t. Σ_T (denoted by $C \preceq^n_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of Σ_T, $\forall d \in \Delta^{\mathcal{I}}, |C|^t(d) \leq |D|^t(d)$ and $|C|^f(d) \geq |D|^f(d)$ holds.

Finally, given a neutrosophic KB Σ and an assertion α, we define the *greatest lower bound* of α w.r.t. Σ (denoted by $glb(\Sigma, \alpha)$) to be $\langle \sup\{n : \Sigma \models^n \langle \alpha :\geq n, \leq m \rangle\}, \inf\{m : \Sigma \models^n \langle \alpha :\geq n, \leq m \rangle\} \rangle$. Similarly, we define the *least upper bound* of α with respect to Σ (denoted by $lub(\Sigma, \alpha)$) to be $\langle \inf\{n : \Sigma \models^n \langle \alpha :\leq n, \geq m \rangle\}, \sup\{m : \Sigma \models^n \langle \alpha :\leq n, \geq m \rangle\} \rangle$ ($\sup \emptyset = 0, \inf \emptyset = 1$). Determing the lub and the glb is called the *Best Truth-Value Bound* (BTVB) problem.

References

[1] F. Bacchus, *Representing and Reasoning with Probabilistic Knowledge*. The MIT Press, 1990.

[2] D. Dubois and H. Prade, "Approximate and commonsense reasoning: From theory to practice," in *Proceedings of the 9th Int. Sym. on Methodologies for Intelligent Systems (ISMIS-96)*. Springer-Verlag, 1996, pp. 19–33.

[3] R. Kruse, E. Schwecke, and J. Heinsohn, *Uncertainty and Vagueness in Knowledge Based Systems*. Springer-Verlag, Berlin, Germany, 1991.

[4] J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann, Los Altos, 1988.

[5] C. Peltason, "The back system - an overview," *ACM SIGART Bulletin*, vol. 2, no. 3, pp. 114–119, 1991.

[6] R. Brachman, "Reducing classic to practice: Knowledge representation meets reality," in *Proc. of the 3rd Int. Conf. on the Principles of Knowledge Representation and Reasoning (KR-92)*. Morgan Kaufmann, Los Altos, 1992.

[7] F. Baader and B. Hollunder, "Kris: Knowledge representation and inference system, system description," *ACM SIGART Bulletion*, vol. 2, pp. 8–14, 1991.

[8] I. Horrocks, "Using an expressive description logic: Fact or fiction?" in *Proc. of the 8th Int. COnf. on the Principles of Knowledge Representation and Reasoning (KR-98)*, 1998.

[9] U. Straccia, "A fuzzy description logic," in *Proc. of 15th National Conference on Artificial Intelligence (AAAI-98)*, 1998.

[10] ——, "A framework for the retrieval of multimedia objects based on four-valued fuzzy description logics," in *Soft Computing in Information Retrieval: Techniques and Applications*, F. Crestani and G. Pasi, Eds. Heidelberg, Germany: Springer Verlag, 2000.

[11] ——, "Reasoning within fuzzy description logics," *Journal of Artificial Intelligence Research*, vol. 14, pp. 137–166, 2001.

[12] R. da Silva, A. Pereira, and M. Netto, "A system of knowledge representation based on formulae of predicate calculus whose variables are annotated by expressions of a fuzzy terminologicval logic," in *Proc. of the 5th Int. COnf. on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU-94)*. Springer-Verlag, 1994.

[13] C. Tresp and R. Molitor, "A description logic for vague knowledge," in *Proc. of the 13th European COnf. on Artificial Intelligence (ECAI-98)*, 1998.

[14] J. Yen, "Generalizing term subsumption languages to fuzzy logic," in *Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI-91)*, 1991, pp. 472–477.

[15] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*. American Research Press, Rehoboth, 2003.

[16] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*. Hexis, Arizona, 2005. [Online]. Available: www.gallup.unm.edu/ smarandache/INSL.pdf

[17] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.