Equations for generalized n-point information with extreme and not extreme approximations in the free Fock space

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Abstract

The general n-point information (n-pi) are introduced and equations for them are considered. The role of right and left invertible interaction operators occurring in these equations together with their interpretation is dicussed. Some comments on aproximations to the proposed equations are given. The importance of positivity conditions and a possible interpretation of n-pi in the case of their non-compliance, for essentially nonlinear interactions (ENI), are proposed. A language of creation, annihilation and projection operators which can be applied in classical as well as in quantum case is used. The role of the complex numbers and functions in physics is also a little elucidated.

Contents

1 Introduction

In many cases, whether due to too sensitivity of the systems to small changes of initial and/or boundary conditions or its complexity, are not important their detailed description. In such cases we are using averages and higher and higher correlations among averages, which we called the n-pi (n-point information), see Eq.3.

For the first time, the general n-pi with multiple-times instead of one time were introduced by Kraichnan and Lewis in [1] in order to describe in a more complete way the random properties of the system S characterized by the 'field' $\varphi[\tilde{x}; \alpha]$. Having similar intensions, we generalize n-pi even more through introduction multiple-initial times and multiple-additional (initial and/or boundary) conditions. In other words, the **tensor products** of the fields concern not only space time variables but also additional conditions imposed on the fields, see 7. It is interesting that **all these generalizations satisfy the same equations**, 20, and hence - similar formulas can be derived with their help!

In the paper, we discuss again, from the point of view of an expected new and more fundamental theory, certain problems concerning a role and inerpretation

of the left and right invertible operators, Sec.4 . In Secs 4 and 5 we consider perturbation expansions with the two canonical extreme assumptions concerning zeroth order approximations. In Sec.7 we avoid such extreme assumptions. In Sec.8 we give some remarks concerning operators describing essentially nonlinear (ENI) interactions. Some general remarks are given in Sec 9-10. In Sec.11 an example of n-pi satisfying the positivity conditions is given.

We also show how - by the relevant averaging - a role of the equations describing in detail the system is gradually diminished to eventually replace them by the energy integral, Sec.12,. Below, in subsections, we put the two basic conditions that must be satisfied by the n-pi.

To avoid excess of indexes, $\tilde{x}$ contains not only the space-time variables (space-time localization) including the initial time t^0 but also subscripts and superscripts indexes denoting components of the field φ. In other words, $\tilde{x}$ may describe the *external and internal particle or field variables*. So we have:

$$\tilde{x} = (\vec{x}, t, t^0, {}^{\eta,\theta,\vartheta,\cdots,}_{\mu,\nu,\xi,\cdots,},) \in R^{\tilde{d}} \tag{1}$$

It means that the field φ is defined (lives) at the larger space which rather should be called a set in which the addition operation is not defined for all components of the 'vector' $\tilde{x}$. The usual space-time space of physics is denoted by $R^d \ni (\vec{x}, t)$ or M^d. In fact we assume a discrete versions of these spaces, hence, in this and other papers, we consider and have been considered the **difference equations**. The initial and/or boundary conditions are denoted by α. For the n point particles system, we usually use another notation and then

$$\varphi \equiv q, \ and \ \tilde{x} = (t, i, j) \tag{2}$$

where t is the time, $i = 1, ..., n$ is numbering the individual particles and $j = 1, 2, 3$ refers to a particle components. This case is called somewhat artificially the *zero-dimentional (0-D) field*. In fact, by assigning to different components and their values of the vector $\tilde{x}$ - the corresponding components of the physical fields, one can consider only the one superfield φ describing the whole physical system.

The field φ at points $\tilde{x}$ may have no physical meaning if it is very sensitive to even small changes of α. Then, physical meaning may have some averaged or smoothed values of the field φ with respect to α or some components of $\tilde{x}$, which in all cases are denoted by $< \varphi(\tilde{x}) >$. However, the problem with averaged quantities is such that equations for them are not closed and to get a complete system of equations, we are forced to introduce other quantities than averaged or smooth solutions of the starting equations which are denoted as $< \varphi(\tilde{x}_1)...\varphi(\tilde{x}_n) >$, $n = 1, \cdots, \infty$ and which are called the n-pi whose particular cases are correlation functions. The symbol $< ... >$ means in fact an *averaging* or *a smoothing* of products $\varphi[\tilde{x}_1; \alpha] \cdots \varphi[\tilde{x}_n; \alpha]$ with respect to the variables or functions α. For example, in the case of averaging

$$< \varphi(\tilde{x}_1)...\varphi(\tilde{x}_n) > = \int \delta\alpha \left\{ \varphi[\tilde{x}_1; \alpha] \cdots \varphi[\tilde{x}_n; \alpha] \right\} P[\alpha] \tag{3}$$

where $P[\alpha]$ is a probability density, but in the case of smoothing, $P[\alpha]$ is some weight function or functional. $\int \delta\alpha$ means the formal functional integration or ordinary integrations or a sum $\sum$ in the case of discrete variables.

1.1 Positivity conditions

The positivity of the $P[\alpha] \geqq 0$ leads to the *positivity conditions* upon the even n-pi:

$$< \varphi(\tilde{x}_1)...\varphi(\tilde{x}_{2n}) > \geqq 0 \qquad (4)$$

if for every $\tilde{x}_i$ there is at least one $\tilde{x}_j$ with $j \neq i$ such that $\tilde{x}_i = \tilde{x}_j$, see , eg., one of the author's papers. This is the *positivity principle* which expresses integral structure 3 of the n-pi.

1.2 Permutation symmetry condition

From 3 it is seen that these n-pi are permutation symmetric:

$$V(\tilde{x}_{(n)}) \equiv < \varphi(\tilde{x}_1)...\varphi(\tilde{x}_n) > = < \varphi(\tilde{x}_{i_1})...\varphi(\tilde{x}_{i_n}) > \qquad (5)$$

where $(i_1, ..., i_n)$ denotes arbitrary permutation of the number sequence $(1, ..., n)$ and $\tilde{x}_i \in R^{\tilde{d}}$. The above symmetry takes place 'iff' we are exchanging among themselves all components of 'vectors', e.g, $\tilde{x}_i$ and $\tilde{x}_j$. However, if you do not exchange at least one pair of the components, e.g, t_i^0 and t_j^0 , then the n-pi may not be symmetric with respect to permutation of the space-time variables and at this point even the classical n-pi are not permutatin symmetrical.

From formula 3 it is also directly seen that any approximation to the field $\varphi[\tilde{x}; \alpha]$ preserves the positivity conditions and the permutation symmetry of the approximated n-pi.

One can get even more general n-pi, if every α describing initial and/ or boundary conditions is the sum of two terms:

$$\alpha = \beta + \gamma \qquad (6)$$

where β can describe different initial and boudary conditions in every field $\varphi[\tilde{x}; \alpha]$ entering the n-point smoothing integral:

$$< \varphi(\tilde{x}_1)...\varphi(\tilde{x}_n); \beta_{(n)} > = \int \delta\gamma \left\{ \varphi[\tilde{x}_1; \beta_1 + \gamma] \cdots \varphi[\tilde{x}_n; \beta_n + \gamma] \right\} P[\gamma] \qquad (7)$$

In the case that all β_i will not be equal to each other, the permutation symmetry 5 and positivity conditions 4 are not satisfied in general.

