A merge of the Rideout-Sorkin growth process with Quantum Field Theory on causal sets.

Johan Noldus[*]

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Abstract

I raise some issues when one combines the dynamical causal sequential growth dynamics with the static approach towards quantum field theory. A proper understanding of these points is mandatory before one attempts to unite both approaches. The conclusions we draw however appear to transcend causal set theory and apply to any theory of spacetime and matter which involves topology change.

1 Introduction.

The classical problem of time in general relativity is that there is no dynamics in history space; hence the theory can be conceived as a topological space where every point represents a *spacetime* whereas for ordinary (gauge) theories one disposes of a phase space which correspond to the configurations on space at a given time. The dynamics then provides a trajectory into that space; for relativity this trajectory is a point (equivalence class under the constraints). Moreover, there is in general no procedure by which one can retrieve a *physical* notion of space and time; motivated by this Sorkin and Rideout developed a new dynamics for causal sets which is generally covariant in a *weaker* sense than relativity is allowing for time to flow and space to exist. Both authors appear to hope that general relativity will come out of their procedure but this author does not agree for the following reasons: (a) the symmetry group of Rideout and Sorkin is dynamical whereas the one of Einstein is static and (b) the dynamics is inherently stochastic whereas Einstein's theory is deterministic and one cannot speak about an average universe with stochastic fluctuations in an invariant way. In order for the reader to appreciate my comments, let me expose some of the details of their reasoning [10]; the dynamics is defined as a Markovian process where an n-element causal set grows to an $n+1$-element causal set by adding one maximal element. That is, one adds one element as well as new causal relations to the existing elements such that the new element is not to the past of some other one. The newborn element gets label $n+1$ and as such the labeling of elements in the resulting causal set by means of the growth process is a *natural* one meaning that an element to the future of another has necessarily a larger label. In Einstein's theory of relativity, all

[*]email: johan.noldus@gmail.com

labelings are a priori allowed for since a natural labelling depends already on the solution of the theory. Rideout's principle of general covariance is that the probability for a particular n-element causal set to exist does not depend upon the path according to which it has been growing; this is obviously weaker than the statement of general covariance in relativity. There is a third demand, apart from temporality and general covariance, which is the so called Bell causality condition. This one states the following: take an n-element causal set C and let it grow to either C_1 or C_2, both $n+1$-element causal sets, by adjoining a new element in both cases. Let B_i denote the past of the newborn element (including the new element) in C_i and $B = (B_1 \cup B_2) \cap C$; then

$$\frac{P(C \to C_1)}{P(C \to C_2)} = \frac{P(B \to B'_1)}{P(B \to B'_2)}$$

where P denotes the probability of the transition and $B'_i = (B_1 \cup B_2) \cap C_i$. These three demands have interesting consequences in the sense that the dynamics is fully specified by one new parameter at every growth stage; obviously, the demand of Bell causality does not hold in the real world and we shall see in this paper that there is indeed something very non-causal going on. Before we come to that however, let me briefly review the appraoch towards quantum field theory originally developed by Johnston for scalar fields [1, 2] and later by this author for Fermi fields [7]. We will concentrate on scalar fields since this is easier to present, does not contain additional physical difficulties inherent in the Fermi theory and still contains the very difficulty this paper is about. Here, we take a *static* n-element causal set C and construct a causal set analogon of the Pauli-Jordan function $\Delta(x, y)$ where x, y are elements in C. We shall explain first how this Pauli-Jordan function is constructed: an n-element causal set is a set of n elements with a partial order $\prec$ defined on it. A partial order is irreflexive, asymmetric and transitive; since the causal set is finite, one can define a relation $\prec\star$ where $x \prec \star y$ if and only if there exists no z different from x, y such that $x \prec z \prec y$. $\prec\star$ defines an incidence matrix I where $I(x, y) = 1$ if $x \prec \star y$ and zero otherwise; in a natural labeling for the causal set, one has that I is upper triangular with all zeroes on the diagonal. From I, one can in general define the advanced Green's kernel as

$$A = \sum_{n>0} (ab)^{n-1} aI = aI(1 - abI)^{-1}$$

where a, b are real coeffients such that b depends upon the inertial mass m of the field as well as a spacetime (sprinkling) density ρ and a just depends upon that density. Hence, in the massless case $A = aI$; the Pauli-Jordan function is then defined as

$$\Delta = i(A - A^T)$$

where T denotes the transpose. This matrix is Hermitian and complex in the sense that its complex conjugate equals $-\Delta$. Let $i\lambda$ be an eigenvalue of $A - A^T$ with eigenvector v_λ; then, $\overline{v_\lambda}$ is an eigenvector of $A - A^T$ with eigenvalue $-i\lambda$. Hence, eigenvalues of Δ come in pairs $\pm\lambda$ with eigenvectors $v_\lambda, \overline{v_\lambda}$ respectively where we may assume that $\lambda > 0$. The only exception to this rule is the eigenvalue zero, where eigenvectors may be real; hence

$$\Delta = \sum_{\lambda>0} \lambda (v_\lambda v_\lambda^\dagger - \overline{v}_\lambda v_\lambda^T)$$

where † denotes the Hermitian conjugate. Real scalar quantum field theory on C is now defined as follows: one looks for an operator valued field $\phi(x)$ such that

$$\begin{aligned} \phi^{\dagger}(x) &= \phi(x) && (1) \\ [\phi(x), \phi(y)] &= \Delta(x,y) && (2) \\ \Delta v &= 0 \Rightarrow \sum_{x \in C} v(x)\phi(x) = 0. && (3) \end{aligned}$$

