An SO(4) Yang-Mills Description of Quantum Gravity

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Abstract

The $U(1) \times SO(4)$ covariant derivative produces an action where the SO(4) generators do not commute with the Dirac matrices because the generators themselves are constructed from those matrices. This yields additional interactions absent in SU(2) and SU(3) Yang Mills theories. The contributions from these interactions produce T-matrix elements consistent with the Newtonian and post Newtonian interactions found in the low energy limit of classical general relativity theory for both matter-matter and matter-photon interactions. The theory here proposed not only reproduces the observed experimental results of general relativity, but it is also renormalizable and more importantly it can be coupled to the standard model in a trivial way. Thus, $SO(4) \times SU(3) \times SU(2) \times U(1)$ Yang Mills best describes all interactions in nature.

1 Introduction

In 1974 Yang [1] proposed a theory based on Yang Mills gauge fields that yielded equations of motion similar to those obtained from general relativity. However, the theory did not receive a very wide audience since at the time general relativity reproduced the observed experimental results. That same year the problem of the absence of renormalization in general relativity was raised in [2], but Yang Mills gravity was not revisited as a possible solution and instead string theory became the favorite candidate. Despite its progress, string theory has not been able to couple properly to the standard model and suffers from an embarrassment of riches in terms of parameters and Calabi-Yau manifolds, as well as all its different limits.

Here we revisit the concept proposed in [1] which recently was shown to have black hole solutions for $SU(2)$ gauge group [3]. We propose a standard Yang Mills theory coupled to matter so that dimensional analysis shows that it is renormalizable. However, we use a SO(4) gauge group constructed with Dirac gamma matrices [4]:

$$T^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b]. \tag{1}$$

This construction has the peculiarity that it does not commute with the Dirac gamma matrices. This feature is absent in the standard model where the gauge group always commute with the gamma matrices. The non-commutation produces terms which enrich the theory with additional couplings which are absent in the standard model. In turn vertices arise which, using the same formalism used in [5], produce tree-level two-body scattering amplitudes that reproduce the born terms for the Newtonian interaction. In addition, the 1-loop correction reproduces the post-Newtonian interaction which is needed to reproduce the observed experimental precision of perihelia. In addition, the theory also yields at tree-level the correct factor due to Einstein for the deflection of light and gravitational redshift, after suitable choice of parameter.

A connection exists between QED and the low energy limit of classical electrodynamics. This connection requires two steps, each with its own map. The first step maps the low energy limit contributions to the scattering amplitude of QED with the Born approximation for the Schrodinger equation with the Coulomb potential [6]. The second step requires taking the limit $\hbar \to 0$ of the Feynman's path integral for this Schrodinger equation that then yields the non relativistic limit of classical electrodynamics. QED then succeeds because its low energy non relativistic limit coincides with the low energy limit of electromagnetism and therefore with low energy experimental results. This composition of two maps differs from the most salient string theory map where only a single map connects the theory directly, through a beta-function calculation, to the classical Hilbert Einstein action. As opposed to QED, that map avoids altogether the non relativistic limit of the quantum theory.

However, also in string theory a connection can be established between the fundamental theory and the classical theory by matching the scattering amplitude of the fundamental theory to the deflection angle derived from general relativity [7][8][9][10]. The Schrodinger equation bridges QED and the low energy classical of electromagnetic interactions. The classical limit exists because the nature of the interaction allows particles to interact with each other even over macroscopic distances. Only long range forces have a classical limit. Short ranged interactions, such as Strong and weak interactions, have no such a limit, because particles do not interact with each other over large distances. It is then unclear how string theory would distinguish short range and long range interactions when such limiting procedure is used.

The long range nature of the gravitational interactions suggests the existence of a classical limit. Therefore, a map between the fundamental quantum gravity theory and its low energy classical limit should exit and even better the Schrodinger equation bridges these extremes. In [5], the non renormalizable action for general relativity was quantized and the scattering amplitude between two massive particles found to correspond with that of the Schrodinger equations with Newtonian and post Newtonian potential. Thus, through the Feynman path integral, the quantized Hilbert Einstein action maps to its expected low energy classical limit. The mapping followed the same procedure used for QED and extended by [5] to general relativity without renormalization. In [5] the map scattering for a photon interacting with a massive particle were not explored. However, necessary vertices to obtain the classical limit through the Schrodinger equation exist and suggest that that such a limit also then exists when the photon energy represents the mass of a particle. However, the scattering amplitudes must be corrected by 1-loop contributions. These 1-loop contributions diverge in the ultraviolet limit. Thus, although its success, use of dimensional analysis shows that the starting quantum action and the needed scattering amplitudes cannot be mapped to its low energy classical limit after renormalization.

Here we apply the same procedure used in QED and [5] to the renormalizable action derived from the covariant derivative D_μ acting on a fermionic matter field ψ

$$D_\mu\psi = (\partial_\mu + \frac{ie}{\sqrt{1+6\alpha^2}}A_\mu + ig(\omega_\mu^{ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ab\nu}A_\nu)T_{ab})\psi. \qquad (2)$$

The field ω, the connecton, has SO(4) gauge symmetry, but constructed from Dirac gamma matrices which has the effect of preventing the commutation of the SO(4) generators with the gamma matrices. The action obtained from this covariant derivative guarantees both gauge invariance under U(1) and SO(4) local transformations. Matching the T-matrix tree and 1-loop contributions with the Born amplitudes as in [5, 6] requires a Schroedinger equation with a Newtonian potential along with the post Newtonian correction expected from general relativity as derived in [4]. While the tree level contribution to the T-matrix reproduces the Newtonian interactions, the post Newtonian interactions come from the 1-loop contribution. The action derived from (2) closely relates to Yang Mills gravity [1] which yields equations very similar to those of Einstein gravity, including the Bianchi identity and even solutions which resemble Schwarzschild black hole solutions [3].

