Impulse solutions in optimization problems

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Abstract

The author considers the optimization problem named ‘the impulse regime’, when the control can have for a short time an instantaneous infinity value and the phase variables have gaps. In mathematics these mean: the variables are not continuous, not differentiable. The variable calculation and Pontryagin principle are not applicable. These problems are in space trajectories, theory of corrections, nuclear physics, economics, advertising and other real control tasks. We need a special theory and special methods for solution of these problems.

Author offers the following method, which simplifies and solves these tasks.

Introduction

Optimization methos are widely used in solving of technical problems. However, there are important classes of problems where they have great difficulties in the application. For example, in problems of space travel. The fact that the operational time of conventional rocket liquid propulsion is small (minutes), while the passive time of the interplanetary flight is large (months). In the result, we can consider the rocket work as an impulse, the speed as a jump which must expend minimum fuel. In mathematics, this means: the control is at an infinity value, the phase variables have a gap, and the variables are not continuous, not differentiable. The variable culculation and the Pontryagin principle are not applicable.

In 1968 the author offered the special methods [1] (see also [2 – 3]) for solution of the difference cases the impulse regime. In book [4] he applied this method to aerospace problems. Authors of work [5] developed the impulse theory for a particular case (linear version of control) using the theory of δ-functions. But his solutions are very complex and not acceptable in many practical problems.

In the given article the author offers a simpler method for solution of these problems: he shows the known impulse problems can be reduced to the special Pontryagin problem. Their solution can be simpler than existing methods.

Statement of the problem

1. **Statement of the conventional Optimization Problem**. Assume the state of system is described by conventional differential equations:

$$I = F[x(t_1), x(t_2)] + \int_{t_1}^{t_2} f_0(t,x,u)dt, \quad \dot{x}_i = f_i(t,x,u) \quad i = 1,2,...,n, \tag{1}$$

where I is the objective function, x is n – dimentinal continuous piece-difference function of phase coordinates; u is r – dimentional piece-continuous, piece-diffrence functions of control, $a_i \le u_i \le b_i$, i = 1,2,…,r , a,b = const; t is time. End values of $x(t_1)$, $x(t_2)$ are given or mobile. F is function of the end values $x(t)$.

We must find the control u, which gives the minumum the objective function I.

In our case (impulse problem) the control (or some its components) is at infinity (a very short time), the some (or all) phase variables have the gaps, and the variables are not continuous, not differentiable. The variable calculation and Pontryagin principle are not applicable.

2. Impulse Optimization Problem. Method of Solution.

The author offer the following method for solution of impulse problems.
We enter the special constants (unknown limited values) of impulses

$$v_i \quad i = 1,2,...,m. \tag{2}$$

These values may be binded the contions

$$x_i^+ = x_i^- + v_i\ , \quad \varphi_i(t,x,u,v) = 0 \quad i = 1,2,...,s\ , \quad s < m \tag{3}$$

and limitations

$$c_{i,1} \le v_i \le c_{im2} \quad i = 1,2,...,m\ , \tag{4}$$

where x_i^-, x_i^+ are x_i is phase coordinate on left and on right from point of impulse (gap), $c_{i,1}, c_{i,2}$ are consts. In particule, *v* can be unknown constant or zero.
The optimal problem is written in fo

$$I = F[x(t_1), x(t_2)] + \int_{t_1}^{t_2} f_0(t,x,u,v)dt, \quad \dot{x}_i = f_i(t,x,u,v) \quad i = 1,2,...,n,$$
$$\varphi_i(t,x,u,v) = 0, \quad i = 1,2,...,m \tag{5}$$

Where *v* are unknown limited impulses (gaps). End values of $x(t_1)$, $x(t_2)$ are given or mobile.
According [2], [3], we can write the generalized functionality introdused in form

$$J = I + \alpha, \tag{6}$$

where *J* - the generalized functionality introduced in [2],[3] p. 42, α is so named α – function introduced in [2],[3] (function equals zero on acceptable set, for example, on curves satisfying the equations (1) – (4)).
In our case we take

$$\alpha = \int_{t_1}^{t_2} \left[\sum_{i=1}^{i=n} \lambda_i(t,x)[\dot{x} - f_i(t,x,u,v)] + \sum_{i=n+i}^{i=n+m} \lambda_i(t,x)\varphi_i(t,x,u,v) \right] dt \tag{7}$$