The third condition, together with the second one, is a substitute for the equations of motion and has been discussed in [2]. The second condition is usually called the causality condition albeit it contains most dynamical information about the quantum field as well. Given the spectral decompostion of Δ, the first, second and third condition imply that

$$\phi(x) = \sum_{\lambda > 0} \sqrt{\lambda} \left(a_{\lambda} v_{\lambda}(x) + a^{\dagger}_{\lambda} \overline{v_{\lambda}}(x) \right)$$

where $[a_{\lambda}, a_{\mu}] = 0$ and $\left[a_{\lambda}, a^{\dagger}_{\mu}\right] = \delta_{\lambda\mu}$. One defines now a natural Fock representation by means of a cyclic vacuum state $|0\rangle$ satisfying $a_{\lambda}|0\rangle = 0$. In the following section, we give an expose of the tension between both views explained in this introduction.

2 The tension between the static and dynamic worldview.

Let us now pose the question what happens in causal set theory to quantum field theory if we grow the n-element causet C to an $n+1$-element causet C' by means of the Rideout-Sorkin growth process. For a transition n even to $n+1$ odd, the number of independent oscillators does in general remain the same, but for a process n odd to $n+1$ even, the number of oscillators grows in general with one. The dimension of the Hilbert space is in both cases $\aleph_0$ unless there are no oscillators at all such as is the case for any causal set which is an antichain (meaning no element is related to another); in case of Fermions, the dimension of the Hilbert space varies as the number of oscillators increases. This begs the question of how to embed the fields $\phi_C(x)$ on $\mathcal{H}_C$ as linear operators on $\mathcal{H}_{C'}$; the canonical mapping is of course $\phi_C(x) \Rightarrow \phi_{C'}(x)$, this one preserves conditions one and two but not condition three. Indeed suppose that $\Delta_C v = 0$, then with $v(n+1) = 0$ one does not necessarily have that $\Delta_{C'} v = 0$ meaning that $\sum_{x \in C} v(x)\phi_{C'}(x) \neq 0$. Hence, the canonical mapping is not linear albeit it preserves causality[1]. Also, this mapping of fields is *not* causal in the sense that $\phi_{C'}(x)$ only depends upon the $\phi_C(y)$ where y is in the past of (or coincides with) x and the dependency is analytic in nature. This is most easily seen by growing a one element causal set C_1 to the two causal element set C_2 with incidence matrix

$$I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[1]This is obvious since by construction $\Delta_C(x,y) = \Delta_{C'}(x,y)$ for all $x, y \in C$.

Here, the massless Pauli-Jordan function is given by

$$\Delta_{C'} = a \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and therefore

$$\phi_{C_2}(1) = \sqrt{\frac{a}{2}} \left(a_1 + a_1^\dagger \right)$$

which cannot in any way be written as an analytic function of 0. Since, the canonical mapping is not linear, one wonders how states in $\mathcal{H}_C$ correspond to states in $\mathcal{H}_{C'}$. For n even, the general eigenvalues of the Pauli-Jordan matrix are of the form $\pm\lambda_i$ with

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{\frac{n}{2}}$$

while for $n+1$ odd one has to supply this series with $\lambda_{n+1} = 0$; this suggests in general the rule $a^\dagger_{i,C} \Rightarrow a^\dagger_{i,C'}$ and $|0\rangle_C \Rightarrow |0\rangle_{C'}$ at the level of states where there is no loss in dimension from n even to $n+1$ odd while for n odd to $n+1$ even one new oscillator appears. This *linear* mapping is by no means unique and it is because local operators are not diagonal in the particle basis that the new modes get activated by the act of measurement. Here, a measurement at x happens when x is born, otherwise it cannot occur; this is how multiple modes can get "born" if spacetime grows. This mapping of states of course induces a mapping of the field operators $\phi(x) \rightarrow \phi'(x)$ but it impossible to construct a field operator $\phi'(n+1)$ in this way such that $[\phi'(x), \phi'(n+1)] = \Delta_{C'}(x, n+1)$ is still satisfied; it is this fundamental difficulty which leads to the following problem. Let us ask the fundamental question if *causality* (in the standard sense) really holds in the dynamical sense; that is, suppose that at growth stage n a measurement occurs for the first time at x and at stage $n' > n$ a second measurement occurs at y spacelike to x - the initial state of course is the vacuum state since the Hilbert space for a one element causal set is one dimensional since there are no oscillator modes. Is this process the same in some sense[2] as a first measurement at y at growth stage m and later on a measurement at x at growth stage $m' > m$ where it is of course the field $\phi_{C_m}(y) \ldots$ which is being measured. The answer is *no* given the above rules by which states are mapped into higher Hilbert spaces and we shall illustrate this by means of the following two processes $C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_4$ and $C_1 \rightarrow C_2 \rightarrow C_3' \rightarrow C_4$. C_2 is given by the incidence matrix

$$I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

C_3 by

$$I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

C_3' by

$$I_3' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[2]The final states possibly live in different Hilbert spaces, therefore one needs to identify them in some way.