The structure of n-pi, 7, shows also surprising property of n-pi. If some systems do not interacting with one another, and in spite of that we want to describe it as a one system, then the corresponding n-pi do not need to be the products of corresponding n'-pi and n''-pi, (n'+n''=n), where n' is related to one

system and n" is related to another system, Sec.11. All depends on the weight $P[\gamma]$ in the above averaging integral. From the point of view of n-pi, it looks as if the knowledge of the systems included in the $P[\gamma]$, for example, that:

$$P[\gamma] \neq P[\gamma'] \star P[\gamma''] \tag{8}$$

imitates an interaction between not interacting systems!!?

2 Generating vectors for n-pi. Cuntz relations and the unit operator. Free Fock space (FFS),

In the Dirac notationa, a generating vector $|V>$, for the infinite collection of n-pi $V(\tilde{x}_{(n)})$, is defined as follows:

$$|V> = |0> + \sum_{n=1}^{\infty} \int d\tilde{x}_{(n)} V(\tilde{x}_{(n)})|\tilde{x}_{(n)} > \tag{9}$$

where the orthonormal base vectors $|\tilde{x}_{(n)}>$ are introduce. Here $\tilde{x}_{(n)} \equiv \tilde{x}_1, ..., \tilde{x}_n$ and so on. They saytisfy the following relations

$$< \tilde{y}_m, ..., \tilde{y}_1|\tilde{x}_1, ..., \tilde{x}_n >= \delta_{mn}\delta(\tilde{y}_1 - \tilde{x}_1) \cdots \delta(\tilde{y}_n - \tilde{x}_n) \tag{10}$$

where $m, n = 0, 1, 2...$, and δ_{mn} is the Kronecker delta, and $\delta(\tilde{y}_1 - \tilde{x}_1) \cdots \delta(\tilde{y}_n - \tilde{x}_n)$ are products of the Dirac and Kronecker delta functions or only Kronecker functions if all components of 'vectors' $\tilde{y}, \tilde{x}$ are discrete. $|\tilde{x}_0> \equiv |0>$.

In contrast to the canonical, classical n-pi V, the base vectors are not permutation symmetric, nevertheless we get:

$$< \tilde{y}_{(n)}|V >= V(\tilde{y}_{(n)}) \tag{11}$$

also for permutation symmetric n-pi V.

If such base vectors are constructed by means of the 'creation' and 'annihilation' operators (Here and below the Asian hat over the letter indicates that the letter represents an operator in the space under consideration)

$$|\tilde{x}_{(n)} >= \hat{\eta}^{\star}(\tilde{x}_1) \cdots \hat{\eta}^{\star}(\tilde{x}_n)|0 > \tag{12}$$

and

$$< \tilde{x}_{(n)}| =< 0|\hat{\eta}(\tilde{x}_n) \cdots \hat{\eta}(\tilde{x}_1) \tag{13}$$

which satsfy the Cuntz relations:

$$\hat{\eta}(\tilde{x})\hat{\eta}^{\star}(\tilde{y}) = \delta(\tilde{x} - \tilde{y})\hat{I} \tag{14}$$

and additional restrictions are imposed:

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$$\hat{\eta}(\tilde{x})|0> = 0, \ <0|\hat{\eta}^{\star}(\tilde{y}) = 0 \tag{15}$$

where the star over the operator means a linear operation, called also involution, with the following properties:

$$(\hat{\eta}^{\star}(\tilde{y}))^{\star} = \hat{\eta}(\tilde{y}) \tag{16}$$

$$(\hat{\eta}(\tilde{x}\hat{\eta}(\tilde{y}))^{\star} = \hat{\eta}^{\star}(\tilde{y})\hat{\eta}^{\star}(\tilde{x}) \tag{17}$$

then the space formed by the vectors 9 is called the *free Fock space* (FFS). In fact, the notions of creation and annihilation operators were also used by Dirac, see [11], who was assuming commutation or antycommutation properties for the creation operators among themselvs. The Cuntz operators, without these properties but only with: 14 and 15, allows to use the notion of creation and annihilation operators in classical as well as in quantum physics and the corresponding Fock space (FS) is called the free Fock space (FFS). An equivalent description of the FFS without using the creation and annihilation operators is given by author in [14] by means of the linear functionals.

In the FFS, the unit operator can be expressed as follows

$$\hat{I} = |0><0| + \int d\tilde{x}\hat{\eta}^{\star}(\tilde{x})\hat{\eta}(\tilde{x}) \tag{18}$$

Remember, $\int d\tilde{x}$ is a summation and integration, or summation only as we usually assume.

3 Equations for n-pi and their structure

With the help of unique 'field' φ (discrete indexes are also contained in the symbol $\tilde{x}$), we consider the following integro-differential or rather the **difference equations**:

$$L[\tilde{x}; \varphi] + \lambda N[\tilde{x}; \varphi] + G(\tilde{x}) = 0 \tag{19}$$

for the field φ which can describe an electromagnetic or hydromechanic or other equations of motion. Here the terms L and N describe a linear and nonlinear dependence on the field φ, the term G does not depend on φ and can describe a source of the field φ or an external force. With the help of Eq.19 and definition of n-pi: 3,5, one can derive, see ,e.g., [4], the following equation for the generating vector |V>:

$$\left(\hat{L} + \lambda\hat{N} + \hat{G}\right)|V> = \hat{P}_0\hat{L}|V> + \lambda\hat{P}_0\hat{N}|V> \equiv |0>_{info} \tag{20}$$

with the following operators:

$$\hat{L} = \int \hat{\eta}^{\star}(\tilde{x})L[\tilde{x}; \hat{\eta}]d\tilde{x} + \hat{P}_0, \tag{21}$$

$$\hat{N} = \int \hat{\eta}^{\star}(\tilde{x}) N[\tilde{x}; \hat{\eta}] d\tilde{x} + \hat{P}_0, \tag{22}$$

$$\hat{G} = \int \hat{\eta}^{\star}(\tilde{x}) G(\tilde{x}) d\tilde{x} \tag{23}$$

where the projector

$$\hat{P}_0 = |0><0| \tag{24}$$

λ is is a theory parameter which can describe the strenth of interaction reffered to as the coupling constant. We remind you that $\int$ means integration and summation or, in the case of all discrete variables - only summation. If L and N are given by the Volterra series, then

$$\hat{L}[\tilde{x}; \hat{\eta}] = \int d\tilde{y} L(\tilde{x}; \tilde{x}_1) \hat{\eta}(\tilde{x}_1) \tag{25}$$

and

$$\hat{N}[\hat{x}; \hat{\eta}] = \sum_{n=2} \int d\tilde{x}_{(n)} N(\tilde{x}; \tilde{x}_{(n)}) \hat{\eta}(\tilde{x}_1) \cdots \hat{\eta}(\tilde{x}_n) \tag{26}$$

For the *local interaction*, for example:

$$N(\tilde{x}; \tilde{x}_{(n)}) = \delta(\tilde{x} - \tilde{x}_1) \cdots \delta(\tilde{x} - \tilde{x}_n) \tag{27}$$

In the case of quantum fields φ, when the field φ appearing in Eq.19 is an operator, the n-pi as Wightman functions satisfy identical equations as Eq.20. This means that the **operator properties of the field φ and appropriate definition of the n-pi** must be only used in the zero approximation to the considered equations for n-pi. (For different transformations of Eq.20, see below). In other words, the FFS is a space in which quantum and statistical properties of a system can be described by the same equations.

In the case of Green's n-pi it is possible another relation between fields and operators $\hat{L}, \hat{N}, \hat{G}$ in which $\hat{G} \equiv \hat{C}$ is still lower triangular operator, but it has other projective properties, see [5]. However, the remaining operators: $\hat{L}$ and $\hat{N}$ have similar structure in classical and quantum cases.

