# New tests of multipartite entanglement in Bell-type experiments

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In trial, we especially consider inequalities for confirming multipartite entanglement from experimental data obtained in Bell-type experiments. We present new entanglement witness inequalities. Some physical situation is that we measure $\sigma_x$, $\sigma_y$, and $\sigma_z$ per side. Our analysis discovers a new multipartite entangled state and it is experimentally feasible. If the reduction factor $V$ of the interferometric contrast observed in a $N$-particle correlation experiment is $V > 0.4$, then a measured state is full $N$-partite entanglement in a significant specific case. It is not revealed by previous Bell-type experimentally feasible methods presented in [17], which states if $V > 0.5$ then the significant specific type state is full $N$-partite entanglement.

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## I. INTRODUCTION

Since the Svetlichny inequality, it has been a problem how to confirm multipartite entanglement experimentally [1]. And we have been given precious experimental data by efforts of experimentalists [2–6]. Proper analysis of these experimental data then becomes necessary, and as a result of such analysis [7], the experimental data obtained by Pan and co-workers [5] confirms the existence of genuinely three-particle entanglement in 2000. More recently, experimental violation of multipartite Bell inequalities with trapped ions is reported [8]. Device-independent tomography of multipartite quantum states is reported [9]. Demonstration of genuine multipartite entanglement with device-independent witnesses is also reported [10].

There have been many researches on the multipartite entanglement problem, providing inequalities for functions of experimental correlations [1, 7, 11–18]. Uffink introduced a nonlinear inequality aimed at giving stronger tests for full $N$-partite entanglement than previous formulas. It was also discussed that when the two measured observables are assumed to precisely anticommute, a stronger quadratic inequality can be used as a witness of full $N$-partite entanglement [17].

After that there are many researches of multipartite entanglement (cf. [19, 20]). We do not know the inequality presented in [17] is the optimal way in detection of multipartite entanglement in Bell-type experiment. In fact it is not so if we introduce measuring $\sigma_z$ per side. Here, we study more efficient way in this case.

In this paper, we investigate inequalities for confirming multipartite entanglement from experimental data obtained in Bell-type experiments. We present new inequalities to do so. Some physical situation is that we measure $\sigma_x$, $\sigma_y$, and $\sigma_z$ per side. Our analysis discovers a new multipartite entangled state and it is experimentally feasible. If the reduction factor $V$ of the interferometric contrast observed in a $N$-particle correlation experiment is $V > 0.4$, then a measured state is full $N$-partite entanglement in a significant specific case. It is not revealed by previous Bell-type experimentally feasible methods presented in [17], which states if $V > 0.5$ then the significant specific type state is full $N$-partite entanglement.

## II. TESTS OF MULTIPARTITE ENTANGLEMENT

We want to know if the following multipartite state is full $N$-partite entanglement experimentally. The value of $V$ can be interpreted as the reduction factor of the interferometric contrast observed in a $N$-particle correlation experiment.

$$\rho = V|GHZ\rangle\langle GHZ| + (1-V)|1...1\rangle\langle 1...1|, \quad (1)$$

where $|GHZ\rangle = \frac{|1...1\rangle + |0...0\rangle}{\sqrt{2}}$ is the $N$-partite Greenberger-Horne-Zeilinger (GHZ) state [21].

### A. Lemma

In what follows, we use the following lemma.

*Lemma [17]: Let $-\mathbf{1} \le A, B \le \mathbf{1}$ be Hermitian operators satisfying $\{A, B\} = \mathbf{0}$. Then*