and finally C_4 by

$$I_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reader can easily calculate their massless Pauli-Jordan kernels. Here, the final causal sets are identical so there is no problem in identifying the final states; they simply ought to be the same. We are interested in computing the embedding (according to the *linear* maps) of $\phi_{C_2}(x), \phi_{C_3}(y), \phi_{C_4}(x)$ and finally $\phi_{C_4}(y)$ in $\mathcal{H}_{C_4}$, which we will denote with primes. Here x and y represent the two maximal points in C_4; the result is the following:

$$\begin{aligned}
\phi'_{C_2}(x) &= -i\sqrt{\frac{a}{2}}\left(a_1 - a_1^\dagger\right) \\
\phi'_{C_3}(y) &= -\frac{\sqrt{\sqrt{2}a}}{2}\left(a_1 + a_1^\dagger\right) \\
\phi_{C_4}(x) &= \frac{\sqrt{\sqrt{\frac{3-\sqrt{5}}{2}}a}}{2\sqrt{10-4\sqrt{5}}}\sqrt{6-2\sqrt{5}}\left(a_1 + a_1^\dagger\right) + \frac{\sqrt{\sqrt{\frac{3+\sqrt{5}}{2}}a}}{2\sqrt{10+4\sqrt{5}}}\sqrt{6+2\sqrt{5}}\left(a_2 + a_2^\dagger\right) \\
\phi_{C_4}(y) &= -i\frac{\sqrt{\sqrt{\frac{3-\sqrt{5}}{2}}a}}{2\sqrt{10-4\sqrt{5}}}\left(1-\sqrt{5}\right)\left(a_1 - a_1^\dagger\right) - i\frac{\sqrt{\sqrt{\frac{3+\sqrt{5}}{2}}a}}{2\sqrt{10+4\sqrt{5}}}\left(1+\sqrt{5}\right)\left(a_2 - a_2^\dagger\right)
\end{aligned}$$

as the reader may wish to verify. One immediately notices that $\phi'_{C_2}(x)$ commutes with $\phi_{C_4}(y)$ but not with $\phi_{C_4}(x)$ and likewise $\phi'_{C_3}(y)$ commutes with $\phi_{C_4}(x)$ but not with $\phi_{C_4}(y)$. Obviously, the spectrum of all four operators constitutes the entire real line and in particular 0 belongs the continuous spectrum; hence, for example, the projection operators $\pi_{0,2}(x)$ and $\pi_{0,4}(y)$ on the zero eigenspace are not well defined and the common improper eigenvector $|\psi\rangle_0$ of $\phi'_{C_2}(x)$ and $\phi_{C_4}(y)$ (corresponding to the eigenvalue zero) is given by

$$|\psi\rangle_0 \sim \sum_{n,m=0}^{\infty} \frac{1}{2^{n+m}n!m!}\left(a_1^\dagger\right)^{2n}\left(a_2^\dagger\right)^{2m}|0\rangle$$

where normalization is impossible since the norm of the right hand side equals infinity. It is amusing to notice that for $\phi_{C_4}(x), \phi'_{C_3}(y)$ the common improper eigenvector $|\psi'\rangle_0$ corresponding to the eigenvalue zero is given by

$$|\psi'\rangle_0 \sim \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}}{2^{n+m}n!m!}\left(a_1^\dagger\right)^{2n}\left(a_2^\dagger\right)^{2m}|0\rangle$$

so that the projection of the vacuum state $|0\rangle$ on $|\psi\rangle_0$ and $|\psi'\rangle_0$ is not identical (albeit the coefficients $\langle 0|\psi\rangle_0$ and $\langle 0|\psi'\rangle_0$ are identical.). Therefore, it is clear that making a measurement at x between the values $(-\epsilon, \epsilon)$ and at y between the values $(-\delta, \delta)$ for $\epsilon, \delta > 0$ small enough, on the vacuum state is going to give a different answer. This proves our assertion that causality is not preserved; the reader might think we just have chosen a bad mapping, however, the point is that there does noit exist any linear mapping which preserves causality. Hence, our conclusion is a generic one.

Physically, this result resembles the situation in semiclassical gravity where a measurement changes the future of spacetime and therefore influences the evolution of operators even if they were to be spacelike to one and another. As long as no measurement occurs here and one considers the non-linear embeddings of the operator algebra (restricted to the self-adjoint and causality condition) one would also say causality is preserved. The reason why the third condition does not hold under this mapping is because on causal sets the d'Alembertian is a non-local operator and appearantly *our* hidden definition[3] of this operator also involves the future and not only the past. There exist proposals for the d'Alembertian operator on causal sets which are only retarded in nature and quantum field theories based upon such proposition would not suffer from this drawback. However, it is easily seen that an exclusive retarded procedure would lead one into conflict with the "causality condition" in the static sense; consider for example the causal set given by the elements x, y, v, w, z where $x \prec \star v$, $x \prec \star w$, $y \prec \star v$, $y \prec \star w$ and finally $v \prec \star z$; moreover, consider as "initial conditions" $\phi(x), \phi(y), \phi(v), \phi(w)$, such that the "causality condition" is satisfied, supplemented with the equation $\left(\Box + m^2\right)\phi(z) = 0$. Then, the latter must be of the form

$$\phi(z) = \alpha\phi(v) + \beta(\phi(x) + \phi(y))$$

with α, β nonzero real numbers. But then,

$$[\phi(z), \phi(w)] = 2\beta\,[\phi(x), \phi(w)] \neq 0$$

which we needed to prove. This is in sharp contrast to the continuum where "quantum causality" and in particular causality is automatically satisfied for suitable initial conditions, and where the operator *is* causal in the sense that it only depends upon the initial values in the past of some future event.