4 Right and left invertible operators, physical interpretation, and transformed Eq.20

For a definition and properties of right and left invertible operators, see Danuta Przeworska-Rolewicz book: "Introduction to Algebraic Analysis" (1988), see also author's papers.

4.1 Right invertible self-interaction and interaction operators

Let us take the operator $\hat{L} \equiv \hat{L}_r$ related to the linear part of Eq.19

$$\hat{L}_r = \int \hat{\eta}^\star(\tilde{y}) L_r(\tilde{y}, \tilde{x}) \hat{\eta}(\tilde{x}) d\tilde{y} d\tilde{x} + \hat{P}_0 \tag{28}$$

with a function L_r that there is such a function L_r^{-1} that

$$\int L_r(\tilde{x}, \tilde{z}) L_r^{-1}(\tilde{z}, \tilde{y}) d\tilde{z} = \delta(\tilde{x} - \tilde{y}) \tag{29}$$

We define the operator

$$\hat{L}_r^{-1} = \int \hat{\eta}^\star(\tilde{x}) L_r(\tilde{x}, \tilde{y}) \hat{\eta}(\tilde{y}) d\tilde{x} d\tilde{y} + \hat{P}_0 \tag{30}$$

Then we get

$$\hat{L}_r \hat{L}_r^{-1} = \int d\tilde{y} \hat{\eta}^\star(\tilde{y}) \hat{\eta}(\tilde{y}) + \hat{P}_0 = \hat{I} \tag{31}$$

see Sec.2. This means that $\hat{L}_r$ is the *right invertible operator* in the FFS and $\hat{L}_r^{-1}$ is a *right inverse operator*.

For a special important case of

$$L_r(\tilde{x}, \tilde{z}) = \delta(\tilde{x}' - \tilde{z}') L_r(\tilde{x}'', \tilde{z}'') \tag{32}$$

where, e.g., $\tilde{x}' + \tilde{x}'' = \tilde{x}$, and $\tilde{x}' \cdot \tilde{x}'' = 0$, then

$$\hat{L}_r = \int \hat{\eta}^\star(\tilde{x}) \delta(\tilde{x}' - \tilde{y}') (L_r(\tilde{x}'', \tilde{y}'') \hat{\eta}(\tilde{y}) d\tilde{x} d\tilde{y} + \hat{P}_0 \tag{33}$$

if

$$\int L_r(\tilde{x}'', \tilde{z}'') L_r^{-1}(\tilde{z}'', \tilde{y}'') d\tilde{z}'' = \delta(\tilde{x}'' - \tilde{y}'') \tag{34}$$

Now, let take us an example of the *local operator* $\hat{N}$ related to the nonlinear part of Eq.19 describing *self-interaction* of the system elements :

$$\hat{N} \equiv \hat{N}_{r(self)} = \int d\tilde{y} \hat{\eta}^\star(\tilde{y}) \hat{\eta}^2(\tilde{y}) + \hat{P}_0 \tag{35}$$

For discrete $\tilde{y}$, it is easy to show, Sec.2, that a right inverse to $\hat{N}$ is

$$\hat{N}_{r(self)}^{-1} = \int d\tilde{y} \hat{\eta}^\star(\tilde{y})^2 \hat{\eta}(\tilde{y}) + \hat{P}_0 \tag{36}$$

An example of the right invertible operator describing an interaction of system elements is the operator

$$\hat{N}_{r(int)} = \int d\tilde{y} \hat{\eta}^{\star}(\tilde{y}) \hat{\eta}(\tilde{y}) \hat{\eta}(\tilde{y} + \tilde{a}) + \hat{P}_0 \tag{37}$$

where for the **local interaction** the components of vector $\tilde{a}$ related to the space time variables: $(\vec{a}, t_a) \equiv 0$. Some components of the vector $\tilde{a}$ related to identification of the system elements **should be** different from zero. A right inverse for this operator is:

$$\hat{N}_{r(int)}^{-1} = \int d\tilde{y}' \hat{\eta}^{\star}(\tilde{y}' + \tilde{a}) \hat{\eta}^{\star}(\tilde{y}') \hat{\eta}(\tilde{y}') + \hat{P}_0 \tag{38}$$

At this point, and we will continue to assume that the indexes identifying the individual elements of the system under consideration run an infinite set of values, and variables, which do not correspond to any elements we put identically equal to zero.

4.2 Left invertible self-interaction operators

It turns out that the local operators $\hat{N}$ can also be left invertible. As an example let take a formal operator

$$\hat{N} \equiv \hat{N}_{l(self)}(\lambda') = \int d\tilde{x} \hat{\eta}^{\star}(\tilde{x}) \frac{H(\tilde{x})\hat{I}}{\hat{I} - \lambda'\hat{\eta}(\tilde{x})} + \hat{P}_0 \tag{39}$$

where λ' is a new coupling constant usually called the *minor coupling constant* and H is an arbitrary auxiliary function. In particular, it can be equal to one. A left inverse to this operator, denoted by $\hat{N}_l^{-1}$ and satisfying equation:

$$\hat{N}_{l(self)}^{-1} \hat{N}_{l(self)} = \hat{I} \tag{40}$$

is

$$\hat{N}_{l(self)}^{-1}(\lambda') = \int d\tilde{y} E(\tilde{y}) \left(\hat{I} - \lambda'\hat{\eta}(\tilde{y}) \right) \hat{\eta}(\tilde{y}) + \hat{P}_0 \tag{41}$$

with

$$\int d\tilde{x} E(\tilde{x}) H(\tilde{x}) = 1 \tag{42}$$

Other examples of the left invertible self-interaction operators are

$$\hat{N}_{l(self)}(\lambda') = \int d\tilde{x} \hat{\eta}^{\star}(\tilde{x}) \frac{H(\tilde{x})\hat{I}}{\hat{I} - \lambda'\hat{\eta}^m(\tilde{x})} + \hat{P}_0 \tag{43}$$

where e.g.,$m \in R' \subset R$, see also [3]. They can be used for modeling the system S with self-interaction of its 0-D elements, which interact between each other in a linear way described by the functional L, see Eq.19. It worth noting that the above nonlinear term (the linear operator $\hat{N}$ in FFS depending on the operator $\hat{\eta}$ in a nonlinear way) can be related to an external field acting on the system

S. We assume that a kinematic term of the system S is also contained in the operator $\hat{L}$. In this case the "external" interaction related to the operator $\hat{G}$ depends only on the time t and indexes i, j, see 2.