We apply the methods in [5, 6] to SO(4) Yang Mills to obtain the same results in the matter-matter sector as in [5]. However, this paper goes a step further and also produces the desired results for photon-matter interactions using the same prescription applied to the matter-matter interactions. Thus the theory here proposed not only reproduces the experimental results, but also is renormalizable and more importantly it can be coupled to the standard model in a trivial way. However, the gauge group elements in (2), as opposed to those considered in [1], do not commute with the Dirac gamma matrices. This property, absent in other theories like SU(2) and SU(3) gauge theories, generates terms which couple the connecton to matter fields ψ with couplings absent in the Yang-Mills sector. While the construction of standard gauge theories involves only terms of the form $\{\gamma^\mu, D_\mu\}$ the present theory also includes terms of the form $[\gamma^\mu, D_\mu]$. While re-balancing the former by a constant circles back to redefine the coupling of the theory due to the presence of a kinetic term $\bar{\psi}\partial\psi$, the re-balancing of the latter structure does not circle back to redefine the couplings of the theory. This frees the amount of couplings needed to accommodate the full matter spectrum.

Section 2, details the symmetries and action derived from the covariant derivative (2) and for a single matter field. Section 3 calculates the propagators, vertices and T-matrix elements as well as their relation to the Schroedinger equations with and without post Newtonian corrections. Section 4 details the description of the photon interaction with a single matter field and reproduces the gravitational lensing by a mass point particle. Section 5 extends the covariant derivative to include multiple matter fields which requires multiple connecton fields in order to reproduce both matter and photon Newtonian and post Newtonian interaction in the non relativistic limit. Section 6 presents the conclusions.

2 Symmetries and Actions

We consider the following derivative operator D_μ acting on a Dirac spinor field ψ in 4 dimensions:

$$D_\mu \psi = (\partial_\mu + \frac{ie}{\sqrt{1+6\alpha^2}} A_\mu + ig(\omega_\mu^{ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}} \epsilon_\mu^{ab\nu} A_\nu) T_{ab})\psi. \tag{3}$$

The space-time metric, $g^{\mu\nu}$, is flat, does not carry any dynamics and has signature $(+,-,-,-)$. $\epsilon^{\mu\nu\rho\sigma}$ represents the Levi-Civita tensor in 4 dimensions. The vielbeins e_ρ^a connect space-time coordinates (Greek indices) and fiber bundle coordinates (Latin indices). In particular $e_\rho^a e^{b\rho} = \eta^{ab}$ and $e^{a\rho} e_a^\sigma = g^{\rho\sigma}$. The metric η^{ab} has signature $(+,+,+,+)$. A_μ represents a U(1) gauge field while the T_{ab} generate the SO(4) group transformations. The couplings e, g and α denote coupling constants. The connecton, ω_μ^{ab}, has SO(4) gauge symmetry and relates to $\tilde{\omega}_\mu^{ab}$, with antisymmetric indices a and b, in the following manner

$$\omega_\mu^{ab} = \tilde{\omega}_\mu^{ab} + e_c^\mu e_a^\nu \tilde{\omega}_\nu^{cb}. \tag{4}$$

The antisymmetric indices a and b in $\tilde{\omega}_\mu^{ab}$ ensure that

$$\omega_\mu^{ab} e_{\nu a} e_{\rho b} \epsilon^{\mu\nu\rho\sigma} = \omega_{\mu ab} \epsilon^{\mu ab\sigma} = 0. \tag{5}$$

U(1) transformations with $\Omega = exp(i\lambda)$, with λ a scalar function, act on the following fields

$$\begin{aligned}
\psi' &= \Omega\psi && (6)\\
A'_\mu &= A_\mu - \frac{\sqrt{1+6\alpha^2}}{e}\partial_\mu\lambda && (7)\\
\omega_\mu^{\prime ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ab\nu} A'_\nu &= \omega_\mu^{ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ab\nu} A_\nu && (8)
\end{aligned}$$

Eq. (8) implies

$$\omega'^{ab}_{\mu} = \omega^{ab}_{\mu} + \frac{\alpha}{e}\epsilon^{ab\nu}_{\mu}\partial_{\nu}\lambda \tag{9}$$

Equations (7) and (9) guarantee the covariance property

$$D'_{\mu}\psi' = \Omega D_{\mu}\psi. \tag{10}$$

SO(4) transformations with $\hat{\Omega} = exp(i\Lambda^{ab}T_{ab})$ and Λ a 2-Tensor, act on the following fields

$$\begin{aligned}
\psi' &= \hat{\Omega}\psi && (11)\\
A'_{\mu} &= A_{\mu} && (12)\\
\omega'^{ab}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ab\nu}_{\mu}A'_{\nu} &= \omega^{ab}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ab\nu}_{\mu}A_{\nu} - \frac{1}{g}\partial_{\mu}\Lambda^{ab} \\
&\quad + C^{ab}{}_{cdef}\Lambda^{cd}(\omega^{ef}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ef\nu}_{\mu}A_{\nu}) && (13)
\end{aligned}$$

Eq. (13) implies

$$\omega'^{ab}_{\mu} = \omega^{ab}_{\mu} - \frac{1}{g}\partial_{\mu}\Lambda^{ab} + C^{ab}{}_{cdef}\Lambda^{cd}(\omega^{ef}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ef\nu}_{\mu}A_{\nu}) \tag{14}$$

Equations (12) and (14) ensure the covariance property

$$D'_{\mu}\psi' = \hat{\Omega}D_{\mu}\psi. \tag{15}$$

$D_{\mu}\psi$ also transforms covariantly under SO(1,3) transformations. The vielbeins e^{μ}_{a} assist in expressing $\hat{\Omega}$ in terms of the generators $T_{\rho\sigma}$ of the SO(1,3) Lie group

$$\hat{\Omega} = e^{i\Lambda^{ab}T_{ab}} = e^{-i\Lambda^{ab}T_{cd}e^{\mu}_{a}e^{\nu}_{b}e^{c}_{\mu}e^{b}_{\nu}} = e^{i\Lambda^{\mu\nu}T_{\mu\nu}}. \tag{16}$$