3 What now?

The question now is, how should we proceed? The main requirement of the static worldview was condition number two, which we called "quantum causality", and a seemingly innocuous third axiom which served to eliminate some unphysical operators which commute with everything and hence provide supplementary quantum numbers if the representation is not irreducible. The result we obtained here simply is that both conditions are incompatible with a growing spacetime if measurements actually occur, at least this is so in the discrete setting of causal sets. As mentioned before, this result also holds on the continuum semi-classically and nobody knows how to make sense of causality in quantum gravity: as the failure of perturbative quantum gravity has shown, the notion of causality of a background spacetime (in this case Minkowski) simply doesn't work either. The correct conclusion however is to accept that the "quantum dynamics" of a field cannot correspond to the local dynamics of a quantum field

[3]Hidden, because we never made such definition explicit; the d'Alembertian is already present in the Pauli-Jordan kernel.

theory[4] coupled to gravity (at least not for quantum field theories which are of second order in time in the standard Fock space representation). This is a major conclusion and *suggests* that the correct dynamics for ϕ is a non-local one: speaking in the ordinary language of quantum field theory, this implies that one has to include all non-renormalizable interactions. Indeed, for scalar field theory one obtains that every higher derivative term comes with a coupling constant of negative mass dimension so that renormalization theory implies that they all need to be included in the action if one is. This was already accepted to be the case for the gravitational field itself, but it appears to be mandatory for any field: we will provide more evidence of this later on. Another possibility, to which we shall return now, is that there is nothing wrong with the dynamics but that the fault lies in the standard Fock space representation: that is, condition number three needs to be dropped.

One might indeed guess that the most innocent way to solve this problem would be to drop the third condition; in that case the quantum field can be written as

$$\phi_C(x) = \sum_{\lambda>0} \sqrt{\lambda}\left(v_\lambda(x)a_\lambda + \overline{v_\lambda}(x)a_\lambda^\dagger\right) + \sum_i v_i(x)a_i$$

where all v_i are real eigenvectors corresponding to the eigenvalue 0, $a_i^\dagger = a_i$ and they commute with every operator. So, we do not impose the condition anymore that the a_i vanish; let an n-element causal set C grow to an $n+1$-element causal set C' and define for $\lambda > 0$ in the spectrum of Δ_C the operators

$$a_\lambda'^C = \frac{1}{\sqrt{\lambda}} \sum_{x\in C} \overline{v_\lambda^C}(x)\phi_{C'}(x)$$

and

$$a_i'^C = \sum_{x\in C} v_i^C(x)\phi_{C'}(x)$$

in case Δ_C has eigenvalue zero. Then, one can verify these satisfy the standard commutation algebra $\left[a_\lambda'^C, a_\mu'^C\right] = 0$ and $\left[a_\lambda'^C, \left(a_\mu'^C\right)^\dagger\right] = \delta_{\lambda\mu}$ and the $a_i'^C$ commute with everything. Hence,

$$\phi_C(x) = \sum_{\lambda>0} \sqrt{\lambda}\left(v_\lambda^C(x)a_\lambda'^C + \overline{v_\lambda^C}(x)\left(a_\lambda'^C\right)^\dagger\right) + \sum_i v_i^C(x)a_i'^C$$

and we have obtained a generalization of a linear Bogoliubov transformation from the creation and annihilation operators on C into the creation and annihilation operators and commuting operators on C'. Specifically,

$$a_\lambda'^C = \sum_{0<\mu\in\sigma(\Delta_{C'})} \sqrt{\frac{\mu}{\lambda}}\left(\left(\sum_{x\in C} \overline{v_\lambda^C}(x)v_\mu^{C'}(x)\right)a_\mu^{C'} + \left(\sum_{x\in C}\overline{v_\lambda^C}(x)\overline{v_\mu^{C'}}(x)\right)\left(a_\mu^{C'}\right)^\dagger\right) +$$

[4] Albeit we have proven this only for the free field theory, our result can be generalized to the interacting case by means of a perturbative expansion in the interaction picture generalized to causal sets :

$$\phi_H(\vec{x},t) = e^{iHt}e^{-iH_0t}\phi_F(\vec{x},t)e^{iH_0t}e^{-iHt}$$

where ϕ_F denotes the free quantum field. As is well known $e^{iHt}e^{-iH_0t}$ can be expanded in terms of the free field between times 0 and t, where on finite causal sets one uses a unique notion of time.