4.3 Left invertible, essentially nonlinear interaction (ENI) and self-interaction operators (ENSI)

The next step is to construct the left invertible operators in the case of nonlinear interactions (not self-interaction). A simple modification of the formula 43:

$$\hat{N}_{l(int)}(\lambda') = \int d\tilde{x}\hat{\eta}^{\star}(\tilde{x}) \frac{H(\tilde{x})\hat{I}}{\hat{I} - \lambda'\hat{\eta}(\tilde{x} + \tilde{a}_1)\cdots\hat{\eta}(\tilde{x} + \tilde{a}_m)} + \hat{P}_0 = \hat{I} + \lambda'\hat{N} + \cdots + \hat{P}_0 \tag{44}$$

where 'vectors' $\tilde{a}_i$ are defined as $\tilde{a}$ in subsection 4.2. In this case

$$\hat{N}_{l(int)}^{-1}(\lambda') = \int d\tilde{y}E(\tilde{y})\left(\hat{I} - \lambda'\hat{\eta}(\tilde{y} + \tilde{a}_1)\cdots\hat{\eta}(\tilde{y} + \tilde{a}_m)\right)\hat{\eta}(\tilde{y}) + \hat{P}_0 \tag{45}$$

One can also use the left invertible formal operator:

$$\hat{N}_l(\lambda') = \hat{N}_l \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r} \tag{46}$$

with a left invertible $\hat{N}_l$ and 'any' operator $\hat{N}_r$. A left inverse to the operator $\hat{N}_l(\lambda')$ is

$$\hat{N}_l^{-1}(\lambda') = \left(\hat{I} - \lambda'\hat{N}_r\right)\hat{N}_l^{-1} \tag{47}$$

with a left inverse $\hat{N}_l^{-1}$ to the operator $\hat{N}_l$. It is easy to check that the operator

$$\hat{N}_l'(\lambda') = \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r}\hat{N}_l \tag{48}$$

is also a left invertible operator with left inverse

$$\hat{N}_l'^{-1}(\lambda') = \hat{N}_l^{-1}\left(\hat{I} - \lambda'\hat{N}_r\right) \tag{49}$$

If all $\tilde{a}_i = 0$ in the formula 44, then the operator $\hat{N}_{l(int)}(\lambda')$ desribes self-interaction.

4.4 Transformed equations 20

Let us assume first that in Eq.20 the $\hat{L} \equiv \hat{L}_r$ is a right invertible operator with its right inverse $\hat{L}_r^{-1}$. Multiplying Eq.20 by $\hat{L}_r^{-1}$ we get an equivalent equation:

$$\left(\hat{I} + \hat{L}_r^{-1}\hat{G} + \lambda\hat{L}_r^{-1}\hat{N}\right)|V> = \hat{L}_r^{-1}|0>_{info} + \hat{P}_{L_r}|V> \tag{50}$$

with any projection $\hat{P}_{L_r}|V>$ where the projector $\hat{P}_{L_r} = \hat{I} - \hat{L}_r^{-1}\hat{L}$.

Now let us assume that the operator $\hat{N}$ in Eq.20 is a right invertible operator with its right inverse $\hat{N}_r^{-1}$. We get similar equation to Eq.50:

$$\left(\lambda^{-1}\hat{N}_r^{-1}(\hat{L} + \hat{G}) + \hat{I}\right)|V> = \lambda^{-1}\hat{N}_r^{-1}|0>_{info} + \hat{P}_{N_r}|V> \qquad (51)$$

with the projector $\hat{P}_{N_r} = \hat{I} - \hat{N}_r^{-1}\hat{N}$. This equation with any projection $\hat{P}_{N_r}|V>$ is also equivalent to Eq.20. Now the problem arises how to choose vectors $\hat{P}_{L_r}|V>$ and $\hat{P}_{N_r}|V>$, which for the point of view Eq.20 may be any of the vectors of the respective subspace! In the perturbation theory we solve this problem by choosing zero approximations in such a way as **to satisfy the symmetry conditions, 59, and the positivity principle, 4**. See also a symmetrized version of the equations as above. Then, at least, for small values of parameters $|\lambda|$ or $|1/\lambda|$, one can fulfill the above conditions. In all these considerations we assume that vectors $\hat{L}_r^{-1}|0>_{info}$, $\lambda^{-1}\hat{N}_r^{-1}|0>_{info}$ are trivial and are

$$\sim |0> \qquad (52)$$

Let us assume that the operator $\hat{N}$ is a left invertible operator with the left invers operator $\hat{N}_l^{-1}$. Then, by multiplying Eq.20 by this left inverse operator, we get

$$\left(\lambda^{-1}\hat{N}_l^{-1}(\hat{L} + \hat{G}) + \hat{I}\right)|V> = \lambda^{-1}\hat{N}_l^{-1}|0>_{info} \qquad (53)$$

It is also easy to see that in this case the operator

$$\hat{P}_{N_l} = \hat{I} - \hat{N}\hat{N}_l^{-1} \qquad (54)$$

is a projector.

We do not consider the case of left invertible $\hat{L}_l$ because of additional conditions which are usually related to the linear part of Eq.19. In the case of Eq.53 would be interesting whether the positivity principle can be easy satisfied.

5   Matter disappears in zeroth order approximation

We assume that the generating vector $|V>$ can be presented as the power series with respect to the positive or negative powers of the coupling constant λ

$$|V> \equiv |V(\lambda)> = \sum_{\mu} \lambda^{\mu}|V>^{(\mu)} \qquad (55)$$

where μ runs through integers from zero to plus infinity, or from zero to minus infinity. The first expansion we will use in the case of Eq.50, the second expansion will be used in the cases of Eq.51 and Eq.53. The vectors $|V>^{(\mu)}$ are called the μ *approximations*.

In this Section we use Eq.50 in which in the 0-approximation the term with the operator $\hat{N}$ disappears. We may say that matter, a cause of any interaction, disappears. We get:

$$\left(\hat{I} + \hat{L}_r^{-1}\hat{G}\right)|V>^{(0)} = \hat{L}_r^{-1}|0>_{info} + \hat{P}_{L_r}|V>^{(0)} \tag{56}$$

For $\hat{G} = 0$, and so when external influence, which is produced by material objects disappears too, zero approximation to the full generating vector |V> for all n-pi is given by the equation:

$$|V>^{(0)} = \hat{L}_r^{-1}|0>_{info}^{(0)} + \hat{P}_{L_r}|V>^{(0)} = |0> + \hat{P}_{L_r}|V>^{(0)} \tag{57}$$

The permutation symmetry of n-pi imposes on the generating vector $|V>^{(0)}$ the condition:

$$|V>^{(0)} = \hat{S}|V>^{(0)} \tag{58}$$

where the projector $\hat{S}$ is constructed in, e.g., [2]. A construction in the zeroth approximation of n-pi satisfying these two equations and the positivity priciple 4 one can find in [4].

Taking into account the permutation symmetry for all n-pi

$$|V> = \hat{S}|V> \tag{59}$$

and the *perturbation principle:*

$$|V(0)> = |V>^{(0)} = \hat{S}\hat{P}|V(0)>^{(0)} \tag{60}$$

where $\hat{P}|V>$ represents the projection occurring in Eq.50, one can transform this equation in such a way that every higher approximation to the generating vector |V>, even in the case of $\hat{G} \neq 0$, is completely expressed by the zeroth order approximation, see also Secs 6 and 7. So, multiplying Eq.50 by the projector $\hat{S}$, we can get:

$$\left(\hat{I} + \hat{S}\hat{L}_r^{-1}\hat{G} + \lambda\hat{S}\hat{L}_r^{-1}\hat{N}\right)|V> = \hat{S}\hat{L}_r^{-1}|0>_{info} + \hat{S}\hat{P}_{L_r}|V> \tag{61}$$

Assuming in addition to 60:

$$\hat{S}\hat{L}_r^{-1}|0>_{info} = \hat{S}\hat{L}_r^{-1}|0>_{info}^{(0)} = |0> \tag{62}$$

we get the following perturbation formulas:

$$\left(\hat{I} + \hat{S}\hat{L}_r^{-1}\hat{G}\right)|V>^{(n)} + \left(\hat{S}\hat{L}_r^{-1}\hat{N}\right)|V>^{(n-1)} = 0 \tag{63}$$

for n=1,2,...which can be further transform as follows:

$$|V>^{(n)} + \left(\hat{I} + \hat{S}\hat{L}_r^{-1}\hat{G}\right)^{(-1)}\left(\hat{S}\hat{L}_r^{-1}\hat{N}\right)|V>^{(n-1)} = 0 \tag{64}$$

6 Space and time disappear in zeroth approximation

We can achieve this situation in the Eq.20 if you divide it by λ, and assuming that in the $\lambda \to \infty$ solutions are bounded. In this case we get from Eq.51 and 59 the following equation:

$$|V(\infty) >= |V >^{(0)} = \hat{S}\hat{P}_{N_r}|V(\infty) >^{(0)} \tag{65}$$

which leads, for local interactions, to the product of the Kronecker's deltas for n-pi. Higher approximations are given by similar formulas as 64.