The commutator $[D_{\mu}, D_{\nu}]$ determines the field strength of the action:

$$\begin{aligned}
[D_{\mu}, D_{\nu}] &= ieF_{\mu\nu} + igG^{ab}_{\mu\nu}T_{ab} && (17)\\
F_{\mu\nu} &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} && (18)\\
G^{ab}_{\mu\nu} &= (\partial_{\mu}\omega^{ab}_{\nu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ab\rho}_{\nu}\partial_{\mu}A_{\rho}) - (\partial_{\nu}\omega^{ab}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ab\rho}_{\mu}\partial_{\nu}A_{\rho}) \\
&\quad - gC^{ab}{}_{cdef}(\omega^{cd}_{\mu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{cd\rho}_{\mu}A_{\rho})(\omega^{ef}_{\nu} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ef\sigma}_{\nu}A_{\sigma}) && (19)
\end{aligned}$$

The purely bosonic Lagrangian density requires use of equations (18) and (19)

$$\begin{aligned} L_{gauge} &= L^{(U(1))} + L^{(SO(4))} && (20) \\ L^{(U(1))} &= \frac{1}{4} F_{\mu\nu} F^{\mu\nu} && (21) \\ L^{(SO(4))} &= \frac{1}{4} G^{ab}_{\mu\nu} G^{ab}_{\mu\nu}. && (22) \end{aligned}$$

The Lorenz gauge fixings

$$\begin{aligned} \partial_\mu A^\mu &= 0 && (23) \\ \partial_\mu \omega^{\mu ab} &= 0 && (24) \end{aligned}$$

and property (5) produce the quadratic bosonic Lagrangian density

$$L^{(2)}_{gauge} = \frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu + \frac{1}{2} \partial_\mu \omega^{\nu ab} \partial^\mu \omega_{\nu ab}. \quad (25)$$

The structure constant defined in [4]

$$C^{abcdef} = -h^{af} h^{ce} h^{bd} + h^{ad} h^{ce} h^{bf} - h^{ae} h^{cb} h^{df} + h^{ac} h^{eb} h^{df}, \quad (26)$$

simplifies the relevant terms of the cubic Lagrangian density

$$\begin{aligned} L^{(3)}_{gauge} &= 2g(\partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu) \omega^\mu{}_{ca}\, \omega^{\nu c}{}_b \\ &\quad - g \frac{4\alpha^2}{1+6\alpha^2} A_\mu \partial_\rho A_\nu e^\nu_a e^\rho_b \omega^{\mu ab} - g \frac{4\alpha^2}{1+6\alpha^2} \partial_\nu A_\mu A_\rho e^\mu_a e^\rho_b \omega^{\nu ab}. \quad (27) \end{aligned}$$

The terms in (27) take the form $\omega\omega\partial\omega$ and $\partial A A \omega$. In the low energy limit the photon momentum, r, and the connecton momentum, k, satisfy

$$k << r \quad (28)$$

or equivalently

$$\partial A >> \partial \omega. \quad (29)$$

This condition accounts for the omission of all other terms. For example, for matter-photon scattering the vertex generated by $A\omega\partial\omega$ would only participate at 1-loop level, would necessarily appear twice and thus be proportional to the square of the connecton momentum. Instead, the 1-loop contribution of the third or fourth term in (27) would appear first at tree and at 1-loop level and would be proportional to twice the photon momentum.

2.1 Unitarity

We first analyze the well known free Dirac spinor in a unitary theory. The propagator relates to the Green function only after implementing an analytic continuation which requires the infinitesimal shift of the on-shell poles away from the real axis. Although the literature implements this mathematical artifact after the quantization of the Hamiltonian, it nevertheless amounts to having a Hamiltonian with complex energy levels. Automatically, the action looses hermiticity and, more importantly, also unitarity. With hermiticity lost the probability density decreases towards zero as time evolves which guarantees the absence of unitarity. However, the loss of unitarity due to the pole's infinitesimal shift away from the real axis does not pose a serious problem because any significant decrease in probability density takes place long after the experiment's conclusion, or even better, several orders of magnitude greater than the universe's age.

QED has a real coupling, the electron charge. However, even when the charge becomes purely imaginary, the theory still avoids loss of unitarity. This follows because the self energy of both the electron and the photon from any order of perturbation theory always remains real since only a single vertex, $e^- \to e^- + \gamma$, exists. Conservation of momentum requires these vertices appear in pairs and therefore it does not matter if the vertex itself remains purely imaginary or real because their product will always yield a real contribution to the self energy, albeit with different signs.

With Yang-Mills coupled to fermions, the situation changes. We assume for the sake of argument that the coupling in the covariant derivative is independent of the coupling in the pure bosonic sector. The loss of gauge invariance, which only affects this argument and not the remainder of the paper, allows rewriting the coupling constant for the covariant derivative without affecting the gauge self interaction. We consider the contribution to the fermion self energy from the diagram with the 3-point bosonic vertex and 3 vertices obtained from the covariant derivative. When the coupling from the covariant derivative rotates from real to imaginary, then the contribution becomes imaginary. Then

$$\Sigma \sim \mathcal{O}(g^4) \tag{30}$$

In strongly coupled QCD, $g \sim 1$, the probability density vanishes well before any particle reaches a detector. Thus, the rotation of the coupling in the covariant derivative removes unitarity.

On the other hand the gravitational coupling g satisfies

$$g \sim \mathcal{O}(10^{-5}) \tag{31}$$

which implies an infinitesimal shift of the pole away from the real axis. In fact, in the gravitational case, the $\mathcal{O}(g^4)$ contribution dents the probability density well after 10^{34} years or many orders of magnitude the estimated age of the universe. Therefore such a rotation of the coupling in the covariant derivative effectively preserves unitarity in the same amount as the usual infinitesimal shift away from the real axis required to connect the propagator with the Green's function. In fact, quantum gravity as described here assigns physical meaning to the mathematical artifact needed to relate the propagator with the Green function.

