The Principle of Least Action

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In classical mechanics, this article obtains the principle of least action for a single particle in a didactic and simple way.

Introduction

Let us consider the following tautological equation for a single particle:

$$\frac{d(\mathbf{v} \cdot \delta \mathbf{r})}{dt} = \delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r}$$

Now, integrating with respect to time from t_1 to t_2, yields:

$$\int_{t_1}^{t_2} \left[\frac{d(\mathbf{v} \cdot \delta \mathbf{r})}{dt} \right] dt = \int_{t_1}^{t_2} \left[\delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r} \right] dt$$

The left side of the equation is zero, therefore, we obtain:

$$0 = \int_{t_1}^{t_2} \left[\delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r} \right] dt$$

In classical mechanics, this last tautological equation is the mathematical basis of the principle of least action for a single particle.

Now, multiplying by m (mass of the particle) the following tautological equation is obtained:

$$0 = \int_{t_1}^{t_2} \left[\delta \frac{1}{2} m \left(\mathbf{v} \cdot \mathbf{v}\right) + m\, \mathbf{a} \cdot \delta \mathbf{r}\right] dt$$

Substituting $\mathbf{a} = \mathbf{F}/m$ (Newton's second law) the following empirical equation is obtained:

$$0 = \int_{t_1}^{t_2} \left[\delta \frac{1}{2} m \left(\mathbf{v} \cdot \mathbf{v}\right) + \mathbf{F} \cdot \delta \mathbf{r}\right] dt$$

If the particle is only subject to conservative forces then $\delta V = -\mathbf{F} \cdot \delta \mathbf{r}$ and since $T = \frac{1}{2} m \left(\mathbf{v} \cdot \mathbf{v}\right)$ yields:

$$0 = \int_{t_1}^{t_2} \left[\, \delta T - \delta V \,\right] dt$$

That is:

$$0 = \delta \int_{t_1}^{t_2} \left[\, T - V \,\right] dt$$

Or else:

$$\delta \int_{t_1}^{t_2} \left[\, T - V \,\right] dt = 0$$

Finally we obtain:

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

Since $L = T - V$.

Annex

$$\frac{d}{dt}(\mathbf{v}\cdot\delta\mathbf{r}) = \ldots$$

$$\frac{d}{dt}(m\,\mathbf{v}\cdot\delta\mathbf{r}) = \ldots$$

$$\sum_i \frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\delta\mathbf{r}_i) = \ldots$$

$$\sum_{i,j} \frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j) = \ldots$$

$$\sum_{i,j} \frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j) = \sum_{i,j} m_i\,\mathbf{v}_i\cdot\frac{d}{dt}(\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j) + \sum_{i,j} m_i\,\mathbf{a}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j$$

$$\sum_{i,j} \frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j) - \sum_{i,j} m_i\,\mathbf{v}_i\cdot\frac{d}{dt}(\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j) = \sum_{i,j} m_i\,\mathbf{a}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j$$

$$\sum_{i,j} \left[\frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}) - m_i\,\mathbf{v}_i\cdot\frac{d}{dt}(\frac{\partial\mathbf{r}_i}{\partial q_j})\right]\delta q_j = \sum_{i,j} m_i\,\mathbf{a}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j$$

$$\sum_{i,j} \left[\frac{d}{dt}(m_i\,\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}) - m_i\,\mathbf{v}_i\cdot\frac{d}{dt}(\frac{\partial\mathbf{r}_i}{\partial q_j})\right]\delta q_j = \sum_{i,j} \mathbf{F}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\ \delta q_j$$

Bibliography

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