$$\sum_i \frac{1}{\sqrt{\lambda}} \left(\sum_{x \in C} \overline{v^C_\lambda}(x) v^{C'}_i(x) \right) a^{C'}_i$$

and likewise so for the a'^C_i if they are present. In case a'^C_i exists, then this transformation shows that the spectrum of the transformed a'^C_i is continuous. In order for our Bogoliubov transformation to define a homomorphism (which then maps the identity operator on $\mathcal{H}_C$ to the identity operator on $\mathcal{H}_{C'}$) implying that the spectrum of the mapped operator is a subset of the spectrum of original one, it must be so that an infinite number of quantum numbers are attributed to standard bosonic particles something which is not observed in nature. If it were a homomorphism, then it might be possible to define a unitary mapping $U_{CC'} : \mathcal{H}_C \to \mathcal{H}_{C'}$, $U^\dagger_{CC'} U_{CC'} = 1_{\mathcal{H}_C}$ and $U_{CC'} U^\dagger_{CC'} = 1_{\mathcal{H}_{C'}}$, such that $U_{CC'} a^C_\lambda U^\dagger_{CC'} = a'^C_\lambda$ and $U_{CC'} a^C_i U^\dagger_{CC'} = a'^C_i$. Suppose this can be done at all stages, then the reader notices that for relativistic causality to hold it is necessary and sufficient that

$$U_{C_{n-1}C_n} \ldots U_{C_2 C3} U_{C_1 C_2}$$

only depends upon C_1 and C_n and not upon the particular growth process: this is how a condition of "general covariance" is intertwined with causality. Indeed, suppose one has two growth processes from C_1 to C_n and that at two intermediate stages of these processes a measurement is made at x and y which are spacelike related to one and another (it doesn't matter if x and y appear in the same order or not). Then, starting out from a state $|\psi\rangle$ in $\mathcal{H}_{C_1}$ one obtains that the relevant sequence of operators is in both cases of the form

$$U_{C_{n-1}C_n} \ldots U_{C_k C_{k+1}} \phi_{C_k}(z_1) U_{C_{k-1}C_k} \ldots U_{C_l C_{l+1}} \phi_{C_l}(z_2) U_{C_{l-1}C_l} \ldots U_{C_1 C_2}$$

which equals

$$\phi_{C_n}(z_1) \phi_{C_n}(z_2) U_{C_{n-1}C_n} \ldots U_{C_1 C_2}$$

where $z_1 \neq z_2 \in \{x, y\}$; this proves then causality, since it holds on C_n. We shall now study if it is possible to find such operators satisfying all these conditions; let us begin by looking at a simple example explained at the bottom of page three where the one element causal set C_1 grows to a two element causal set C_2 given by the incidence matrix

$$I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, the massless quantum field on C_1 is given by $\phi_{C_1}(1) = a^{C_1}$ where a^{C_1} is self-adjoint and

$$\phi_{C_2} = \sqrt{\frac{a}{2}} \left(\begin{pmatrix} 1 \\ -i \end{pmatrix} a^{C_2} + \begin{pmatrix} 1 \\ i \end{pmatrix} \left(a^{C_2}\right)^\dagger \right).$$

Now, we must find Hilbert spaces $\mathcal{H}_{C_1}$ and $\mathcal{H}_{C_2}$ and a unitary mapping $U_{C_1C_2}$ such that $U_{C_1C_2} a^{C_1} U^\dagger_{C_1C_2} = \sqrt{\frac{a}{2}} \left(a^{C_2} + \left(a^{C_2}\right)^\dagger \right)$. So, unlike what is usually thought, we come to the physical worldview that the Hilbert space must be static and contain all future possibilities, while it is the operator algebra which is dynamic. This might not be surprising since the space of possibilities of the causal growth process is static also, it is given by all finite causal sets. The reader notices that within the operator algebra generated by a^{C_2} and $(a^{C_2})^\dagger$, one can find exactly one Heisenberg conjugate to $a'^{C_1} = \sqrt{\frac{a}{2}}\left(a^{C_2} + \left(a^{C_2}\right)^\dagger\right)$ which is given by

$$b'^{C_1} = -i\sqrt{\frac{1}{2a}}\left(a^{C_2} - \left(a^{C_2}\right)^\dagger\right)$$

and indeed

$$\left[a'^{C_1}, b'^{C_1}\right] = i1_{\mathcal{H}_{C_2}}.$$

This shows that the spectrum of a'^{C_1} is continuous (and more in particular the entire real line) and that exactly one generalized eigenstate can be found for any eigenvalue *if* the representation of the operator algebra on $\mathcal{H}_{C_2}$ is irreducible: we will shortly arrive to the conclusion that this cannot be. More in general, assume that the vacuum state $|0,\alpha\rangle_{\mathcal{H}_{C_2}}$ carries other quantum numbers given by α, then the Hilbert space $\mathcal{H}_{C_2}$ is spanned by the improper eigenstates $|\lambda,\alpha\rangle_{\mathcal{H}_{C_2}}$ where λ belongs to the spectrum of a'^{C_1}. Hence, we have a natural identification of $|\lambda,\alpha\rangle_{\mathcal{H}_{C_2}}$ with the improper vacuum states $|0,\lambda,\alpha\rangle_{\mathcal{H}_{C_1}}$ where one has that

$$a^{C_1}|0,\lambda,\alpha\rangle_{\mathcal{H}_{C_1}} = \lambda|0,\lambda,\alpha\rangle_{\mathcal{H}_{C_1}}.$$