6.1 Higher approximations. Exact solutions?

Taking into account the symmetrized Eq.51:

$$\left(\lambda^{-1}\hat{S}\hat{N}_r^{-1}(\hat{L} + \hat{G}) + \hat{I}\right)|V >= \lambda^{-1}\hat{S}\hat{N}_r^{-1}|0 >_{info} + \hat{S}\hat{P}_{N_r}|V > \tag{66}$$

we can derive the following formula:

$$|V >\equiv |V(\lambda) >= \left(\lambda^{-1}\hat{S}\hat{N}_r^{-1}(\hat{L} + \hat{G}) + \hat{I}\right)^{-1}\left(\lambda^{-1}\hat{S}\hat{N}_r^{-1}|0 >_{info} + \hat{S}\hat{P}_{N_r}|V >\right) \tag{67}$$

in which the inverse operator is an inverse to the unit and the lower triangular operator $\hat{S}\hat{N}_r^{-1}(\hat{L} + \hat{G})$. It is interesting to notice that in the extreme case $(\lambda \to \infty)$ traces of interaction are stuck only in the projector $\hat{P}_{N_r} = \hat{I} - \hat{N}_r^{-1}\hat{N}_r$. Now, the perturbation principle gives the following identification

$$|V(\infty) >= |V >^{(0)} = \hat{S}\hat{P}_{N_r}|V(\infty) > \tag{68}$$

If the formula 68 satisfies the positivity conditions 4, then the generating vector $|V >$ satisfies these conditions at least for appropriate small values of $|1/\lambda|$.

Using the **perturbation principle**, we identify the zero approximation with the projection:

$$|V >^{(0)} = \hat{S}\hat{P}_{N_r}|V(\infty) > \tag{69}$$

In this way, for the polynomial interactions, in a finite number of steps, **one can receive exact expressions for the n-pi**!! For the projection properties of operators describing the polynomial interactions and their inverses, see, e.g., [5, 2].This is not the case for the theory of perturbations considered in Sec.5, for which in zero approximation matter disappears. A common feature of both theory is, however, a similar structure of zero approximations given respectively by formulas 60 and 68, see [4].

7 Equations with left invertible interaction; matter, space and time do not disappear

In Eq.20 three kind of operators appear: $\hat{L}$, $\hat{N}$ $\hat{G}$ which are related to different physical and mathematical situations. The operator $\hat{L}$ is related to the linear part of Eq.19 which describes *kinematics of the theory*; In other words, it describe changes, not causes of changes. The causes of changes are related to the second and third of listed operators even if they have different characters. $\hat{N}$ describes rather causes (interaction) among the system elements, e.g., particles. $\hat{G}$ describes causes (sources of fields or external forces) acting on the system. Looking at the transformed Eq.20 given in Sec.4 in three forms: Eqs 50, 51 and 53, and using as in Sec.5 rather unused language, we see that when the matter disappears in the zeroth approximation then we are getting an expansion of the generating vector |V> in the positive powers of the coupling constant λ, but when the time and space disappear, we are getting an expansion in the inverse powers of the coupling constatnt.

In spite of well established habit of assuming that $\hat{N}$ in the FFS is a right-invertible operator (polynomial interactions), there are several reasons to give up such assumptions. First of all, for polynomial interactions the perturbation expansion can be only used in the two extreme cases: a small and a large absolute value of the coupling constant λ. Moreover, even the exact solutions obtained in Sec.6.1 in the inverse coupling constant λ, for which the space and time disapears in the limit $\lambda \to \infty$, have no simple physical interpretations.

7.1 Expansion in the minor (scale) coupling constant λ' ; the left invertible operator $\hat{N}(\lambda') \equiv \hat{N}_l(\lambda')$

The situation may change, when $\hat{N}$ is a left invertble operator with a left inverse denoted by $\hat{N}_l^{-1}$, see Sec.4. In this case, Eq.53 can be used:

$$\left(\lambda^{-1}\hat{N}_l^{-1}(\lambda')(\hat{L}+\hat{G})+\hat{I}\right)|V> = \lambda^{-1}\hat{N}_l^{-1}|0>_{info} = \lambda^{-1}|0> \qquad (70)$$

See, however, Subsec.8.2. Taking into account that the operator $\hat{N} = \hat{N}(\lambda')$, related to the inner and/or external interaction of the system, depends on the constatnt λ' (minor coupling constant) and that the following expansion is possible:

$$\hat{N}_l^{-1}(\lambda') = \hat{N}_l^{-1(0)} + \sum_{n=1} \lambda'^n \hat{N}_l^{-1(n)} \qquad (71)$$

see Sec.4, Eq.41 and other, we get, for the zeroth approximation $|V>^{(0)}$ in the minor coupling constant λ',

$$|V> = |V;\lambda,\lambda'> = |V;\lambda>^{(0)} + \sum_{n=1} \lambda'^n |V;\lambda>^{(n)} \qquad (72)$$

the following equation:

$$\left(\lambda^{-1} \hat{N}_l^{-1(0)} (\hat{L} + \hat{G}) + \hat{I} \right) |V; \lambda >^{(0)} = \lambda^{-1} \hat{N}_l^{-1} |0 >_{info}^{(0)} = \lambda^{-1} |0 > \qquad (73)$$

In spite of the branching character of this equation, it can be solved at least for interaction type such as those given by Eq.39, for which

$$\hat{N}_l^{-1(0)} = \hat{N}_l^{-1}(0) = \int d\tilde{y} E(\tilde{y}) \hat{\eta}(\tilde{y}) + \hat{P}_0 \qquad (74)$$

In this case, the zeroth order approximation to Eq.20

$$\left(\hat{L} + \hat{G} + \lambda \hat{N}(0) \right) |V; \lambda >^{(0)} = |0 > \qquad (75)$$

is unbranching because

$$\hat{N}^{(0)} \equiv \hat{N}(0) = \int d\tilde{x} \hat{\eta}^{\star}(\tilde{x}) H(\tilde{x}) + \hat{P}_0 \qquad (76)$$

is a lower triangular operator with respect to projectors $\hat{P}_n$ projecting on the n-point vectors given by Eqs12, see e.g. [4]. The Eq.75 is described by the sum of diagonal operator $\hat{L}$ and two lower triangular operators $\hat{G}$ and $\lambda \hat{N}(0)$. Eq.75, and so the Eq.73 can be easy solved in the case of right invertible operator $\hat{L}$ which usually takes place. In this case we can describe its solutions in the following form:

$$|V; \lambda >^{(0)} = \left[\hat{I} - \hat{S} \hat{L}_r^{-1} (\hat{G} + \lambda \hat{N}(0)) \right]^{-1} \left(\hat{S} \hat{P}_{L_r} |V; \lambda >^{(0)} + |0 > \right) \qquad (77)$$

where the symmetry condition 58 was taken into account. Here the projector $\hat{P}_{L_r} = \hat{I} - \hat{L}_r^{-1} \hat{L}$. Assuming the perturbation principle 60 in the form

$$\hat{S} \hat{P}_{L_r} |V; \lambda >^{(0)} = \hat{S} \hat{P}_{L_r} |V; 0 >^{(0)} \qquad (78)$$

we can guarantee that at least for a small external field $\hat{G}$ and a small coupling constant λ, the coefficients of the zeroth order of the generating vector $|V; \lambda >^{(0)}$ satisfies the positivity condition 4. For a more explicit description of the above formula, see [4].

