The Worldline of an Inertial Observer on the Kruskal-Szekeres Coordinate Chart

Christopher A. Laforet
claforet@gmail.com

Abstract

The (proper) transit time of a light signal emitted by an observer at rest at r_0 in a Schwarzschild gravitational field, reflected by body at some distance $r > r_0$ and received by the observer is shown to decrease as the observer is moved closer to the event horizon. This suggests that observers at rest close to the event horizon will see an increasingly length contracted Universe. We then seek to discover whether the free falling observer will also see the Universe similarly contracted as he/she falls toward the horizon. Using the definition of the Kruskal-Szekeres coordinates (*T*, *X*) in terms of Schwarzschild coordinates (*r*, *t*) and expanding the expression for *dT/dX*, it is shown that the worldlines of both timelike and spacelike observers must have slopes approaching 1 on the Kruskal-Szekeres coordinate chart implying that the free falling observer should indeed see an increasingly length contracted Universe as he/she falls toward the horizon.

Length Contraction for an Observer at Rest

Let us begin by considering an observer in a Schwarzschild gravitational field at r_0 sending a light signal to another observer at rest at $r > r_0$ who then reflects the signal back to a radius $r_1 \leq r_0$. We want to find the total time measured by the first observer's clock between emission of the first signal and receipt of the reflected signal. To do this we can use the definition of Kruskal-Szekeres (KS) coordinates [1] (throughout the paper, we are using units where the Schwarzschild radius is equal to 1):

$$X = \sqrt{(r-1)e^r}\cosh\left(\frac{t}{2}\right)$$
$$T = \sqrt{(r-1)e^r}\sinh\left(\frac{t}{2}\right) \tag{1}$$

Furthermore, we know that light signals travel on 45 degree lines on KS charts such that, for a light signal, $|\Delta T| = |\Delta X|$. For clarity, we define $R \equiv \sqrt{(r-1)e^r}$. For the first outgoing signal we have:

$$\Delta T = R\sinh\left(\frac{t_1}{2}\right) - R_0\sinh\left(\frac{t_0}{2}\right),\;\; \Delta X = R\cosh\left(\frac{t_1}{2}\right) - R_0\cosh\left(\frac{t_0}{2}\right) \tag{2}$$

Setting $\Delta T = \Delta X$ and using $\sinh\left(\frac{t}{2}\right) - \cosh\left(\frac{t}{2}\right) = -e^{-\frac{t}{2}}$, we get

$$t_1 = t_0 + (r - r_0) + \ln\left(\frac{r-1}{r_0-1}\right) \tag{3}$$

And for the reflected signal:

$$\Delta T = R_1 \sinh\left(\frac{t_2}{2}\right) - R\sinh\left(\frac{t_1}{2}\right), \ \ \Delta X = R_1 \cosh\left(\frac{t_2}{2}\right) - R\cosh\left(\frac{t_1}{2}\right) \tag{4}$$

Setting $\Delta T = -\Delta X$ and using $\sinh\left(\frac{t}{2}\right) + \cosh\left(\frac{t}{2}\right) = e^{\frac{t}{2}}$, we get

$$t_2 = t_1 + (r - r_1) + \ln\left(\frac{r-1}{r_1-1}\right) \tag{5}$$

Giving a total transit time of:

$$\Delta t = t_2 - t_0 = \left[(r - r_0) + \ln\left(\frac{r-1}{r_0-1}\right)\right] + \left[(r - r_1) + \ln\left(\frac{r-1}{r_1-1}\right)\right] \tag{6}$$

An observer at rest in Schwarzschild coordinates experiences proper time via the relationship $\Delta\tau = \sqrt{\frac{r-1}{r}}\Delta t$. Thus, the time measured by the clock of an observer at rest ($r_1 = r_0$) between emission and receipt of the signal is given by:

$$\Delta\tau = 2\left[(r - r_0) + \ln\left(\frac{r-1}{r_0-1}\right)\right]\sqrt{\frac{r_0-1}{r_0}} \tag{7}$$

A plot of (7) is given in Figure 1 below with $r = 100$:

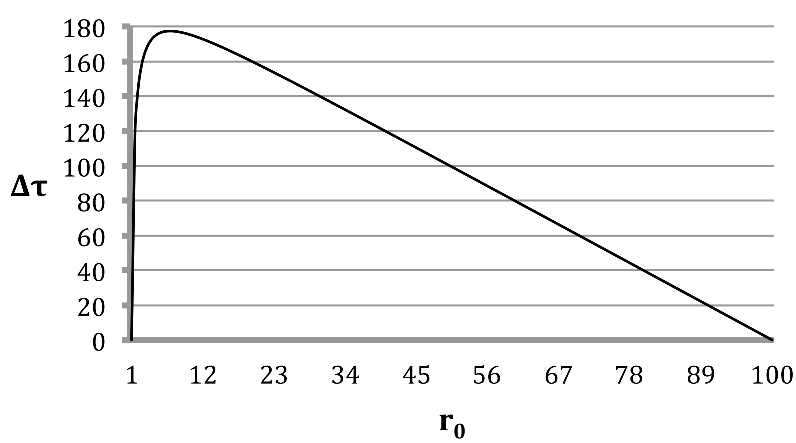

Figure 1 – $\Delta\tau$ vs. r_0 for $r = 100$

At the far right side of Figure 1, the two observers are at the same location and so the transit time of the light is zero. As we move left on the Figure, the reflecting observer remains at $r = 100$ while the measuring observer is at a reduced radius. At first, the transit time increases because the space between the observers increases until the transit time reaches a maximum, after which it begins to decrease again. When r is less than that radius where the transit time is maximum, we find that the deeper that observer is in a gravitational field (the closer r_0 is to 1), the more length contracted the Universe appears to him. This is because the amount of time (measured by that observer's clock) that it takes for light to travel to r (100 in this case) and back decreases as he moves deeper into the gravitational field. What we would like to know is whether or not a free falling observer will also see a length contracted Universe as he or she falls toward the event horizon.

The Worldlines of Falling Observers in Kruskal-Szekeres Coordinates

Consider Figure 2 below:

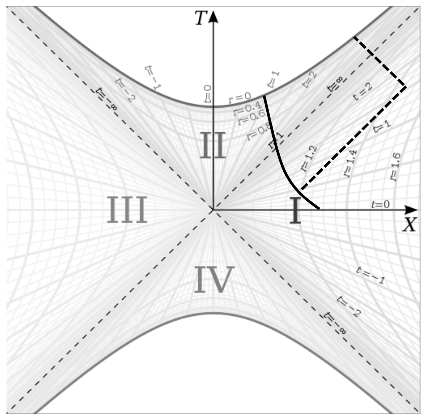

Figure 2 – Commonly Shown Inertial Worldline on KS Chart[1]

In Figure 2, the solid black line represents the worldline (only a rough depiction of the major attributes of the worldline) commonly attributed to an inertial observer [2,3,4,5]. But is it possible that the inertial observer follows a different path to the singularity?

We can find an explicit description of the worldlines of different observers on the KS coordinate chart by taking the differentials in (1):

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial t}dt = \frac{re^r dt}{2\sqrt{(r-1)e^r}}\left[\frac{dr}{dt}\cosh\left(\frac{t}{2}\right) + \frac{(r-1)}{r}\sinh\left(\frac{t}{2}\right)\right]$$
$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial t}dt = \frac{re^r dt}{2\sqrt{(r-1)e^r}}\left[\frac{dr}{dt}\sinh\left(\frac{t}{2}\right) + \frac{(r-1)}{r}\cosh\left(\frac{t}{2}\right)\right] \qquad (8)$$

The worldline we are considering here is one in which $\frac{dr}{dt} \to 0$ as $t \to \infty$. We see that this is true even in the case of Figure 2 because $dt \to \infty$ as the worldline approaches the horizon but for clarity, consider the metric in Schwarzschild coordinates as one approaches the horizon:

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[1] Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

$$d\tau^2 = \frac{r-1}{r}\left[1 - \left(\frac{r}{r-1}\frac{dr}{dt}\right)^2\right] dt^2 \tag{9}$$

