A Simple Argument Supporting Twin Prime Conjectures

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Abstract

Prime numbers are infinite since the time when Euclid gave his one of the most beautiful proof of this fact! Prime number theorem (PNT) reestablishes this fact and further it also gives estimate about the count of primes less than or equal to x. PNT states that as x tends to infinity the count of primes up to x tends to x divided by the natural logarithm of x. Twin primes are those primes p for which $p+2$ is also a prime number. The well known twin prime conjecture (TPC) states that twin primes are (also) infinite. Related to twin primes further conjectures that can be made by extending the thought along the line of TPC, are as follows: Prime numbers p for which $p+2n$ is also prime are (also) infinite for all n, where n = 1(TPC), 2, 3, …, k, …. In this paper we provide a simple argument in support of all twin prime conjectures.

1. **Introduction:** The celebrated prime number theorem (PNT) gives exact estimate for cardinality of primes up to x. If $\pi(x)$ denotes the number of primes less than or equal to x then

$$\pi(x) = \frac{x}{\ln(x)} \text{ as } x \to \infty .$$

Let us denote by $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$ the number of primes, p, less than or equal to x for which $p+2n$ is also a prime number. $\pi_{2n}(x)$ may be appropriately called as twin primes of n-type. Clearly, twin primes of 1-type are usual twin prime numbers.

2. **All kinds of Twin Primes are Infinite:** According to prime number theorem (PNT) when x is tending to infinity the cardinality of primes up to x, $\pi(x)$, is given by $\pi(x)=\dfrac{x}{\ln(x)}$.

Let us consider all possible distinct prime pairs made out of all prime numbers up to x. So, let $\{p_1, p_2, \cdots, p_k\}$ be the primes up to x. Then initially we will form following types of pairs of primes $\{(p_1,p_1),(p_2,p_2),\cdots,(p_k,p_k)\}$. The count of these pairs will be obviously equal to $\pi(x)$. We then will form the pairs of primes $\{(p_1,p_2),(p_2,p_3),\cdots,(p_{k-1},p_k)\}$, then in continuity we will form the pairs of primes $\{(p_1,p_3),(p_2,p_4),\cdots,(p_{k-2},p_k)\}$,….., finally we form $\{(p_1,p_k)\}$. Obviously, if we will consider the count of all these pairs of distinct prime together then it will be clearly equal to $\begin{pmatrix}\pi(x)\\2\end{pmatrix}$. Thus, the total count of all prime pairs will be will be

$$\pi(x)+\begin{pmatrix}\pi(x)\\2\end{pmatrix}=\frac{1}{2}\pi(x)(1+\pi(x))=\frac{1}{2}[\pi(x)+(\pi(x))^2] \quad \text{….(A)}$$

It is very interesting to observe that actually the count of these pairs is also equal to

$$\pi(x)+\pi_2(x)+\pi_4(x)+\cdots\pi_{2n}(x)+\cdots \quad \text{…..(B)}$$

thus, equating these two counts, (A) and (B), and dividing by both sides by $\pi(x)$ we have

$$\frac{1}{2}[1+\pi(x)]=[1+\frac{\sum_{n=1}^{\infty}\pi_{2n}(x)}{\pi(x)}] \quad \text{….(C)}$$

as $x \to \infty$.

Example: Let us consider primes up to first 50 integers.

Primes: {3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

Therefore, $\pi(50) = 14$.

So, $\frac{1}{2}[\pi(50)][1+\pi(50)] = 105$.

Also, .

$$\left[\pi(50) + \pi_2(50) + \pi_4(50) + \pi_6(50) + \cdots + \pi_{44}(50)\right] = 105.$$

As it is equal to:

$$\begin{bmatrix} 14+6+6+9+5+6+7+5+4+6+4+2+ \\ 6+4+3+4+2+3+3+2+2+1+1 \end{bmatrix} = 105.$$

It is clear to see that the LHS of above equation (C) goes to infinity $x \to \infty$, since as per PNT $\pi(x) = \frac{x}{\ln(x)}$ which clearly goes to infinity as $x \to \infty$. On RHS of equation (C) we have collection of terms made up of densities of twin primes of various types, $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$ divided by density of primes, $\pi(x)$. So, if the count of each term on RHS of equation (C), $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$, is finite and converges to some finite value as $x \to \infty$ then RHS of equation (C) will be made up of terms which are all equal to zero (except the first term which is equal to 1). Therefore RHS of equation (C) can't diverge to infinity. Actually, RHS of equation (C) should diverge to infinity to avoid contradiction since LHS of equation (C) contains $\pi(x)$ and so is diverging to infinity when $x \to \infty$. Now on RHS of equation (C) since we have a divergent quantity, namely, the density of total primes, $\pi(x)$, present in the denominator as mentioned above, therefore, each $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$ must be diverging to infinity, or some infinitely many of $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$, for some infinitely many *n*, must be diverging to infinity. But since count of primes counted in the each of the twin prime densities $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$ is actually a subset of $\pi(x)$ therefore each, or, at least some infinitely many of $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$, though should diverge to infinity still must be diverging slowly than $\pi(x)$ and therefore as $x \to \infty$ each ratio $\frac{\pi_{2n}(x)}{\pi(x)}, n = 1,2,\cdots,k,\cdots$ must be converging to a (nonzero) positive number, λ_n, such that series $1 + \sum_{n=1}^{\infty} \lambda_n$ should form a divergent series as required for maintaining consistency with the divergent LHS of equation (C)! Thus, at least some infinitely many of $\pi_{2n}(x), n = 1,2,\cdots,k,\cdots$, though slowly than $\pi(x)$ must be diverging to infinity as $x \to \infty$!!

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