For the higher approximations occurring in Eq.72, the Eq.53 can be used, which we rewrite in the symmetrical form as:

$$\left(\lambda^{-1} \hat{S} \hat{N}_l^{-1} (\lambda') (\hat{L} + \hat{G}) + \hat{I} \right) |V > = \lambda^{-1} \hat{S} \hat{N}_l^{-1} |0 >_{info} \qquad (79)$$

See also Subsec.8.2. It should be noted that to implement the expansion 72 we do not need to know explicitly the operator $\hat{N}(\lambda')$. We only need to assume that $\hat{N}(0)$ has the form 76 which allows us to compute the zero approximation $|V; \lambda >^{(0)}$ given implicitly by formula 77 and we have to know the expicit form of well defined operator $\hat{N}_l^{-1}(\lambda')$, see Sec.4 and Final Remarks.

8 Essentially nonlinear interaction (ENI)

Equations for n-pi with polynomial interactions are typically branched equations, for which the closure problem arises with different ad hock amputation methods proposed for its solutions. In such situation the ENI with no closure problems are sought which would approximate in some way the models with the closure problem, [7]. There are also physical reasons to consider such interactions, [8]. Let us now consider Eq.20 with the left invertible interaction 46:

$$\left(\hat{L} + \lambda \hat{N}_l \frac{\hat{I}}{\hat{I} - \lambda' \hat{N}_r} + \hat{G} \right) |V> = |0>_{info} \tag{80}$$

The interaction operator $\hat{N} = \hat{N}_l \frac{\hat{I}}{\hat{I} - \lambda' \hat{N}}$ appearing in the above equation can be essentially nonlinear (it means that it corresponds to essentially nonlinear interaction) for two reasons, namely that $\hat{N}_l$ would be essentially nonlinear and that $\frac{\hat{I}}{\hat{I} - \lambda' \hat{N}_r}$ is essentially nonlinear even if the operator $\hat{N}_r$ is related to a polynomial interaction. Multiplying Eq.80 by the left inverse operator 47, we get the equation

$$\left(\left(\hat{I} - \lambda' \hat{N}_r \right) \hat{N}_l^{-1} (\hat{L} + \hat{G}) + \lambda \hat{I} \right) |V> = |0>_{info} \tag{81}$$

which is conditionally equivalent to Eq.80, see Subsec.8.2.

Assuming as usually that

$$(\hat{L} + \hat{G}) \equiv (\hat{L} + \hat{G})_r \tag{82}$$

is a right invertible operator and introducing the operator

$$\hat{A}_r = \lambda' \hat{N}_r \hat{N}_l^{-1} (\hat{L} + \hat{G})_r \tag{83}$$

one can describe Eq.81 in an equivalent way:

$$\left(\hat{I} - \hat{S} \hat{A}_r^{-1} [\hat{N}_l^{-1} (\hat{L} + \hat{G})_r + \lambda \hat{I}] \right) |V> = -\hat{S} \hat{A}_r^{-1} |0>_{info} + \hat{S} \hat{P}_A |V> \tag{84}$$

where a right inverse to the operator $\hat{A}_r$ is

$$\hat{A}_r^{-1} = \lambda'^{-1} (\hat{L} + \hat{G})_r^{-1} \hat{N}_l \hat{N}_r^{-1} \tag{85}$$

and the projector $\hat{P}_A = \hat{I} - \hat{A}_r^{-1} \hat{A}_r$. In the Eq.84 we also took into account the permutation symmetry 59 of solutions.

For $\hat{N}_l = \hat{I}$ and the whole set of lower triangular operators $\hat{N}_l$, the Eq.84 is not branching equation in spite of the fact that Eq. 81 are branched equations. To fix the arbitrary projection $\hat{P}_A |V>$ one can use the symmetry condition and the perturbation principle which relies on assumption that the arbitrary projection $\hat{P}_A |V>$ is constructed by means of the generating vectors $|V>$ solving Eq.80 with $\lambda = 0$. It means that

$$\hat{S}\hat{P}_A|V> = \hat{S}\hat{P}_{\hat{L}+\hat{G}}|V>_{\lambda=0} \tag{86}$$

We also have to satisfy the positivity conditions for the n-pi which can be fulfilled by a proper choice of the projection $\hat{P}_A|V> |_{\lambda=0}$ and the appropriate choice of parameters like λ and λ'.

8.1 Another ENI

That is interaction 48:

$$\hat{N} = \hat{N}'_l(\lambda') = \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r}\hat{N}_l \tag{87}$$

For $\hat{N}_r = \hat{N}'_r\hat{N}_l^{-1}$, we can formally write:

$$\hat{N}'_l(\lambda') = \hat{N}_l + \lambda'\hat{N}'_r + \lambda'^2\hat{N}'_r\hat{N}_l^{-1}\hat{N}'_r + \cdots \tag{88}$$

If the variables entering the interactions $\hat{N}_l$ and $\hat{N}'_r$ are independent of each other, it is already the second approximation to the formula 48 in which interaction between them appears. In this case we have clear separation between ENI and polynomial interactions exemplified in the second and higher powers of the expansion 88.

8.2 A definition of the operator $\frac{\hat{I}}{\hat{I}-\lambda'\hat{N}_r}$ in the FFS

We define this operator as a righ inverse operator to the operator $(\hat{I} - \lambda'\hat{N}_r)_r$. Here subscript r has to remind us that the relevant operators are left reversible. This is much weaker assumtion then an assumption that it is a non singular operator. This means that in the adopted notation:

$$\frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r} \equiv (\hat{I} - \lambda'\hat{N}_r)_r^{-1} \equiv \hat{Y} \tag{89}$$

So to that specific operator we have to solve the equation:

$$(\hat{I} - \lambda'\hat{N}_r)_r\hat{Y} = \hat{I} \tag{90}$$

In fact this equation is like an operator version of equations for Green's functions. Multiplying this equation by a right inverse to the operator $\lambda'\hat{N}_r$, we get equivalent equation:

$$(\hat{I} - \lambda'^{-1}\hat{N}_r^{-1})\hat{Y} = \hat{P}\hat{Y} - \lambda'^{-1}\hat{N}_r^{-1} \tag{91}$$

from which we get

$$\hat{Y} \equiv \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r} = (\hat{I} - \lambda'^{-1}\hat{N}_r^{-1})^{-1}\left(\hat{P}\hat{Y} - \lambda'^{-1}\hat{N}_r^{-1}\right) \tag{92}$$

Here the projector $\hat{P} = \hat{I} - \hat{Q}$ where the projector $\hat{Q} = \hat{N}_r^{-1}\hat{N}_r$ and the projection $\hat{P}\hat{Y}$ describes an arbitrary term which has to be specified by means of additional conditions to Eq.91, see [9].