In order for the worldline to remain timelike as the observer approaches the horizon, we see that $\frac{dr}{dt}$ must approach $\frac{r-1}{r}$, which goes to zero at the horizon (while these coordinates may not apply *at* the horizon, they can tell us what happens in the limit of approach). We see from (9) that not only must $\frac{dr}{dt}$ go to zero for the inertial observer, but also for an observer in *any* state of motion (this condition is merely the statement that the coordinate speed of an observer cannot exceed the local coordinate speed of light at a given *r*). Furthermore, we see that by combining the equations in (8):

$$\frac{dX}{dT} = \frac{\frac{dr}{dt}\cosh\left(\frac{t}{2}\right) + \frac{(r-1)}{r}\sinh\left(\frac{t}{2}\right)}{\frac{dr}{dt}\sinh\left(\frac{t}{2}\right) + \frac{(r-1)}{r}\cosh\left(\frac{t}{2}\right)} \tag{10}$$

For an observer following a timelike or lightlike worldline on approach of the horizon, according to (9) we can substitute $\frac{dr}{dt} = -V(r)\frac{r-1}{r}$, where $V(r)$ ranges from 0 to 1. For a timelike worldline ($V(r) < 1$), we can take the limit of (10) as $t \to \infty$:

$$\lim_{t\to\infty}\left(\frac{dX}{dT}\right) = \frac{-V(r)\cosh\left(\frac{t}{2}\right) + \sinh\left(\frac{t}{2}\right)}{-V(r)\sinh\left(\frac{t}{2}\right) + \cosh\left(\frac{t}{2}\right)} = \frac{\tanh\left(\frac{t}{2}\right) - V(r)}{1 - V(r)\tanh\left(\frac{t}{2}\right)} = 1 \tag{11}$$

So when $V(r)$ is less than 1, the limit goes to 1 as can be seen in (11). It is notable that for the case where V is 1, corresponding to an infalling lightlike worldline, $\frac{dX}{dT}$ is -1 for any t. Also, for a spacelike worldline ($V(r) > 1$), we see that the limit also goes to 1, except in this instance, both dX and dT are negative, meaning the worldline approaches the horizon in the same way as the timelike worldline, but it curves in the opposite direction. Thus, as the worldlines of timelike and spacelike observers approach the horizon, their slopes must approach 45 degrees on the KS coordinate chart as per (11). The form of timelike, lightlike and spacelike trajectories on approach of the horizon are shown in Figure 3 below:

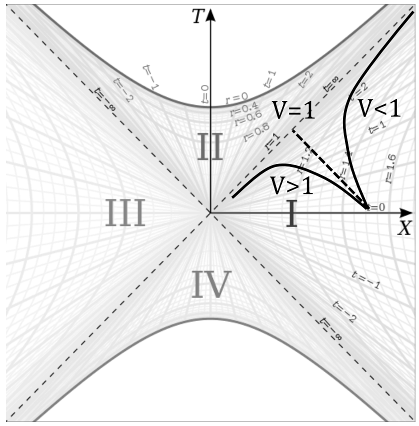

Figure 3 – Proposed Timelike, Lightlike, and Spacelike Trajectories on KS Chart[2]

The timelike worldline depicted in Figure 3 will reach the horizon (and singularity) at infinite coordinate time in both KS and Schwarzschild coordinates.

The Schwarzchild solution in KS coordinates is given by the metric:

$$d\tau^2 = \frac{4}{re^r}\left[1 - \left(\frac{dX}{dT}\right)^2\right] dT^2 \qquad (12)$$

We can modify this metric by substituting (11) into (12), noting that $\tanh\left(\frac{t}{2}\right) = \frac{T}{X}$ for the region outside the black hole:

$$d\tau^2 = \frac{4}{re^r}\left[1 - \left(\frac{T - V(r)X}{X - V(r)T}\right)^2\right] dT^2 \qquad (13)$$

