Measurement theory in Deutsch's algorithm based on the truth values

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We propose a new measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement are either 0 or 1. To implement Deutsch's algorithm, we need both observability and controllability of a quantum state. The new measurement theory can satisfy these two. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

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I. INTRODUCTION

The projective measurement theory is indeed a successful quantum measurement theory. The projective measurement theory (cf. [1–6]) gives approximate and at times remarkably accurate numerical predictions. Much experimental data approximately fits to the projective measurement theory for the past some 100 years. We do not doubt the correctness of the project measurement theory. It is one of consistent quantum measurement theories.

As for the foundations of the projective measurement theory, Leggett-type non-local variables theory [7] is experimentally investigated [8–10]. The experiments report that the projective measurement theory does not accept Leggett-type non-local variables interpretation. As for the applications of the projective measurement theory, implementation of a quantum algorithm to solve Deutsch's problem [11, 12] on a nuclear magnetic resonance quantum computer is reported firstly [13]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [14]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [15]. Single-photon Bell states are prepared and measured [16]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such a single-photon and by using two logical qubits [17]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [18].

Rolf Landauer says that *Information is Physical* [5]. We cannot create any computer without physical phenomena. This fact motivates us to investigate the relation between physical phenomena and quantum computing. Especially, we investigate what measurement theories meet quantum computing.

In this paper, we propose a new measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement are either 0 or 1. To implement Deutsch's algorithm, we need both observability and controllability of a quantum state. The new measurement theory can satisfy these two. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

This paper is organized as follows:

In Sec. II, we investigate the relation between the double-slit experiment and the new measurement theory. We can measure a spin observable by using the new measurement theory. The new measurement theory can satisfy observability.

In Sec. III, we discuss the fact that the new measurement theory can satisfy controllability.

Section IV concludes this paper.

II. THE NEW MEASUREMENT THEORY CAN SATISFY OBSERVABILITY

In this section, by using the double-slit experiment, we consider the relation between the new measurement theory and observability. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are either 1 or 0 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is 0. This is an easy detector model of a single Pauli observable.

A. A wave function analysis

Let σ_x be a single Pauli observable. We assume that a source of a spin-carrying particle emits them in a state ρ. We consider a quantum expected value $\mathrm{Tr}[\rho\sigma_x]$. If we consider only a wave function analysis, the possible values of the square of the quantum expected value are

$$0 \leq (\mathrm{Tr}[\rho\sigma_x])^2 \leq 1. \tag{1}$$

We define $\|E_{\mathrm{QM}}\|^2$ as

$$\|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \tag{2}$$

Then we have

$$\|E_{\mathrm{QM}}\|^2_{\mathrm{min}} = 0 \text{ and } \|E_{\mathrm{QM}}\|^2_{\mathrm{max}} = 1. \tag{3}$$

$\|E_{\mathrm{QM}}\|^2_{\mathrm{max}}$ and $\|E_{\mathrm{QM}}\|^2_{\mathrm{min}}$ are the maximal and minimal possible values of the product, respectively. We get $\|E_{\mathrm{QM}}\|^2_{\mathrm{min}} = 0$ or $\|E_{\mathrm{QM}}\|^2_{\mathrm{max}} = 1$ if the system is a pure state lying in either the z-axis or the x-axis, respectively.

B. New measurement theory

A mean value E satisfies the new measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}, \tag{4}$$

where l denotes a notation and r is the result of the new measurement of the Pauli observable σ_x. We assume the values of r are either 1 or 0 (in $\hbar/2$ unit). Assume the quantum mean values with the system in a state admits the new measurement theory. One has the following proposition concerning the new measurement theory

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}. \tag{5}$$

We can assume the following by Strong Law of Large Numbers,

$$\mathrm{Tr}[\rho\sigma_x](+\infty) = \mathrm{Tr}[\rho\sigma_x]. \tag{6}$$

We define $\|E_{\mathrm{QM}}\|^2(m)$ as

$$\|E_{\mathrm{QM}}\|^2(m) = (\mathrm{Tr}[\rho\sigma_x](m))^2. \tag{7}$$

We can assume the following by Strong Law of Large Numbers,

$$\|E_{\mathrm{QM}}\|^2(+\infty) = \|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \tag{8}$$

In what follows, we show that we can accept the relation (5) concerning the new measurement theory. Assume the proposition (5) is true. By changing the notation l into l', we have same quantum mean value as follows

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m}. \tag{9}$$

We have the following when the system is in a pure state lying in the x-axis.

$$\begin{aligned} &\|E_{\mathrm{QM}}\|^2(m) \\ &= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \\ &\leq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} |r_l(\sigma_x) r_{l'}(\sigma_x)| \\ &= \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = 1. \end{aligned} \tag{10}$$