2.2 Hermiticity and Antihermiticity

As explained in the last subsection, the gravitational constant smallness ensures that the gravitational theory does not require hermiticity. However, for the sake of minimalism, the theory we consider uses only hermitic and antihermitic terms. Thus, the kinetic terms for the fermions involve the following two candidates. The first candidate

$$L_{fermion} = i\bar{\psi}\gamma^{\mu}D_{\mu}\psi \tag{32}$$

does not have hermitic nor antihermitic properties because $[\gamma_{\mu}, T_{ab}] \neq 0$. Rather, the following hermitic combinations exist

$$L_{f1} = \frac{i}{2}\bar{\psi}\{\gamma^{\mu}, D_{\mu}\}\psi = i\bar{\psi}(\not{\partial} + \frac{ie}{\sqrt{1+6\alpha^2}}\not{A} + \frac{3ig\alpha}{\sqrt{1+6\alpha^2}}A^{\mu}\gamma^5\gamma_{\mu})\psi, \tag{33}$$

$$L_{f2} = i\bar{\psi}[\gamma^{\mu}, D_{\mu}]\psi = g\bar{\psi}\bar{\omega}^{\mu a}{}_{\mu}\gamma_a\psi, \tag{34}$$

after using the relation

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda} = \eta^{\mu\nu}\gamma^{\lambda} + \eta^{\nu\lambda}\gamma^{\mu} - \eta^{\mu\lambda}\gamma^{\nu} - i\epsilon^{\sigma\mu\nu\lambda}\gamma_{\sigma}\gamma^5. \tag{35}$$

Note that L_{f2} does not depend on ∂_{μ}. This means that gauge invariance constrains L_{f1} but not L_{f2}. Thus the rescalling of L_{f1} by a constant redefines the coupling. However, the redefinition guarantees gauge invariance only after a subsequent redefinition of the spinor ψ. On the other hand, a recalling by a constant, even an imaginary one, of L_{f2} does not affect gauge invariance.

This should be contrasted with SU(2) of SU(3) couplings where the Dirac matrices commute with those elements and therefore L_{f2} necessarily vanishes.

The following Lagrangian density has gauge invariance

$$L_f = i\bar{\psi}(\not{\partial} + \frac{ie}{\sqrt{1+6\alpha^2}}\not{A} + \frac{3ig\alpha}{\sqrt{1+6\alpha^2}}A^\mu\gamma^5\gamma_\mu + g'\bar{\psi}\bar{\omega}^{\mu a}{}_\mu\gamma_a - m)\psi. \tag{36}$$

Here the real coupling constants e, g and g' adjust the theory to observation. In particular e represents the electric charge and gauge invariance constrains g and coincides with the coupling constant found in the purely bosonic sector (27). Instead, gauge invariance does not constrain g' which can take any value, real or purely imaginary. As shown below, antihermiticity ensures that the term proportional to g' contribute real rather than imaginary post Newtonian corrections. As discussed above, this does not pose a problem to unitarity because the first imaginary contribution to the self energy, of $\mathcal{O}(g^4)$, does not significantly dent the probability density until 10^{34} years from the start of the experiment.

3 Propagators and Vertices

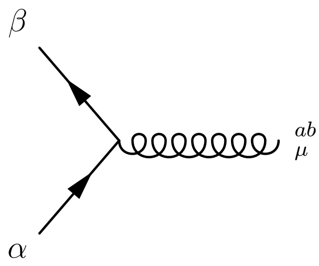
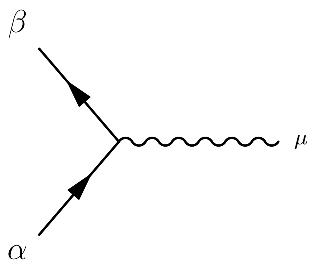
The Lagrangian densities (25) and (36) yield the following propagators

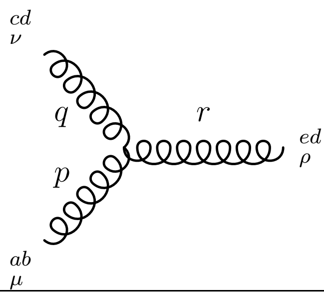
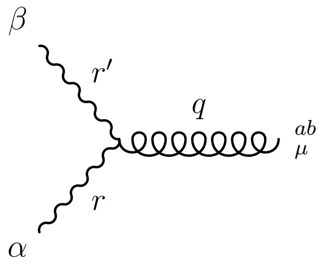
$^{ab}_{\mu}$  $^{cd}_{\nu}$ k	$-\frac{ig^{\mu\nu}\eta^{ab}\eta^{cd}}{k^2+i\epsilon}$
α  β p	$\frac{i\delta^{\alpha\beta}}{\not{p}-m+i\epsilon}$
μ ν r	$-\frac{ig^{\mu\nu}}{r^2+i\epsilon}$

For the avoidance of doubt, in the limit considered here

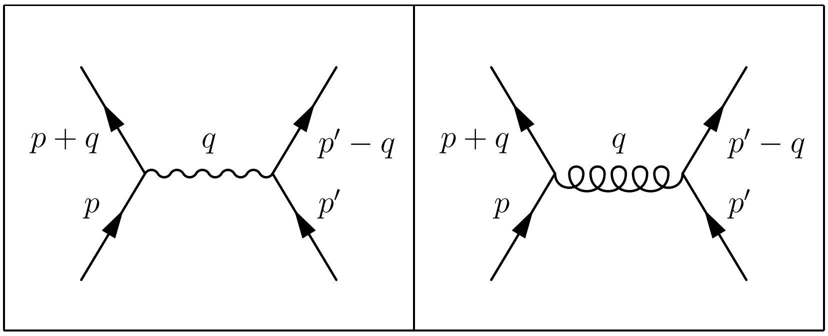
$$m >> \partial A >> \partial\omega \tag{37}$$

or equivalently $m >> r >> k$. The Lagrangian densities (27) and (36) generate the following vertices:

	$-g'\ e^{\mu a}e^{\nu b}\gamma_{\nu}$
	$-\frac{i}{\sqrt{1+6\alpha^2}}(e\gamma_{\mu}+3g\alpha\gamma_5\gamma_{\mu})$

	$g(h^{af}h^{ce}h^{bd}+h^{ad}h^{ce}h^{bf}-h^{ae}h^{cb}h^{df}+h^{ac}h^{eb}h^{df})\cdot$ $\cdot((p-q)^{\rho}g^{\mu\nu}+(q-r)^{\mu}g^{\nu\rho}+(r-p)^{\nu}g^{\mu\rho})$
	$g\frac{4\alpha^2}{1+6\alpha^2}(g_{\mu\alpha}e^a_{\beta}r'^b\ +\ g_{\mu\beta}e^a_{\alpha}r^b\ +\ e^a_{\alpha}e^b_{\beta}r_{\mu}\ +\ e^a_{\beta}e^b_{\alpha}r'_{\mu})$

The matter-matter T-matrix elements at tree level come from the following diagrams



The contribution from the photon exchange due to the electric charge

$$T_{QED} = 4im^2 \frac{e^2/(1+6\alpha^2)}{-q^2} \tag{38}$$

coincides with the Born amplitudes obtained from Schroedinger's equation with a Coulomb potential with an electric charge of $\frac{e}{\sqrt{1+6\alpha^2}}$. See for example [6] for a detailed analysis of this methodology. Thus the classical low energy limit coincides with classical electrodynamics.

The remaining tree-level interactions contribute as follows

$$T_N = -4im^2 \frac{g'^2}{-q^2} + 4im^2 eg \frac{3\alpha}{\sqrt{1+6\alpha^2}} \frac{p^\mu J'_{A\mu} + p'^\mu J_{A\mu}}{-q^2} + 4im^2 g^2 \frac{9\alpha^2}{1+6\alpha^2} \frac{J^\mu_A J'_{A\mu}}{-q^2} \tag{39}$$

with $J_{A\mu} = \bar\psi \gamma_\mu \gamma_5 \psi$ and $J'_{A\mu} = \bar\psi' \gamma_\mu \gamma_5 \psi'$. In the low energy quantum limit the axial charge, $J_{A\mu}$ vanishes. Then the property

$$p^\mu J'_{A\mu} + p'^\mu J_{A\mu} \sim \mathcal{O}(m) << \mathcal{O}(m^2). \tag{40}$$

suppresses in that limit the second term in (39). A further limiting to the classical limit requires in-states and out-states to have no axial current or equivalently that $< J_{A\mu} >=< J'_{A\mu} >= 0$. Thus in the classical limit of the non-relativistic quantum limit, the second and third term in (39) vanish exactly.

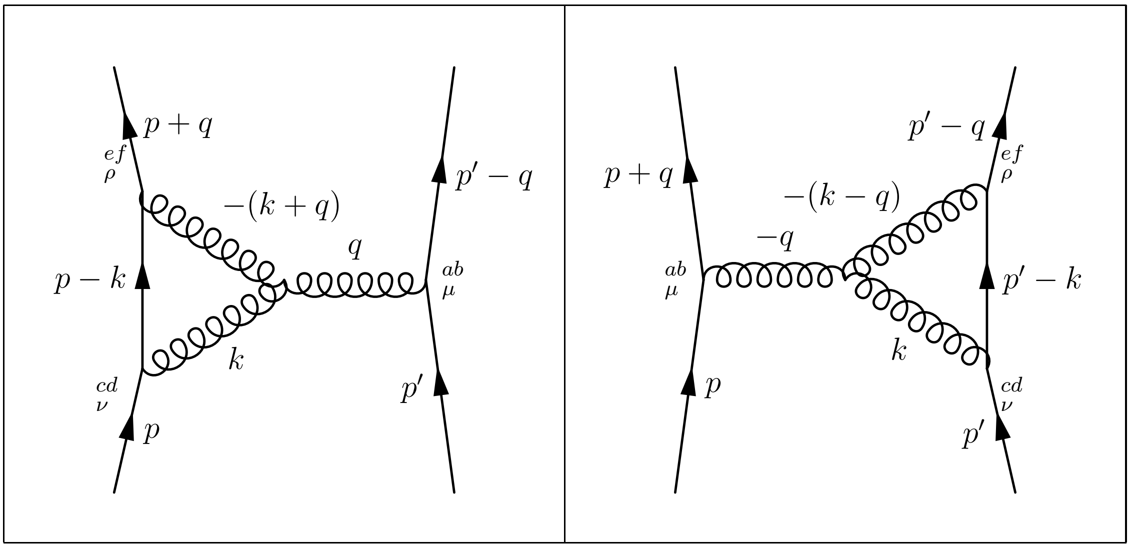
Thus, in the classical limit of the non-relativistic quantum limit, the surviving Born term produces the Newtonian potential

$$V = -G\frac{m^2}{r} \tag{41}$$

when

$$g' = \sqrt{G}m. \tag{42}$$

The post-Newtonian correction comes from considering the following 1-loop diagrams