Therefore, at every level of the construction do we have an infinite degenerate vacuum and every two growth steps obviously increase α by one parameter. This, however, does not imply that we must consider the situation with an infinite number of particles present, something which has been avoided in the usual Fock space representations of quantum field theory. Indeed, those spaces carry an arbitrary number of particles, but never an infinite number of them: this increases the dimension of the Hilbert space to $\aleph_1$ instead of $\aleph_0$. However, one should notice that there does not exist a nondegenerate vacuum for an infinite causal set C_∞; indeed growing C_∞ to $C_{\infty+2}$ by adding a disjoint C_2 leads to one new creation-annihilation pair which commutes with the rest of the algebra adding another quantum number (which takes a countably infinite number of values) to the vacuum state. The distinction with the continuum is that there one can speak about a unique maximal development [13] as being the unique (up to a diffeomorphism) largest globally hyperbolic development of initial data on a hypersurface of fixed topology which is fine since the dynamics can be phrased such that it doesn't extend the initial data as well as the initial surface itself. For paracompact and *connected* spacetimes one gan even go further and speak about inextendible spacetimes [13], those are spacetimes which cannot be isometrically embedded into a larger spacetime (the connectedness assumption is crucial here). Moreover, every connected and paracompact spacetime can be embedded into an inextendible one by means of Zorn's lemma[5]. For example, Minkowski spacetime is inextendible in this sense and hence a maximal element in the class of connected, paracompact spacetimes. In causal set theory however, spatial topology change can always occur, at any stage, and it is this feature which does not allow one to speak about a unique maximal (in the sense that every past finite causal set can embedded into it) past finite causal set in the sense that one cannot speak about an order preserving bijection between them: indeed, suppose that there exists such maximal causal set C; in case C has a minimal point with a finite number of future relations, then

[5]Such inextendible extension however is by no means unique.

we can obviously embed C into C' (and by definition C' is then maximal too) where no minimal element in C' has a finite number of future relations. Since C is maximal, we can embed C' into C but clearly no such embedding can be bijection. So suppose that C has the property that every minimal element has an infinite number of future relations, then growing C to C' where the newborn element is spacelike to C, we again obtain a maximal causal set in the sense that there exists no causality preserving bijection between C and C'. Due to this topology changing character, we cannot appeal to Zorn's lemma to prove the existence of a maximal element: indeed, as shown before it might be that $C \leq C'$ and $C' \leq C$ but $C \neq C'$ where the relation $\leq$ means that the former can be embedded into the latter and equality means that there exists an order preserving bijection between them. This is even the case when one considers only embeddings which map minimal elements to minimal elements; hence, as far as I know, the existence problem of a maximal past finite causal set is an open one. One notices that the unitary mapping $U_{C_1 C_2}$ is not unique at all in the sense that one might consider transformations of the labels α; for example, it might very well be that

$$U_{C_1 C_2} |0, \lambda, \beta\rangle_{\mathcal{H}_{C_1}} = |\lambda, \alpha\rangle_{\mathcal{H}_{C_2}}$$

which is not very encouraging in the sense that one would like to obtain a unique theory. As before, the unitary mapping maps vacua to multiparticle states and this provides a mechanism by which matter is born out of nothing.

It might be that our notion of general covariance for the unitary operators brings a solution to this problem, but computations are bound to be complicated as it is no easy matter to construct even specific unitary operators. We will show explicitely which difficulties arise by commenting upon the two processes at the bottom of page four, where this time we start at C_2; for the process $C_2 \to C_3$ one obtains after some calculation (for massless free fields) that

$$a'^{C_2} = \frac{1}{2\sqrt{\sqrt{2}}} \left((1+\sqrt{2}) a_1^{C_3} + (1-\sqrt{2}) \left(a_1^{C_3} \right)^{\dagger} \right) + \frac{1}{2\sqrt{a}} a^{C_3}$$

and it is clear that the algebra generated by $a'^{C_2}, \left(a'^{C_2}\right)^{\dagger}, a^{C_3}$ is identical to the one generated by $a_1^{C_3}, \left(a_1^{C_3}\right)^{\dagger}, a^{C_3}$. To construct $U_{C_2 C_3}$ we need to study the spectral decomposition of a^{C_3} in the kernel of a'^{C_2}: for every vacuum degeneracy parameter α, there is exactly one generalized eigenvector corresponding to an eigenvalue λ. That is, we consider the states $|\lambda, \alpha\rangle_{\mathcal{H}_{C_3}}$ satisfying

$$a'^{C_2} |\lambda, \alpha\rangle_{\mathcal{H}_{C_3}} = 0$$

and

$$a^{C_3} |\lambda, \alpha\rangle_{\mathcal{H}_{C_3}} = \lambda |\lambda, \alpha\rangle_{\mathcal{H}_{C_3}}.$$

This suggests that we identify the latter with the vacuum states $|0, \lambda, \alpha\rangle_{\mathcal{H}_{C_2}}$ satisfying

$$a^{C_2} |0, \lambda, \alpha\rangle_{\mathcal{H}_{C_2}} = 0.$$

Hence, our unitary mapping is given by

$$U_{C_2 C_3} |0, \lambda, \alpha\rangle_{\mathcal{H}_{C_2}} = |\lambda, \alpha\rangle_{\mathcal{H}_{C_3}},$$

the problem now is that we have to calculate $|\lambda,\alpha\rangle_{\mathcal{H}_{C_3}}$ in terms of the natural basis generated by the standard creation operator $\left(a_1^{C_3}\right)^\dagger$ and a^{C_3}. Indeed, as the reader may verify, this is the natural basis in which $U_{C_3 C_4}$ is expressed; this is quite some laborious work as we have to look now for states of the form