For a local theories, we should rather choose the local operators $\hat{N}_r^{-1}$. The arbitrary projection $\hat{P}\hat{Y}$ can be restricted by the positivity properties of n-pi when at least all arbitrary pairs of variables $\tilde{x}_j; j = 1, ..., n$ are equal.

One can show that for

$$\hat{P}\hat{Y} = 0 \tag{93}$$

$$\hat{Y} = \hat{Y}\hat{Q} \tag{94}$$

and in this case the natural property:

$$\hat{N}_r \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r} = \frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r}\hat{N}_r \tag{95}$$

takes place, see also [2].

PROBLEM:

It is not clear if for the choice 93 the positivity properties of n-pi are satisfied, for all equal pairs of 'points' $\tilde{x}_j; j = 1, ..., n$. In other words, this choice can be treated as the first approximation to a proper definition of the operator $\frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r}$. At this point we would like to notice that a 'proper definition' of the operator depends on the context which in our case is Eq.20. At this point we would also like to notice that in the situation of such a dramatice freedom of solutions in the FFS, the positivity condition of n-pi is a very important tool ensuring a proper relationship of equations 20 with the original equations 19.

COMMENT1

The choice 93 in the case of polynomial operators $\hat{N}_r$ leads to lower triangular operators in Eq.20 which usually do not appear in such equations, see e.g. [5].

COMMENT2

A proper definition of the ENI $\frac{\hat{I}}{\hat{I} - \lambda'\hat{N}_r}$ is also important for such a reason that with its help we can constract other operators, see Hilbert transforms and [2, 3].

COMENT3

However, the positivity condition 4 is required if the values of the field $\varphi(\tilde{x})$ are numbers. If this is not the case and values of the field are operators, then this condition need not occur. Because in both cases we are dealing with the same equations in terms of the form, we have here the same situation as in classical and quantum mechanics, where the same equations in terms of the form describe physical systems differing only in the micro and macro scale. See planetary Bohr atomic model. I would like to stress that in the case of all functions β equal to each other and describing initial and boundary conditions for Eq.19 the lack of positivity conditions for n-pi can be most likely interpreted as a lack of commutativity among values of the field $\varphi(\tilde{x})$ at differen points $\tilde{x}$. In such a case do we need to know commutation relations of the implicitly

appearing operators? Perhaps - no - if we are able to interpret n-pi not satisfying the positivity principle.

9 Overdetermined equations

An equation is overdetermined if its multiplication by the projector $\hat{Q}$ does not change its set of possible physical solutions. In the FFS it is the case when physical solutions have certain symmetries.

Let us multiply Eq.20

$$\left(\hat{L} + \lambda\hat{N}_l + \hat{G}\right)|V> = |0>_{info}$$

by a left invers operator $\hat{N}_l^{-1}$. We get

$$\left(\hat{N}_l^{-1}(\hat{L} + \hat{G}) + \lambda\hat{I}\right)|V> = \hat{N}_l^{-1}|0>_{info} = |0>_{info} \tag{96}$$

Multiplying Eq.96 by $\hat{N}_l$, we get

$$\left(\hat{Q}(\hat{L} + \hat{G}) + \lambda\hat{N}_l\right)|V> = |0>_{info} \tag{97}$$

Assuming that $\hat{N}_l = \hat{Q}\hat{N}_l$ with the projector $\hat{Q} \equiv \hat{N}_l\hat{N}_l^{-1}$ and that $|0>_{info} = \hat{Q}|0>_{info}$, we see that Eq.97 is equivalent to Eq.96 only if this equation is overdetermined in such a way that the multiplication by the projector $\hat{Q}$ does not change its set of possible physical solutions. In the FFS it is the case especially when the physical solutions have certain symmetries, for example: the permutations or Poincare symmetry.

10 Complete and closed sets of n-pi

The set of n-pi is *complete*, if we are able to derive equations for them by means of which one can construct these n-pi. A finite set of n-pi is *closed* if is also complete. Equations as 20, containing an upper triangular operator, usually produce unclosed sets of n-pi. Then we say that the closure problem arises. The closure problem is *apparent* if after appropriate transformation of a considered equation this problem disappears.

11 A construction to Eq.19 the zeroth order n-pi satisfying the positivity conditions 4

In this case Eq.19 looks as follows

$$L[\tilde{x}; \varphi] + G(\tilde{x}) = 0 \tag{98}$$

The general solution to this equation is represented in the form

$$\varphi[\tilde{x};\alpha] = \varphi_0[\tilde{x};\alpha] - \int d\tilde{u} L_r^{-1}(\tilde{x},\tilde{u})G(\tilde{u}) = \int d\tilde{u}\, P_0(\tilde{x},\tilde{u})\alpha(\tilde{u}) - \int d\tilde{u} L_r^{-1}(\tilde{x},\tilde{u})G(\tilde{u})$$
(99)

where the function P_0 is a kernel of projector projecting on the free solutions to Eq.98 ($G = 0$) and L_r^{-1} is a right inverse to the linear operator L. In fact, in the symbolical notation we have

$$P_0 = I - L_r^{-1} L$$
(100)

and

$$\varphi[\tilde{x};\alpha] = P_0[\tilde{x};\alpha] - L_r^{-1}G[\tilde{x}]$$
(101)

c.f 99. Here, all operators without Asian hat, ^ , act in the function space, which are defined on the same manifold which are defined the fields $\varphi(\tilde{x}) \equiv \varphi[\tilde{x};\alpha]$.

To generate n-pi 3 we introduce the generating functional (g.f.)

$$V[j] = \int \delta\alpha\, e^{ij\cdot\varphi[\alpha]} P[\alpha]$$
(102)

from which

$$\frac{\delta^n}{\delta j(\tilde{x}_1)\cdots\delta j(\tilde{x}_n)} V[j]_{j=0} = i^n < \varphi(\tilde{x}_1)...\varphi(\tilde{x}_n); 0 >$$
(103)

where zero in the r.h.s of the above formula means that generating n-pi are permutation symmetrical. In the exponent of formula 102 $\varphi[\alpha]\cdot j \equiv \int d\tilde{x}\varphi[\tilde{x};\alpha]j(\tilde{x})$, where $\varphi[\tilde{x};\alpha]$ is a general solution of Eq.19. In the case of Eq.98, The general solution has the form 99, so we can write the formula 102 as

$$V[j] = e^{iL_r^{-1}G\cdot j} \int \delta\alpha\, e^{iP_0\alpha\cdot j} P[\alpha]$$
(104)

with the help of which, by the n-th order functional differentiations with respect to function $j(\tilde{x})$, one can generate the permutation symmeric n-pi, for a given probability distribution $P[\alpha]$. Of course, to get a final answer we have yet to execute the functional integration $\int \delta\alpha\cdots$, which only can be done, for a Gaussian type of probability distribution $P[\alpha]$, cf. e.g. [12]. In other cases, one can use equations as Eq.20 or equations presented in the paper in which we do not need to know the general solutions to Eq.19. For the Gaussian probability density $P = exp(-1/2 \cdot \alpha^2)$ we get

$$V[j] = e^{iL_r^{-1}G\cdot j} e^{-\frac{1}{2}(P_0 j)^2}$$
(105)

It is also well known that for certain averages we do not need to know general solutions to Eq.19. This happens when the probability distribution P is related to a certain integral of motion to Eq.19, for example the energy integral $H[\alpha(t,\cdot)]$. Of course, it is not the case of theory with $G \neq 0$.