All other 1-loop diagrams contribute to either analytic terms, or quantum corrections of order $\mathcal{O}(ln(-q^2))$ and do not contribute to the low energy limit considered here[5]. In addition, diagrams involving photons and connectons simultaneously do not contribute because the transformation $\omega'^{ab}_\mu = \omega^{ab}_\mu - \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon^{ab\nu}_\mu A_\nu$ leaves the contribution of the above diagram unchanged while all terms but the first two in (27) remain present in the action. These two diagrams contribute

$$T_{PN} = -4im^2 \frac{27c_1 G^2 m^3}{16\sqrt{-q^2}} \tag{43}$$

provided $g = -c_1\sqrt{G}m$ with c_1 a numerical constant. This result as those below require the following steps. First, we focus only on the “electric” form factor which means that

$$\bar{\psi}\gamma_\mu\gamma_\nu\psi = mg_{\mu\nu}. \tag{44}$$

Equivalently, terms proportional to $[\gamma_\mu, \gamma_\nu]$ and that only contribute to the "magnetic" form factor produce diagrams not considered here. Second, the on-shell external momenta imply the following relations:

$$p \cdot q = -\frac{1}{2} q^2 \tag{45}$$

$$p' \cdot q = \frac{1}{2} q^2. \tag{46}$$

Third, we suppress terms of $\mathcal{O}(q^4)$ using the low energy limit property $q << p$. Fourth, we use the approximation $\bar{\psi}\gamma_\mu\psi = 2p_\mu$ and $\bar{\psi}'\gamma_\mu\psi' = 2p'_\mu$. Finally, we use the appendix in [5] with the expressions for the several Feynman integrals. The contribution (43) matches the Born term of the potential

$$V = -a\frac{2G^2m^3}{r^2c^2} \tag{47}$$

given the right choice of the constant a which depends on c_1 and on the precise definition of the potential calculated in the Post-Newtonian expansion[4, 5].

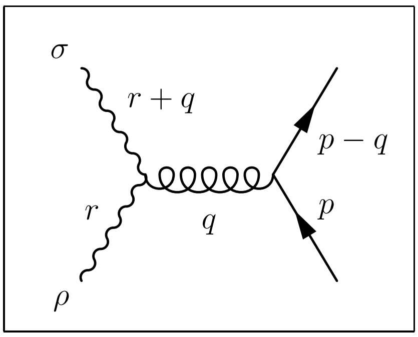
Thus in the non-relativistic quantum limit $q << p$ the contributions T_N and T_{NP} match those of the potential

$$V = -\frac{Gm^2}{r}(1 + a\frac{2Gm}{rc^2}), \tag{48}$$

the post Newtonian potential derived from general relativity in the non-relativistic limit [4][5] and much in the exact same way that QED produces T-matrix elements that match the Schroedinger equation with a Coulomb potential.

4 The photon

The diagram



produces the T-matrix contribution for the photon-matter scattering process. The T-matrix contribution from this diagram simplifies to

$$T_{N,PN} = 2im^2c_1G\frac{mE_{photon}}{-q^2}, \tag{49}$$

where the photon energy E_{photon} satisfies

$$E_{photon} << m, \tag{50}$$

after using the photon on-shell condition

$$r \cdot q \;=\; \frac{q^2}{2} \tag{51}$$

$$r' \cdot q \;=\; -\frac{q^2}{2} \tag{52}$$

along with the properties derived and used in the previous section. The constant c_1 absorbs all the constants and constant parameters. The differential cross section derived from this T-matrix in terms of the Mandelstam variable s reads

$$\left(\frac{d\sigma}{d\Omega}\right)_{C.M.} = \frac{1}{64\pi^2 s}\frac{|p_f^{(c.m.)}|}{|p_i^{(c.m.)}|}|T_{N,PN}|^2 \tag{53}$$

In the small angle approximation with $|p_f^{(c.m.)}| \simeq |p_i^{(c.m.)}|$ and $q = rsin(\theta/2)$, equation (53) simplifies to

$$\left(\frac{d\sigma}{d\Omega}\right)_{C.M.} = \frac{1}{64\pi^2}(2c_1G\frac{m^2E_{photon}}{r^2sin^2(\theta/2)})^2 \tag{54}$$

since $s \simeq m^2$. In the low energy limit, equation (54) describes the Rutherford scattering of a mass m projectile off of a mass E_{photon} target. The exchange of these particles' properties describes the scattering of mass E_{photon} projectile off of a mass m target. The exchange $m \leftrightarrow E_{photon}$ modifies the differential cross section which now takes the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{m\leftrightarrow E_{photon}} = \frac{1}{64\pi^2}(2c_1G\frac{m}{c^2})^2\frac{1}{sin^4(\theta/2)}. \tag{55}$$

The relation between the impact parameter, b, and the differential cross section,

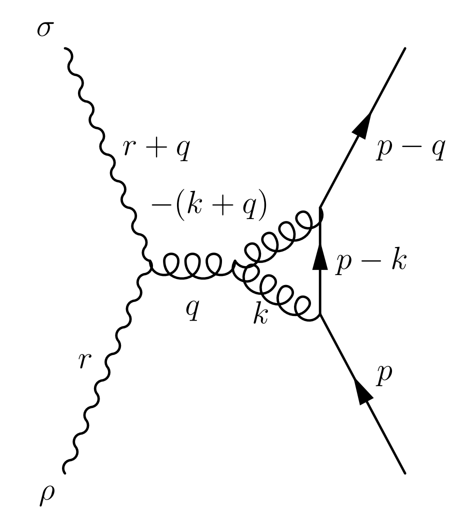
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|, \tag{56}$$

yields after a suitable choice of the constant c_1

$$\theta = \frac{4Gm}{c^2 b}. \tag{57}$$

Equation (57) reproduces the relation between the impact parameter and the deflection angle expected for the gravitational lensing of a photon by a point particle of mass m. See [11] for an overview of that calculation.

For the sake of completeness the diagram below gives the correction to $\mathcal{O}(G^2)$.



$$2im^2\frac{51\alpha^2}{32(1+6\alpha^2)}G^2\frac{m^2E_{photon}}{\sqrt{-q^2}}$$

5 Different kinds of matter

Thus far we considered a single particle, like the electron, and successfully reproduced the necessary post-Newtonian corrections to fit theory to experiment. Expanding the particle spectrum leads to a lack of solution because the number of equations generated by matching T-matrix elements to Born amplitudes is greater than the number of couplings available to the system. An increase in the number of connectons to $\omega^{(i)}$, $i = 1, ..., N$ allows an increase in the spectrum to $j = 1, .., J$ particles to include all the quarks and neutrinos.