$$|0,\lambda,\alpha\rangle_{\mathcal{H}_{C_3}} = \sum_{0\leq n} \alpha_n \left(\left(a'^{C_2}\right)^\dagger\right)^n |\lambda,\alpha\rangle_{\mathcal{H}_{C_3}}$$

satisfying

$$a_1^{C_3}|0,\lambda,\alpha\rangle_{\mathcal{H}_{C_3}} = 0$$

and write the states

$$\left((a'^{C_2})^\dagger\right)^n |\lambda,\alpha\rangle_{\mathcal{H}_{C_3}}$$

in terms of

$$\left((a_1^{C_3})^\dagger\right)^m |0,\lambda,\alpha\rangle_{\mathcal{H}_{C_3}}$$

for all $0 \leq n, m$. Therefore, such programme is very difficult to calculate with or even establish any result in by the very ambiguity in the definition of the operators $U_{CC'}$. Note that in any case the spectral properties of the operators a^C are (mildly) constrained by a potential future and the same holds for the mappings $U_{CC'}$. We will now come to study of a non-local dynamics.

In search for such dynamics, we still need to be guided by some principle of causality and it is this issue which we shall adress in the remainder of this paper. One option is to question general covariance, in particular it may very well be that the operators on a growing spacetime *do* depend upon the natural labeling. Suppose t and t' constitute two natural labelings of an n-element causal set C and consider two spacelike separated points x and y where x comes before y for t' and the other way around for t; then it is necessary that the field operators ϕ, ϕ' defined with respect to t and t' respectively satisfy the following criteria

$$\begin{aligned}
\sigma(\phi(x)) &= \sigma(\phi'(x)) \\
\sigma(\phi(y)) &= \sigma(\phi'(y)) \\
\pi'_y(\lambda-\epsilon,\lambda+\epsilon)\pi'_x(\mu-\delta,\mu+\delta) &= \pi_x(\mu-\delta,\mu+\delta)\pi_y(\lambda-\epsilon,\lambda+\epsilon)
\end{aligned}$$

where σ denotes the spectrum and $\lambda \in \sigma(\phi(y))$, $\mu \in \sigma(\phi(x))$. Such reasoning appears to lead to nowhere and one should insist that $\phi = \phi'$. Here for example, one might use the retarded *non-local* Klein-Gordon operator to formulate the dynamics; this would lead, upon reflection, to a worldview where *all* field operators commute with one and another meaning that time evolution changes the state only in a very limited sense[6]. The reader can see this by studying the example on page six: here, we must conclude that $\phi(w)$ commutes with $\phi(x) + \phi(y)$ but by adding a few points and relations to the causal set such that there exists a new element o which is spacelike to w but which has x but not y in its past, one obtains the conclusion that $[\phi(x),\phi(w)] = 0 = [\phi(y),\phi(w)]$. This is true for all points at all stages and I am not sure if this is in contradiction with

[6]Obviously, the states remain invariant if all eigenvalues of field operators are non-degenerate, but this is not the case for field operators.

experiment or not. In this argument we have assumed that the operators do *not* transform under a (unitary) mapping as we did before; indeed, ideally one would get rid of these transformations of states and operators at every stage of the growth process. This leads me to a second conclusion: the value of a field at a certain spacetime point x cannot only depend upon the values $\phi(y)$ for y in the past of x even if the first two "time slices" *appear* to be only chosen such that spacelike separated operators commute (as pointed out, they are not since all operators in the second time slice must commute with all operators in the first time slice). Strictly speaking, we have proven this for *linear* dependencies, but it is clear that it holds for a *generic* nonlinear functionality as well as the reader may want to figure out. We obtain here again the following result: albeit the dynamics of the field is retarded in nature, the initial values are constrained by a possible *future*. The problem might be traced back to the ambiguity of the meaning of global hyperbolicity for causal sets: (a) it is certainly true that every past inextendible causal curve intersects the initial data "surface" (in this case the first two time slices of the causal set) so in that sense a finite causal set is globally hyperbolic (b) it is also stably causal (since there are no closed timelike curves) and the Alexandrov sets are finite, hence compact, so it is also globally hyperbolic in the sense of Leray [12, 13] but (c) it is not so that the topology of a causal set is of the form $\Sigma \times \mathbb{Z}_k$ where $\mathbb{Z}_k$ represents time and Σ represents space for $k > 0$. Indeed, the number of points may change from slice to slice and only the first two slices do in general define a "Cauchy surface" whereas in the continuum all slices of some foliation have the Cauchy property. Since the canonical formulation of quantum field theory does not even exist on a manifold which is not globally hyperbolic and one can only make some ansatze in the path integral formulation, it is not surprising that the *constraint* of causality in the future is going to have some impact on the past configurations whereas this was sort of automatically implied for globally hyperbolic spacetimes. So, in that sense, our finding is by no means to be interpreted as a critique on discrete approaches since it might very well be that this feature turns out to be generic if one allows for spatial topology change to occur and insist on field theory to hold. These results might have been anticipated already from another point of view as I am unaware if higher derivative Lorentz invariant theories allow for a well posed initial value problem in the sense that the unique solution is causal. Therefore, what we seem to need is a quantum theory of creation, where just as for the classical growth dynamics a new operator is created (with a certain probability) satisfying the non-local causality constraints that spacelike separated operators commute. This appears to be in conflict with the Bell causality condition in the context of the causal sequential growth dynamics since now the newborn operator is constrained beyond the past of the newborn element. Such ideas meet at first sight formidable obstacles in the sense that it is impossible to sample in an infinite dimensional vector space of Hermitian operators on an infinite dimensional Hilbert space. It is here that a new physical principle might come into play: "there are at most as many particles as there are atoms of space[7] at a given instant in a causal set" which sounds logical since there is not more room to stack particles into. This implies that not a single bosonic creation-annihilation pair can be derived from the causal set and more in particular that $[\phi(x), \phi(y)]$ cannot be a scalar multiple of the identity operator. This