Where, again, when $0 \leq V(r) < 1$, we have an infalling timelike worldline. When $V(r)$ = 1, we have an infalling lightlike worldline, and when $V(r) > 1$, we have an infalling spacelike worldline. The key thing to note here is that regardless of the value of $V(r)$, when X and T are equal (corresponding to the event horizon), $d\tau = 0$, which agrees with the results in Schwarzschild coordinates (we can make the substitution $\frac{dr}{dt} = -V(r)\frac{r-1}{r}$ into (9), giving $d\tau^2 = \frac{r-1}{r}[1 - V(r)^2]dt^2$, which shows that $d\tau = 0$ at the horizon for all worldlines). The speed of light in Schwarzschild coordinates is given by $C = \frac{r-1}{r}$ (we find this by setting the RHS of (9) to 0 and solving for $\frac{dr}{dt}$). We can similarly express the speed of light in KS coordinates from (1), which tells us that $X^2 - T^2 = (r-1)e^r$. Therefore, the speed of light in KS coordinates is given by: $C = \frac{X^2 - T^2}{re^r}$. We must notice that if we solve for $\frac{dX}{dT}$ in (12) for RHS=0, we get +/-1, so why do we choose the

[2] Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

Schwarzschild coordinates to say that the speed of light is zero at the horizon? It is because $\frac{dX}{dT}$ is +/-1 over the entire spacetime manifold for light. But in [6], Einstein showed that for reasons of energy conservation the speed of light must vary with the gravitational potential in the way described by the Schwarzschild coordinates. Therefore, $\frac{dX}{dT}$ is useful in determining whether a worldine in KS coordinates is lightlike or not, but it tells us nothing about the relative propagation speed of a light signal over the spacetime manifold. This is unsurprising given that the KS coordinates were chosen such that the speed of light in KS coordinates would be +/-1 over the entire manifold. And this gives us another clue regarding why the horizon is a special place on the KS coordinate chart. As one gets closer to the horizon, the magnitude of the speed of light gets smaller and therefore the magnitude of the function(s) needed to make the transformed speed equal to 1 must grow. But at the horizon, the speed is zero and therefore in order to turn 0 into 1 the function(s) must grow to infinity, which explains why the definitions of T and X in (1) become zero times infinity at the event horizon. This also supports the claim that all points on the $X = T$ line are the same spacetime point because light does not propagate over the manifold on that line. So even though in Figures 2 and 3, the horizon looks like a typical lightlike worldline, since the *speed* of light along that line is zero, the light signal is not actually moving through *spacetime* as one moves along that line. This also implies that in Figure 3, the lightlike worldline ($V(r) = 1$) stops propagating at the horizon, because its speed is zero there. Thus, the event horizon can be viewed as a kind of relativistic stagnation point, a place that not even light can pass through (ingoing or outgoing).

If we consider the possible timelike trajectories in Figure 3, we see that when $V(r)$ is zero, the worldline follows a hyperbola of constant r. When $V(r)$ is 1, the worldline is a straight line as depicted in the figure. For trajectories with constant intermediate values of V, as V is increased from 0 toward 1 the worldline becomes more ‘L-shaped’ as the curve gets ‘pushed’ into the corner formed by the intersection of the horizon and the lightlike worldline.

We might further explain the form of such a worldline for an inertial observer as follows. The inertial worldline is determined by following a path of maximum proper time. With this in mind let us consider the points in Figure 2 lying on the line $r = 1$. What we need to acknowledge is that all the points on that line represent the same point in spacetime. The first thing supporting that argument is that each point on the line corresponds to the same r and t. Also, if we look at the points in Figure 2 not lying on the horizon, we find that there is a unique T and X assigned to each combination of r and t via the definitions of T and X in (1) such that each spacetime point has a unique label in both coordinate systems. But at the horizon, T and X are not unique, rather they are both defined as zero times infinity along that line (as can be seen in (1)), suggesting that the spacetime *point* in Schwarzschild coordinates is in fact a *line* in KS coordinates (because zero times infinity can be any finite number) on which all the points label the *same* spacetime event. Furthermore, if we once again consider the case of the reflected signal, we find that as the free falling observer approaches the horizon, the ends of the light path shown in Figure 2 become more widely separated on the KS coordinate chart, while the proper time separating them in the frame of the free faller converges to zero, suggesting that there is no measurable spacetime separation between the points on that line. For these reasons, we conclude that the entire line at $r = 1$ is equivalent to a single point at some t and $r > 1$. Therefore, a freefalling observer will follow a path that maximizes the proper time it takes to reach the *line* $r = 1$ on the KS chart. Such a path must therefore approach the line asymptotically, rather than head-on (as depicted in Figure 2).