The action also requires a matricial form of the covariant derivative, $\hat{D}_\mu$

$$\hat{D}_\mu = \begin{pmatrix} D_\mu^{(1)} & 0 & \dots & \dots & \dots & 0 \\ 0 & D_\mu^{(2)} & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & D_\mu^{(i)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & D_\mu^{(N)} \end{pmatrix}.$$

Where

$$D_\mu^{(i)} = \partial_\mu + \frac{i\tilde{e}}{\sqrt{1+6\alpha^2}} A_\mu + ig_i\ (\omega_\mu^{(i)ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}} \epsilon_\mu^{ab\nu} A_\nu) T_{ab}. \tag{58}$$

$\hat{D}_\mu$ acts on a vector-spinor $\hat{\psi}^{(j)}$ which now takes a vector form

$$\hat{\psi}^{(j)} = \begin{pmatrix} \psi^{(j)} \\ \psi^{(j)} \\ \vdots \\ \psi^{(j)} \\ \vdots \\ \psi^{(j)} \end{pmatrix}$$

with $\psi^{(j)}$ the usual spinor associated to a single particle j. For the avoidance of doubt, all N components $\psi^{(j)}$ of the vector-spinor $\hat{\psi}^{(j)}$ are equivalent.

As before, $\hat{D}_\mu \hat{\psi}$ transforms covariantly under U(1) and SO(4) transformations both denoted by a generic transformation $\bar{\Omega}$ acting on the i^{th} component of $\hat{\psi}^{(j)}$.

$$\hat{\psi}^{(j)} \rightarrow \begin{pmatrix} \psi^{(j)} \\ \psi^{(j)} \\ \vdots \\ \bar{\Omega}\psi^{(j)} \\ \vdots \\ \psi^{(j)} \end{pmatrix}$$

Transformations of A and ω under U(1) read

$$A'_\mu = A_\mu - \frac{\sqrt{1+6\alpha^2}}{\tilde{e}} \partial_\mu \lambda \tag{59}$$

$$\omega_\mu^{(i)\prime ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}} \epsilon_\mu^{ab\nu} A'_\nu = \omega_\mu^{(i)ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}} \epsilon_\mu^{ab\nu} A_\nu \tag{60}$$

Equation (60) implies

$$\omega_\mu^{(i)'ab} = \omega_\mu^{(i)ab} + \frac{\alpha}{\tilde{e}}\epsilon_\mu^{ab\nu}\partial_\nu\lambda \tag{61}$$

Equations (59) and (61) ensure that $D_\mu\psi$ transforms covariantly under U(1) transformations.

Transformations under SO(4) read

$$\begin{aligned} A'_\mu &= A_\mu & (62)\\ \omega_\mu^{'(i)ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ab\nu}A'_\nu &= \omega_\mu^{(i)ab} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ab\nu}A_\nu - \frac{1}{g_i}\partial_\mu\Lambda^{ab} \\ &\quad + C^{ab}{}_{cdef}\Lambda^{cd}(\omega_\mu^{(i)ef} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ef\nu}A_\nu) & (63) \end{aligned}$$

Equation (63) implies

$$\omega_\mu^{'(i)ab} = \omega_\mu^{(i)ab} - \frac{1}{g_i}\partial_\mu\Lambda^{ab} + C^{ab}{}_{cdef}\Lambda^{cd}(\omega_\mu^{(i)ef} + \frac{\alpha}{\sqrt{1+6\alpha^2}}\epsilon_\mu^{ef\nu}A_\nu) \tag{64}$$

Equations (62) and (64) ensure that $D_\mu\psi$ transforms covariantly under SO(4) transformations.

Thus both U(1) and SO(4) transformations acting on the i^{th} vector component of $\hat{\psi}^{(j)}$ with $\bar{\Omega}$ an arbitrary U(1) or SO(4) transformation imply

$$\hat{D'}_\mu\hat{\psi}' \to \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \bar{\Omega} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 1 \end{pmatrix} \hat{D}_\mu\hat{\psi}$$

Where the element $\bar{\Omega}$ enters in the i^{th} row and i^{th} column. As expected, this convoluted way of writing a $U(1)\times SO(4)^N$ theory reproduces (25), (27) and (36) after equating $\omega^{(i)}$'s to a unique field ω.

This construction permits the derivation of the Lagrangian of the theory in the usual manner. After redefining $A \to A/\sqrt{N}$, The bosonic Lagrangian given by the trace of the commutator $[\hat{D}_\mu, \hat{D}_\nu]$ produces the quadratic component

$$L^{(2)}_{gauge} = \frac{1}{2}\partial_\mu A^\nu\partial^\mu A_\nu + \frac{1}{2}\sum_i \partial_\mu\omega^{(i)\nu ab}\partial^\mu\omega^{(i)}_{\nu ab}. \tag{65}$$

The relevant terms of the cubic Lagrangian density involve again and for the same reasons as before term of the form $\omega\omega\partial\omega$ and $\partial AA\omega$ which take the form

$$
\begin{aligned}
L^{(3)}_{gauge} &= \sum 2g_i(\partial_\mu \omega^{(i)ab}_\nu - \partial_\nu \omega^{(i)ab}_\mu)\omega^\mu_{(i)ca}\ \omega^{(i)\nu c}_{\ \ b} \\
&\quad -g_i\frac{4\alpha^2}{N(1+6\alpha^2)}(A_\mu\partial_\rho A_\nu e^\nu_a e^\rho_b \omega^{(i)\mu ab}\partial_\nu A_\mu A_\rho e^\mu_a e^\rho_b \omega^{(i)\nu ab}). \quad (66)
\end{aligned}
$$