[7] Here, space is the set of all maximal elements in a finite causal set.

seems to suggest the use of nilpotent operators of finite rank as the building blocks for fields with numerical coefficients which depend upon the causal set structure. Such space could be sampled if one assumes additional constraints on the statistics and one could imagine building models based upon this premise: note that for the Fermi theory on a causal set, the maximal number of fermions roughly equals the number of spacetime events.

As a final comment, it might be that the causality question, as it is *posed*, is irrelevant in the sense that in an evolving universe only one observer can make a measurement first given two spacetime positions. Indeed, the whole idea that one can consider the situation where both observers are in position to make a measurement first at specified spacetime events is just theoretical and does not occur in practice. In that case, the causality condition would be given by the following, more realistic, non-local criterion: given two freely falling observers Bob and Alice measuring the quantum field ϕ and suppose that at labeling time $t = m$, Bob measures at spacetime event x labeled by m, $\phi(x)$ and then Alice measures at labelling time $t = n > m$, $\phi(y)$ where the event y labelled by n is spacelike separated to x. Then, the outcome of both acts is almost the same as when Alice measures first at y at time $t = n$ and subsequently Bob measures at z, spacelike separated to y, for $t > n$ at least when x and z are sufficiently close to one another. Obviously, one would expect this to be true in quantum field theory in the continuum, although there one would need to speak about observables defined on spacetime regions so that these operators have a nontrivial domain and continuity properies that can be adressed[8]. However, our causality requirement is weaker and more physical albeit somewhat vaguely phrased at this moment. Appearantly, this does not need to imply that $[\phi(x), \phi(y)] \sim 0$ for x spacelike separated to y since $\phi(x)$ is an unbounded operator.

4 Conclusions.

Although we have assumed very little, we obtained some rather important conclusions about the fundamental assumptions which went into the argument. As I have repeatedly stated, I do not think that these conclusions are inherent to the discreteness assumption but that they would prevail in any continuum framework which allows for topology change as we will repeat in somewhat more detail now. If one were to restrict topology change to connected spaces (so that spacetime is automatically connected) in the sense that no disconnected universe from ours could be born, then one would conclude there exists a continuum of topology classes of such maximal spacetimes. Now, every topology equivalence class will define a distinct quantum sector: this implies that at every stage of the dynamics the natural vacuum state has an (countable) infinite degeneracy since it is unclear into which topological configuration it will evolve[9]. This is precisely the conclusion we reached in the discrete case. Hence, let us formulate our assumptions as follows: (a) topology change (b) a notion of time (c) "general covariance" (d) local quantum field theory and (e) relativistic causality. What

[8]Indeed, in the continuum one has that the domain of $\phi(x)$ is zero, the latter being only well defined as a distribution (that is, only matrix elements $\langle\psi|\phi(x)|\zeta\rangle$ can be computed for suitable $|\psi\rangle$ and $|\zeta\rangle$).

[9]States corresponding to a pure topology would be improper then.

we have shown in the context of causal sets is that insisting on all five of these assumptions inevitably leads to the conclusion that a potential future must influence the present: in case one did not want to give up on (d), we obtained an infinite vacuum degeneracy at any moment in "time" and operators whose spectral properties must depend upon a future growth stage. However, as it stands, we are not even sure that (d) can be substained without giving in upon (c); note that we found it particularly hard to even verify if (c) actually holds, moreover, in that context, we reached the conclusion that (c) and (e) are actually equivalent. As far as I understand relativity theory, I think it must be that (a) and (b) given (c) are actually also equivalent, so therefore if we insist upon (b) and (c) we must include (a) also in the picture. This puts a heavy pressure on (d) or (e) and given the evidence that (d) might endanger (c) and moreover, requires a very unusual representation, I think it is justified to sacrifice (d) as a plausible assumption in spite of its partial succeses in flat spacetime (if we ignore for a moment the reality that the theory is not well defined). We have argued this from different points of view and indeed, for physics not to depend upon the future, it appears we need a whole new conception about dynamics of quantum fields. I have provided some points and hints as in which direction such search might lead us: in any case it requires a modified particle statistics. Also, as far as I am aware, this is the first study at this level of quantum field theory in theories of dynamical spatial topology, not just in a static spacetime where topology changes do occur a few times.

We need to put the constraints coming from the future in some perspective though since precisely the same happens in the causal sequential growth dynamics where it is the constraint of "general covariance" which brings a potential future into the present. These considerations throw a very different light upon the notion of time evolution since the potential futures must already have been considered before an actual evolution takes place: this is certainly a novel point of view on quantum mechanics and not as much as on relativity where we are used to global spacetime considerations. I leave it up to the reader to figure out which conclusion he or she may draw from this preliminary work as it concerns some very basic assumptions about the nature of reality.

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