$$\langle A\rangle^2 + \langle B\rangle^2 \le 1. \quad (2)$$

*Proof*: Suppose that $\{A, B\} = \mathbf{0}$ and $-\mathbf{1} \le A, B \le \mathbf{1}$. Let us take $C = A\cos\theta + B\sin\theta$, and derive the maximum value of $\mathrm{tr}[\rho C]$. Since we are interested only in the maximum, we may assume $A^2 = B^2 = \mathbf{1}$. Then we get $C^2 = \mathbf{1} + (1/2)\{A, B\}\sin 2\theta = \mathbf{1}$. The variance inequality leads to $|\mathrm{tr}[\rho C]|^2 \le \mathrm{tr}[\rho C^2] = 1$. Now take $\cos\theta = \langle A\rangle/\sqrt{\langle A\rangle^2 + \langle B\rangle^2}, \sin\theta = \langle B\rangle/\sqrt{\langle A\rangle^2 + \langle B\rangle^2}$, then we get $\langle A\rangle^2 + \langle B\rangle^2 \le 1$. QED.

### B. Previous methods

Let us consider the following Bell operators [22, 23]

$$X_N = 2^{(N-1)/2}(|1...1\rangle\langle 0...0| + |0...0\rangle\langle 1...1|),$$
$$Y_N = 2^{(N-1)/2}(-i|1...1\rangle\langle 0...0| + i|0...0\rangle\langle 1...1|). \quad (3)$$

We can measure the following operators by Bell-type experiments measuring $\sigma_x$ and $\sigma_y$ per side:

$$X = (2)(|1...1\rangle\langle 0...0| + |0...0\rangle\langle 1...1|),$$
$$Y = (2)(-i|1...1\rangle\langle 0...0| + i|0...0\rangle\langle 1...1|). \quad (4)$$

We may assume $-\mathbf{1} \leq X, Y \leq \mathbf{1}$ when the system is not in full $N$-partite entanglement. In fact, we have the following entanglement witness inequalities [18]

$$|\langle X\rangle| \leq 1, |\langle Y\rangle| \leq 1. \quad (5)$$

A violation of the rerations (5) means full $N$-partite entanglement. Let us consider the quantum state (1). After some algebra, we find that

$$|\langle X\rangle| = 2V, |\langle Y\rangle| = 0. \quad (6)$$

Hence we cannot see if the multipartite state (1) is fully entangled when we only use the formulas (5) and

$$V \leq 1/2. \quad (7)$$

From Lemma described above, we have the following entanglement witness inequality because $\{X, Y\} = \mathbf{0}$ and $-\mathbf{1} \leq X, Y \leq \mathbf{1}$ [17].

$$\langle X\rangle^2 + \langle Y\rangle^2 \leq 1. \quad (8)$$

A violation of the reration (8) means full $N$-partite entanglement. Let us consider the quantum state (1). After some algebra, we find that

$$\langle X\rangle^2 + \langle Y\rangle^2 = (2V)^2. \quad (9)$$

Hence we cannot see if the multipartite state (1) is fully entangled when we only use the formula (8) and

$$V \leq 1/2. \quad (10)$$

### C. New method

Let us consider the following operator.

$$Z_N = 2^{(N-1)/2}(|1...1\rangle\langle 1...1| - |0...0\rangle\langle 0...0|). \quad (11)$$

We can measure the following operators by Bell-type experiments measuring $\sigma_z$ and $I(=+1)$ per side:

$$Z = (|1...1\rangle\langle 1...1| - |0...0\rangle\langle 0...0|). \quad (12)$$

Clearly, we see $-\mathbf{1} \leq Z \leq \mathbf{1}$. We have the following entanglement witness inequalities [18]

$$|\langle X\rangle| \leq 1, |\langle Y\rangle| \leq 1. \quad (13)$$

We have the following quantum inequality

$$|\langle Z\rangle| \leq 1. \quad (14)$$

We see the following anti-commutation:

$$\{X, Y\} = \mathbf{0},$$
$$\{Y, Z\} = \mathbf{0},$$
$$\{Z, X\} = \mathbf{0}. \quad (15)$$

Finally, from Lemma, we derive a set of quadratic entanglement witness inequalities

$$\langle X\rangle^2 + \langle Y\rangle^2 \leq 1,$$
$$\langle Y\rangle^2 + \langle Z\rangle^2 \leq 1,$$
$$\langle Z\rangle^2 + \langle X\rangle^2 \leq 1. \quad (16)$$