The terms

$$
\begin{aligned}
L_{f1} &= \frac{i}{2}\sum_j \bar{\hat{\psi}}^{(j)}\{\hat{\gamma}^\mu, \hat{D}_\mu\}\hat{\psi}^{(j)} \quad (67) \\
L_{f2} &= i\sum_j a_j \bar{\hat{\psi}}^{(j)}[\hat{\gamma}^\mu, \hat{D}_\mu]\hat{\psi}^{(j)} \quad (68)
\end{aligned}
$$

formulate the fermionic Lagrangian, where $\hat{\gamma}_\mu = 1_{N\times N}\gamma_\mu$ and where the arbitrary constants a_j do not break gauge invariance. Thus

$$
L_f = i\sum_{ij}\bar{\psi}^{(j)}(\partial\!\!\!/ + \frac{i\tilde{e}}{\sqrt{N(1+6\alpha^2)}}A\!\!\!/ + \frac{3ig_i\alpha}{\sqrt{N(1+6\alpha^2)}}A^\mu\gamma^5\gamma_\mu + g'_{ij}\bar{\omega}^{\mu a}_{\ \ \ \mu}\gamma_a)\psi^{(j)}, \quad (69)
$$

where the coupling constants $g'_{ij} = g_i a_j$.

For matter-matter scattering between particle j and particle k this theory yields the following tree level and 1-loop T-matrix contributions

$$
\begin{aligned}
T^{jk}_N &= -4im_j m_k \sum_i^N \frac{4g'_{ij}g'_{ik}}{-q^2} \quad (70) \\
T^{jk}_{PN} &= -\frac{12i}{64\sqrt{-q^2}}\sum_i^N g'_{ij}g'_{ik}g_i(g'_{ij}(m_j+17m_k) + g'_{ik}(17m_j+m_k)). \quad (71)
\end{aligned}
$$

where terms of $\mathcal{O}(J_{A\mu}(j)J^\mu_A(k))$ vanish in the classical limit after summation over all spin configurations and we avoided dealing with the well known electric charge contributions in order to simplify the discussion.

These terms must match the Born terms from the Schroedinger equation

$$
\begin{aligned}
T^{(Born)jk}_N &= -iG\frac{m_j m_k}{-q^2} \quad (72) \\
T^{(Born)jk}_{PN} &= -iG^2\frac{m_j m_k(m_j+m_k)}{\sqrt{-q^2}}. \quad (73)
\end{aligned}
$$

Matching

$$\frac{1}{2m_j}T_N^{(Born)jk} = 2m_k T_N^{jk} \tag{74}$$

$$\frac{1}{2m_j}T_{PN}^{(Born)jk} = 2m_k T_{PN}^{jk} \tag{75}$$

generate a set of equations whose solutions exist for any given J given a sufficiently large N.

The tree-level contributions to the T-matrix elements for scattering between particle j and a photon of energy E_r read

$$T_{N,PN}^{jE} = 2i\sum_i^N g'_{ij}g_i\frac{m_jE_r}{-q^2} \tag{76}$$

The contribution (76) must match the Born amplitudes for particles with mass E_{photon}

$$T_N^{(Born)jE} = -iG\frac{m_jE_r}{-q^2}. \tag{77}$$

Thus

$$\frac{1}{2m_j}T_N^{jE} = -i2m_jG\frac{m_jE_r}{-q^2} \tag{78}$$

where the right hand of (78) incorporates the exchange of target by projectile. Equation (78) generates a set of equations whose solutions exist for any given J given a sufficiently large N. Together (74), (75) and (78) generate a system of equations that can be solved for sufficiently large N.

6 Conclusions

We used (2) to reproduce Newtonian and post Newtonian interaction for both matter fields and photon field in the non relativistic limit. As opposed to other attempts at describing quantum gravity, here we opted to go from the fundamental theory to the non-relativistic Schroedinger equation and then from there to the non-relativistic classical limit. This limit coincides with the low energy limit of the Hilbert-Einstein equation.

The fundamental theory, a straight forward Yang-Mills, couples to matter in the same manner as SU(3) and SU(2) theories describe the other fundamental forces. Well known dimensional analysis of this theory guarantees renormalization.

The theory can accommodate for the full particle spectrum currently known and yields in the low energy limit the expected relativistic forces provided a sufficiently large number of connectons exist and whose couplings solve (74), (75) and (78).

The theory here proposed not only reproduces the observed experimental results of general relativity, but it is also renormalizable and more importantly it can be coupled to the standard model in a trivial way. Therefore $SO(4) \times SU(3) \times SU(2) \times U(1)$ best describes all interactions in nature.

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References

[1] C. N. Yang, *Integral Formalism for Gauge Fields*, Phys. Rev. Lett. 33, 445 (1974).

[2] G. 't Hooft, M Veltman, *One-loop divergencies in the theory of gravitation*, Ann. Inst. Henri Poincar, 20 (1974) 69.

[3] D. Singleton, *Exact Schwarzschild-like solution for Yang-Mills theories*, Phys. Rev. D 51, 5911 (1995).

[4] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[5] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, Phys.Rev.D50:3874-3888 (1994).

[6] V. N. Gribov, J. Nyiri, *Quantum Electrodynamics: Gribov Lectures on Theoretical Physics*, Volume 13 of Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (Cambridge University Press, 2001).

[7] D. Amati, M. Ciafaloni, G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, Int.J.Mod.Phys. A3, 1615(1988).

[8] D. Amati, M. Ciafaloni, G. Veneziano, *Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions*, Nuc.Phys. B347, 550 (1990).

[9] D. Amati, M. Ciafaloni, G. Veneziano, *Effective action and all order gravitational eikonal at Planckian energies*, Nuc.Phys. B403, 707 (1993).

[10] D. Amati, M. Ciafaloni, G. Veneziano, *Towards an S-matrix description of gravitational collapse*, JHEP 0802, 049 (2008).

[11] M. Meneghetti, *Introduction to Gravitational Lensing*, in preparation, http://pico.bo.astro.it/m̃assimo/pico/Teaching_files/chapter1.pdf