12 Schwinger equations and some surprising parallels

If the system is complicated and/or is very sensitive to small changes in initial and/or boujndary conditions, then it does not matter any detailed his description, and begin to be of interest averaged descriptions as n-pi. Then we replace Eqs 19, which provide a detailed description of the system, by Eqs 20 for averaged or smoothed quantities called n-pi in which the interactions and kinematics of the system are described by the operators in the FFS. Consider again g.f. 102, but this time with the probability distribution P given by the energy integral H:

$$V[j] = \int \delta\alpha \, e^{ij\cdot\varphi[\alpha(t^0,\cdot)]} e^{-\beta \int dt\, H[\alpha(t^0,\cdot)]} = \int \delta\alpha \, e^{ij\cdot\varphi[\alpha(t^0,\cdot)]} e^{-\beta \int dt\, H[\varphi[\alpha(t^0,\cdot)]]}$$

(106)

Here β is a dimensional constant. Denoting the invertible transformation $\alpha \to \zeta$: $\zeta(\tilde{x}) = \varphi[\tilde{x}; \alpha]$, and assuming that

$$(i) \ \delta\zeta = \delta\alpha$$

(cf. Liouville's theorem) we can write the above g.f. as follows:

$$V[j] = \int \delta\zeta e^{ij\cdot\zeta} e^{-\beta \int dt\, H[\zeta(t,\cdot)]}$$

(107)

Additionally assuming the translational invariance of measure:

$$(ii) \ \delta\zeta = \delta(\zeta + \gamma)$$

for an arbitrary function γ, we can derive the Schwinger equations, see [12], for the g.f. $V[j]$ in the case of averagings given by the probability distribution

$$P[\zeta] = e^{-\beta \int dt\, H[\zeta(t,\cdot)]}$$

(108)

As you see, the above functional integral can be constructed without using any solutions to Eq.19. It satisfies Schwinger equation which we now derive. To do this we have to do a third assumption that the functional derivative commutes with the functional integration:

$$(iii) \ \left[\int, \delta/\delta\gamma(\tilde{x})\right] = 0$$

for an arbitrary $\tilde{x}$ and $\gamma(\tilde{x}) = 0$. We start from

$$V[j] = \int \delta\zeta e^{ij\cdot(\zeta+\gamma)} e^{-\beta \int dt\, H[\zeta(t,\cdot)+\gamma(t,\cdot)]}$$

(109)

which we differentate with respect to γ at $\gamma = 0$. In result we get:

$$\int \delta\zeta \cdot \{ij(\tilde{x}) - \beta H'[\zeta; \tilde{x}]\} e^{ij\cdot\zeta} e^{-\beta \int dt\, H[\zeta(t,\cdot)]} = 0$$

(110)

with $H'[\zeta; \tilde{x}] \equiv \delta/\delta\gamma(\tilde{x}) \int dt\, H[\zeta(t,\cdot) + \gamma(t,\cdot)]_{\gamma=0}$. Using the formula for integration by parts, the last equation can be rewritten as

$$\{ij(\tilde{x}) - \beta H'[\tilde{x}; -i\delta/\delta j]\}\, V[j] = 0 \qquad (111)$$

This is the Schwinger equation, which differs from Eq.20 by the first term corresponding to the lower triangular operator, see [5]. The advantages of using this equation is based on the fact that obtained in this way n-pi satisfy the positivity conditions, which is visible from the formula 107 of the generating functional V. It is also interesting that derived in this way n-pi are multi-time. To these multi-time n-pi correspond averages with the probability distribution given by the energy integral. To construct in this way the n-pi any exact solutions to Eq.19 are not used at all. Moreover, even if assumptions $(i - iii)$ are not correct, obtained in this way n-pi can be interpretowanr as averages describing the systems in whose the total energy is not specific. Such situations could correspond to action on the systems unknown external forces changing randomly their energy.

What is interesting here is that the transition in 107 to the purely imaginary time corresponds to the transition to quantum field theory (Green's functions), in which the phenomena are described in which one member of each pair of coupled variables (position or momentum) are not specified. In some way one can say that a property of the impossibility of getting full information about every pair of conjugate variables, in quantum field theory (QFT), is expressed by introduction of the imaginary time. See also [13], where the imaginary time is also interpreted as an impossibility of happining certain events in QM. See also [6]. However, the introduction of imaginary time leads to the appearance of the Fresnel types integrals, that require appropriate modifications of integrals, see [12].

In QFT, the higher n-pi (Green's functions) are related to a creation and annihilation of more particles in the scattering experiments. In classical mechanics or field theory, the higher n-pi are related to extra information about the system. Is it possible that information also had a dual nature?

To describe the Schwinger Eq.111 in the FFS, see, e.g., [5](?).

13   Remarks

Let us also notice that in the free Fock space (FFS) even in the simple mathematical model with an essentially nonlinear (nonpolynomial) interaction (ENI), Eq.39, which is represented by the left invertible operator, successive approximations to it, given by formulas such as Eq.35, are described by right invertible operators. In fact only the zeroth approximation describing external fields is a lower triangular operator, see Eq.23. Electromagnetic, strong, and weak interactions are polynomial type. It is not out of the question that the essentially nonlinear left invertible operator $\hat{N}$, which generates in the expansion with respect to the parameter λ' the right invertible operators, is the prototype of an

operator describing theory of everything containing in itself also the essentially nonlinear gravitation theory?

We would like to underline once again the importance of the auxiliary parameter λ' whose introduction makes possible to execute the perturbation calculations **without dramatice change of the whole theory at the zeroth order approximation**: "no matter, or no space-time assumptions", see Secs 5 and 6. In fact we do such change only for the appropriate projection, see 86. The introduction of this parameter via ENI allows for multiscale description of the system in which these basic concepts do not disapear at any scale. Furthermore, since in some cases the introduction of essentially nonlinear interaction (ENI) leads to close branching equations, **it is a rare instance when it seems that the complexity issue is preferred**. See also [10]where it is shown that very complicated Lagrangians with ENI lead to closed equations.

In fact, we think that this is a **relict** of the period, when the stable solutions of nonlinear equations 19 were the main instrument for learning about Nature. At presence, when the focus of the study is shifted to the equations with unstable solutions which are described by the **linear equations** discussed inter alia in the submitted paper, the precise or approximate solutions should be constructed on the basis of an one-side invertability of the operators and their corresponding definitions in the FFS, which occur at these equations. Of course, we can not forget about an influence of a concrete choice of averaging process on the n-pi. When we claiming that the polynomial approximation for ENI is possible, it means that the field $|\varphi|$ can not to be too large which limits the choice of the functional $P[\gamma]$ in the formula 7.

I believe that the introduction of general n-pi 7 allows for more complete extraction of the physical content of the theory, but also allows a new look at such issues as the perturbation theory, the choice of additional conditions or definitions of operator-valued functions. This belief stems from the fact that every general n-pi depends only on n independent β functions, see 7, and this leads to significant restrictictions for the structure of equations and solutions considered. From this point of view, for example, the definition (92,93) of the operator $\frac{\hat{I}}{\hat{I}-\lambda'\hat{N}_r}$ seems to be appropriate. On the other hand, it requires reflection the fact that the common source of this or that definition is simple function $\frac{1}{1-x}$. Could it was a one more sign of reflection of the simple fundamental laws underlying the Universe? The fact that for a finite x the function $\frac{1}{1-x}$ is singular may means that something peculiar is happening in a theory with such singularity as, e.g., the creation of particles or other things.

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