Given that the timelike worldline in Figure 3 remains at a radius greater than or equal to 1, we see from this metric that the worldline in Figure 3 becomes lightlike at the singularity because $\frac{dX}{dT}$ goes to 1 there and all radial coordinates 'converge' at infinite coordinate time. This is desirable given our discussion above regarding the time it takes to receive the reflected signal, namely that as the observer falls, signals sent to a given radial coordinate is reflected and received by the free falling observer at shorter and shorter intervals as measured by the free falling observer's clock. Thus, the Figure 3 worldline agrees with the appearance of a length contracted Universe as the observer falls.

This alternate worldline also harmonizes the physics described by the KS and Schwarzschild coordinates, which is important since the choice of coordinates should not change the physics. Consider the metric in Schwarzschild coordinates in (9). In order for the worldline to be timelike or lightlike, we see that $\left|\frac{dr}{dt}\right| \leq \frac{r-1}{r}$. Therefore, as the observer falls, $\left|\frac{dr}{dt}\right|$ must get closer and closer to $\frac{r-1}{r}$, which is the local speed of light. Thus, on approach of the horizon, any timelike worldline must converge to a lightlike worldline (in Schwarzschild coordinates, the light cone is commonly depicted as being 'squeezed' as one approaches the horizon, which is equivalent to what we see here). So with the new worldline, the worldline behavior is the same in *both* sets of coordinates. One can even see that $\left|\frac{dr}{dt}\right|$ goes to zero in Figure 3 by noting that the *r* and *t* coordinates become parallel as *T* goes to infinity. This is also preferable to the worldline in Figure 2 because if we look at the worldline as it approaches the horizon, $\left|\frac{dr}{dt}\right|$ goes to zero because the change in t becomes infinite. Thus the Schwarzschild coordinates suggest the observer is approaching a lightlike worldline. But in KS coordinates, we see that at the horizon, the worldline in Figure 2 appears to get 'less' lightlike as the observer approaches the horizon. This is an important observation because going from a timelike to lightlike worldline is physical, so the fact that changing the coordinates adds or removes this transition suggests there is a flaw in the use of one or the other coordinate system.

The Equivalence Principle

According to the equivalence principle, the worldline of a uniformly accelerated observer is equivalent to the worldline of an observer at rest in a uniform gravitational field. Furthermore, for very small (infinitesimal) changes in position and time, the accelerated observer is equivalent to an observer at rest with constant relative velocity (because over small regions of spacetime, the accelerated observer has a constant velocity). This leads to a quality of General Relativity that states that for very small regions of spacetime, the manifold is Minkowskian. A feature of Minkowski spacetime is that there is no difference between an observer 'at rest' and an 'inertial' observer (these two concepts are indistinguishable in flat spacetime).

If we consider the spacetime at the event horizon, the equivalence principle tells us that in the region of a single spacetime point on the horizon, the manifold must be Minkowskian. But an observer at rest at the horizon is moving on a lightlike worldline (note that the observer at rest here is not even accelerating). However, a lightlike observer is not inertial and therefore an observer at the horizon cannot be both at rest and inertial at any point on it. Thus the existence of the event horizon must be a violation of the equivalence principle since the spacetime at the horizon cannot be Minkowskian in the limit of infinitesimal regions of spacetime.

Conclusion

We therefore conclude that an observer can never cross the horizon in finite coordinate time (Schwarzschild *or* KS). A free falling observer should therefore see the Universe increasingly length contracted as they asymptotically approach the event horizon. This analysis may therefore also suggest that the formation of a black hole, *even in the frame of the free falling observer*, is in fact not possible, implying that black holes cannot be physical objects. This question however will not be pursued in the current paper. However, it is notable that in Figure 3, the free falling observer's worldline converges with the worldline of an observer at rest at the horizon. An observer at rest at or near the horizon will see external radiation from the Universe be increasingly blueshifted the closer the observer is to the horizon. Thus, we should expect that material free falling to form a black hole would likely be 'vaporized' by this blueshift before reaching the critical density, perhaps becoming a source of super high-energy particles.

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