Where $\lambda(t,x)$ is an unknown vector function.
We can re-write (6) as (see [3] p.42)

$$J = I + \alpha = A + \int_{t_1}^{t_2} Bdt, \tag{8}$$

where (for brevity repeated indices are summed):

$$A = F + \lambda_i x_i \,\Big|_{t_1}^{t_2}\ , \quad B = f_0 - \left(x_j \frac{\partial \lambda_j}{\partial x_i} + \lambda_i \right) f_i - x_i \frac{\partial \lambda_i}{\partial t}\ , \tag{9}$$

From Theorem 3.8 [3] we get: if we find at least one solution of particular equation about λ

$$J = \inf A + \inf B\ , \quad \inf_{u,v} \left[f_0 - \left(x_j \frac{\partial \lambda_j}{\partial x_i} + \lambda_i \right) f_i - x_i \frac{\partial \lambda_i}{\partial t} \right], \quad \frac{\partial B}{\partial x} = 0\ , \tag{10}$$

for the end condition inf *A*, we get optimal solution.
Note, the *B* (9) is different from the well-known Gamiltonian. If we will take the different function $\lambda(t,x)$, we will get the different conjugated system of equations $\partial B/\partial x = 0$.
In particular, if we will get $\lambda(t)$ ONLY as function *t*, we get the conventional Pontryagin principle of maximum

$$J = A + \int_{t_1}^{t_2} Bdt\ , \tag{11}$$

where

$$A = F + \sum_{i=1}^{i=n} [\lambda_i(t_2)x_i(t_2) - \lambda_i(t_1)x_i(t_1)]\ , \tag{12}$$

$$B = f_0(t,x,u,v) - \sum_{i=1}^{i=n} \lambda_i(t) f_i(t,x,u,v) - \sum_{i=n+i}^{i=n+m} \lambda_i(t)\varphi_i(t,x,u,v) \tag{13}$$

and equations

$$\dot{x}_i = f_i(t,x,u,v)\,, \quad i = 1,2,...,n$$

$$\dot{\lambda}_i = \frac{\partial B}{\partial x_i} \quad i = 1,2,...,n\,, \quad \inf_u B \quad \text{or} \quad \frac{\partial B}{\partial u_i} = 0 \quad i = ,2,...,r, \tag{14}$$

$$\inf_v B \quad \text{or} \quad \frac{\partial B}{\partial v_i} = 0 \quad i = r+1,2,...,m, \tag{15}$$

The equations

$$\frac{\partial B}{\partial u} = 0, \quad \frac{\partial B}{\partial v} = 0 \tag{16}$$

are used only in the open area. λ_i are unknown multipliers.

Equations (11) - (16) gives the optimal trajectoris (minimum of *I*) of the system (5). We also must solve the boundary value problem – find such $\lambda_i(t_1)$ that to get the given $x_i(t_2)$.

The gap time t_θ and gap v inside interval ($t_1 < t_\theta < t_2$) we can also find the next way. Write the objective function in form

$$I = F[x(t_1), x(t_2)] + \Phi(t_\theta, x_\theta) + \int_{t_1}^{t_2} f_0(t,x,u)dt,$$

$$\dot{x}_i = f_i(t,x,u) \quad i = 1,2,...,n, \tag{17}$$

where Φ is additional condition in t_θ (if they are given).

Write the general function as the sum of two functions in (t_1, t_θ) and ($t_1 < t_\theta < t_2$)

$$J = F + \psi_2 - \psi_1 + \Phi + \psi_\theta^+ - \psi_\theta^- + \int_{t_1}^{t_\theta} Bdt + \int_{t_\theta}^{t_2} Bdt\,,$$

$$\text{where} \quad \psi_\theta(t_\theta) = \lambda_i x_i\,, \quad \psi_2(t_2) = \lambda_i x_i \quad \psi_1(t_1) = \lambda_i x_i. \tag{18}$$

In t_θ the minimal condition are

$$\inf_{t_\theta, x_\theta} [\Phi(t_\theta, x_\theta) + \psi_\theta^+(t_\theta, x_\theta) - \psi_\theta^-(t_\theta, x_\theta)], \quad \inf_{x,u} B = 0\,. \tag{19}$$

Here up "-" and "+" are values from left and right from point t_θ. .

Notes:

1. We can find in form (3) ONLY the phase coordinates which we can aproximate as the impulse (in short time we can change a large value – for example, the speed in long flight, agle of trajectory, laser excitation of atom and so on). We cannot pukes space, distance, time.
2. The λ_i of corresponding coorditate has a gap/jump in moment of impulse. The moment (time) of gap or new λ_i (at right side) we can find (in open area) from the second equation (16). We must also to check up the ends of the intersal [t_1,t_2].
3. In some cases, the optimal value of gap we can find by the selection of v.
4. The λ_i of f_i are functions of t, the λ_i of φ_i are constants.