A violation of one of inequalities (16) implies full $N$-partite entanglement. Here, we use new entanglement witness inequality as follows:

$$\langle Z\rangle^2 + \langle X\rangle^2 \leq 1. \quad (17)$$

Let us consider the quantum state (1). After some algebra, we find that

$$\langle X\rangle^2 + \langle Z\rangle^2$$
$$= (2V)^2 + (1-V)^2. \quad (18)$$

Hence we can see that the multipartite state (1) is fully entangled when

$$(2V)^2 + (1-V)^2 > 1. \quad (19)$$

For example, if $V = 1/2$ then

$$(2V)^2 + (1-V)^2 = 1 + 1/4 > 1. \quad (20)$$

Thus, the multipartite state (1) is fully entangled. It is not revealed by previous Bell-type experimentally feasible methods presented in [17]. In fact, we see

$$(2V)^2 + (1-V)^2$$
$$= 5V^2 - 2V + 1. \quad (21)$$

Thus, if $5V^2 - 2V > 0 \Rightarrow V > 2/5 = 0.4$, then the multipartite state (1) is fully entangled. Therefore we presenred new method to detect full $N$-partite entanglement. Is there more efficient way? This is open.

## III. CONCLUSIONS

In conclusions, we have considered inequalities for confirming multipartite entanglement from experimental data obtained in Bell-type experiments. We have presented new entanglement witness inequalities. Some physical situation has been that we measure $\sigma_x$, $\sigma_y$, and $\sigma_z$ per side. Our analysis has discovered a new multipartite entangled state and it has been experimentally feasible. If the reduction factor $V$ of the interferometric contrast observed in a $N$-particle correlation experiment has been $V > 0.4$, then a measured state has been full $N$-partite entanglement in a significant specific case. It has not been revealed by previous Bell-type experimentally feasible methods presented in [17], which states if $V > 0.5$ then the significant specific type state is full $N$-partite entanglement.


[1] G. Svetlichny, Phys. Rev. D 35, 3066 (1987).
[2] D. Bouwmeester, J. -W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999).
[3] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, Nature (London) 404, 256 (2000).
[4] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. -M. Raimond, and S. Haroche, Science 288, 2024 (2000).
[5] J. -W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) 403, 515 (2000).
[6] J. -W. Pan, M. Daniell, S. Gasparoni, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. 86, 4435 (2001).
[7] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. A 65, 042314 (2002).
[8] B. P. Lanyon, M. Zwerger, P. Jurcevic, C. Hempel, W. Dur, H. J. Briegel, R. Blatt, and C. F. Roos, Phys. Rev. Lett. 112, 100403 (2014).
[9] K. F. Pal, T. Vertesi, and M. Navascues, Phys. Rev. A 90, 042340 (2014).
[10] J. T. Barreiro, J. -D. Bancal, P. Schindler, D. Nigg, M. Hennrich, T. Monz, N. Gisin, and R. Blatt, Nature Physics 9, 559 (2013).
[11] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998).
[12] R. F. Werner and M. M. Wolf, Phys. Rev. A 61, 062102 (2000).
[13] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. 88, 170405 (2002).
[14] M. Seevinck and J. Uffink, Phys. Rev. A 65, 012107 (2002).
[15] M. Seevinck and G. Svetlichny, Phys. Rev. Lett. 89, 060401 (2002).
[16] J. Uffink, Phys. Rev. Lett. 88, 230406 (2002).
[17] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. Lett. 89, 260401 (2002).
[18] K. Nagata, Phys. Rev. A 66, 064101 (2002).
[19] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Reviews of Modern Physics 81, 865 (2009).
[20] O. Guhne and G. Toth, Physics Reports 474, 1 (2009).
[21] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), p. 69.
[22] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[23] A.V. Belinskii and D. N. Klyshko, Phys. Usp. 36, 653 (1993).