Clearly, the above inequality can have the upper limit since the following case is possible:

$$\|\{l|l \in \mathbf{N} \wedge r_l(\sigma_x) = 1\}\| = \|\{l'|l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 1\}\|, \tag{11}$$

and

$$\|\{l|l \in \mathbf{N} \wedge r_l(\sigma_x) = 0\}\| = \|\{l'|l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 0\}\|. \tag{12}$$

And we have the following when the system is in a pure state lying in the z-axis.

$$\begin{aligned} &\|E_{\mathrm{QM}}\|^2(m) \\ &= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \\ &\geq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} (0) \\ &= (0) \left(\frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} \right) = 0. \end{aligned} \tag{13}$$

Clearly, the above inequality can have the lower limit since the following case is possible:

$$\|\{l|l \in \mathbf{N} \wedge r_l(\sigma_x) = 1\}\| = \|\{l'|l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 0\}\|, \tag{14}$$

and

$$\|\{l|l \in \mathbf{N} \wedge r_l(\sigma_x) = 0\}\| = \|\{l'|l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 1\}\|. \tag{15}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the new measurement theory is true (in a spin-1/2 system), that is

$$0 \leq \|E_{\mathrm{QM}}\|^2(m) \leq 1. \tag{16}$$

From Strong Law of Large Numbers, we have

$$0 \leq \|E_{\mathrm{QM}}\|^2 \leq 1. \tag{17}$$

Hence we derive the following proposition concerning the new measurement theory

$$\|E_{\mathrm{QM}}\|^2_{\mathrm{min}} = 0 \text{ and } \|E_{\mathrm{QM}}\|^2_{\mathrm{max}} = 1. \tag{18}$$

We can accept the two relations (3) (concerning a wave function analysis) and (18) (concerning the new measurement theory), simultaneously. The new measurement theory meets the wave function analysis and can measure Pauli observable σ_x correctly. Similar to the argumentations, the new measurement theory can measure Pauli observable σ_z correctly. In short, the new measurement theory meets observability of σ_z and σ_x.

III. THE NEW MEASUREMENT THEORY CAN SATISFY CONTROLLABILITY

In this section, we investigate if the new measurement theory can satisfy controllability of a pure state lying in either the z-axis or the x-axis. These quantum states are used in Deutsch's algorithm.

We introduce the following quantum proposition concerning controllability:

$$\langle 0|0\rangle = 1, \langle 1|1\rangle = 1, \langle 0|1\rangle = 0, \text{ and } \langle 1|0\rangle = 0. \quad (19)$$

The proposition (19) implies

$$|\langle 0|0\rangle|^2 = 1, |\langle 1|1\rangle|^2 = 1, |\langle 0|1\rangle|^2 = 0, \text{ and } |\langle 1|0\rangle|^2 = 0. \quad (20)$$

Clearly, the new measurement theory based on the truth values can satisfy all the expected values in the above proposition (20). And we may assume that the new measurement theory can satisfy the proposition (19).

The proposition (19) implies that

$$\langle \sigma_z\rangle^2 + \langle \sigma_x\rangle^2 = 1 \quad (21)$$

when the system is in a pure state lying in either the z-axis or the x-axis. The reason is as follows: Assume a pure state lying in either the z-axis or the x-axis as

$$|\psi_x\rangle = e^{i\phi}\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \quad (22)$$

and

$$|\psi_z\rangle = |0\rangle, |1\rangle, \quad (23)$$

where ϕ is a phase. These quantum states are used in Deutsch's algorithm. Let us write

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \quad (24)$$

and

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \quad (25)$$

Then we have

$$|\langle\psi_x|\sigma_z|\psi_x\rangle| = 0, |\langle\psi_x|\sigma_x|\psi_x\rangle| = 1. \quad (26)$$

and

$$|\langle\psi_z|\sigma_z|\psi_z\rangle| = 1, |\langle\psi_z|\sigma_x|\psi_z\rangle| = 0. \quad (27)$$

Therefore, we see

$$\langle \sigma_x\rangle^2 + \langle \sigma_z\rangle^2 = 1. \quad (28)$$

We thus see the proposition (19) implies the existence of either the z-axis or the x-axis in the Hilbert space formalism of the quantum theory. From the argumentations presented in Sec. III, the new measurement theory meets the relations (26) and (27). We see the new measurement theory can satisfy controllability of a pure state lying in either the z-axis or the x-axis, that is used in Deutsch's algorithm.

IV. CONCLUSIONS

In conclusion, we have proposed a new measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement have been either 0 or 1. To implement Deutsch's algorithm, we need both observability and controllability of a quantum state. The new measurement theory can have satisfied these two. Especially, we systematically have described our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

Our argumentations imply that we can perform the following Deutsch's algorithm.

- The control of quantum states relies on the wave functional analysis.
- The observation of quantum states relies on the new measurement theory based on the truth values.

Consistency between controllability and observability is necessary for an implementation of Deutsch's algorithm. And desired consistency is established.

Are there other measurement theories which can satisfy both controllability and observability? This is an open problem.

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