Example

Let us to consider the typical problem of space travel - transfer from one space orbit to other. Assume the space ship has circular Earth orbit having the radius r_1 and speed V_0 . We want to reach the ecliptic orbit having the maximal radius $r_2 > r_1$ and spend the minimum of fuel. The liquid rocket engine works some seconds, the space flight is some months. That way we can consider the rocket flight as pulse mode which instant change speed (gap the speed). Our task is to find minimal gap of speed (minimal inpulse) $v = \Delta V$, because the minimal gap of speed is equivalent of the minimal expenditure of the rocket fuel.

Our objective function

$$I = \int_0^t \Delta V dt \qquad (20)$$

The variables (speed V and radius r) of free space flight in the Earth gravitation field is binded by the Law of energy conservation (kinetic + potencial energy equils constant c):

$$\frac{mV^2}{2} - m\mu\left(\frac{1}{r_0} - \frac{1}{r}\right) = c\,, \quad \text{or} \quad V^2 = \mu\left(\frac{2}{r} - \frac{2}{r_1 + r_2}\right), \qquad (21)$$

Where m is mass space ship (satellite) mass, kg; r_0 is initial radius, m; μ is gravity constant. For Earth $\mu = 3.9802 \cdot 10^{14}$ m^3/s^2 , for Sun $\mu = 1{,}3276 \cdot 10^{20}$ m^3/s^2. That is elliptic orbite, r_1 is the radius of perigee; r_2 is the radius of apogee. We want to arrive from the circular orbite having V_0 , the radius $r_0 = r_1$ (the point of perigee) to point of apology r_2.

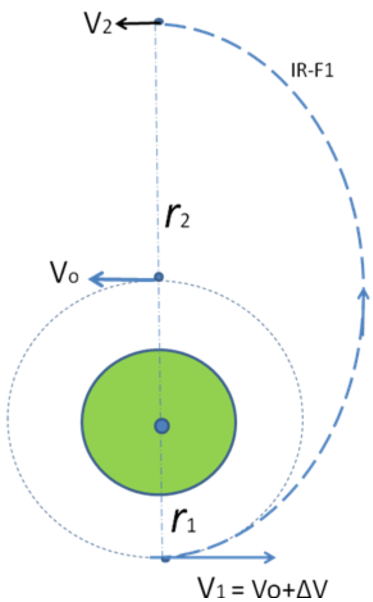

Fig.1. Orbite transver.

For elliptic orbits, the equation (21) may be re-writen in form:

$$(V_0 + \Delta V)^2 - 2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right) = 0 \quad \text{or} \quad V_0 + \Delta V - \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} = 0\,, \qquad (22)$$

where; V_0 is speed on circular orbite having the radius r_1. The speed of circular orbit is

$$V_0 = \sqrt{\frac{\mu}{r_0}}\,, \quad V_1 r_1 = V_2 r_2\,. \qquad (23)$$

Here V_2 is speed in r_2. Last equation in (23) is Law of momentum conservation free flight in the central gravitation field.

Let us the write the function B (13) for left end in right side of point t_1 .

$$B = \Delta V + \lambda\left[V_1 + \Delta V - \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)}\right], \qquad (24)$$

From equation (16) we have

$$\frac{\partial B}{\partial(\Delta V)} = 1 + \lambda = 0 . \qquad (25)$$

The equaetion (25) together with the equations (22),(23) allow to find the λ and the speed gap ΔV :

$$\lambda = -1, \quad \Delta V = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) = V_0\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) = V_0\left(\sqrt{\frac{2\bar{r}}{\bar{r} + 1}} - 1\right) = V_0\left(\frac{V_1}{V_a} - 1\right), \quad (26)$$

where

$$\bar{r} = \frac{r_2}{r_1}, \quad V_a = \sqrt{\frac{V_1^2 + V_2^2}{2}}, \quad V_1 r_1 = V_2 r_2 . \qquad (27)$$

Here V_2 is speed in apogee, V_a is average speed.
We reached the request r_2 by the first impulse. That way we don't need the additional impulse and reseach.

The formula (26) for computation ΔV is known as transfer in Gohman ellipse [6]. New is proof of optimization.

The reader can solve same way the more complex inpulse (gap) problems [4].

Referances

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