Solution to Poisson Boltzmann equation in semi-infinite and cylindrical geometries

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Linearized Poisson-Boltzmann equation (PBE) gives us simple expressions for charge density distribution (ρ_e) within fluids or plasma. A recent work of this author shows that the old boundary conditions (BC), which are usually used to solve PBE, have serious defects. The old solutions turned out to be non-unique, and violates charge conservation principle in some cases. There we also derived the correct formula of ρ_e for a finite, rectangular geometry, using appropriate BCs. Here we consider some other types of geometries and obtain formula of ρ_e, which may be useful to analyse different experimental conditions.

I. INTRODUCTION

Linearized PBE has been used since a long time, almost a century [1], to obtain distribution of free electric charges within ionic solutions or plasma. It has applications in various fields ranging from nanofluidics-microfluidics [2, 3] to astrophysics [4, 5], covering radioscience [6], surface chemistry [7], colloid science [8], laboratory plasma/ thermo-nuclear fusion devices [9, 10], bio-science [11, 12] etc. However, a recent work of this author [13] shows that the Dirichlet or Neumann type BCs that are usually used to solve the PBE, have serious defects, as they produce non-unique solutions. The Dirichlet condition leads to violation of charge conservation principle. We have been able to remove those defects by using different BCs to obtain ρ_e. There we analyzed a 1-D problem using a 'finite' and 'rectangular' geometry to demonstrate the essential ideas. Here, we derive the formula of ρ_e for some other geometries. Firstly, we do an 1-D analysis for a cylindrical geometry; we consider two sub-cases, one with an annular cross-section, and the other with a circular cross-section. Secondly, we do an 1-D analysis for a rectangular, semi-infinite domain, where the fluid is bounded by a single plane and the fluid extends to infinity along the direction normal to that plane. After this, we give an 'alternative' derivation to the distribution formula for a finite rectangular domain that we have derived for the first time in Ref. [14]; a minute correction was done in Ref. [15].

II. CYLINDRICAL GEOMETRY

See Ref. [16] for some formulae that we used here.

For a cylindrical geometry, the position of a point is specified with (R, θ, z), see Ref. [17]. The meaning of most of the other symbols can be found in Ref. ([14]), any exception will be notified. We analyse a straight domain of length L, with annular and circular cross-sections

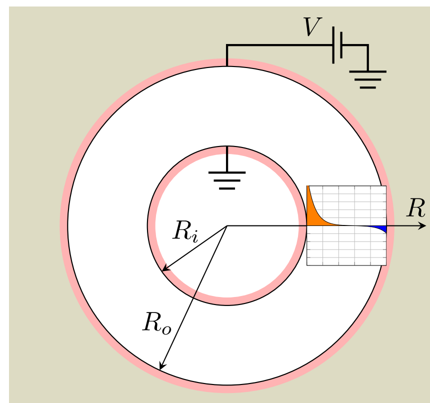

FIG. 1. (Color online) Ionic solution within a cylindrical geometry of annular cross-section. Two circular boundaries have a potential difference. A typical plot that shows how charge density varies in the radial direction, is embedded to show the domain of our interest, see Fig. (2)

respectively. We consider only axi-symmetric problems i.e independent of θ. When L is much greater compared to the radius, the problems become independent of z as well, except some end-effects. Hence, the problems depend only on the radial coordinate R. We use some suitable scales 'a' and 'ζ' (both are positive) for distance and electrostatic potential (ψ) respectively. Let's define a few quantities:

$$r \equiv \frac{R}{a};\ \kappa \equiv \left[\frac{\lambda_D}{a}\right]^{-1};\ \psi^* \equiv \frac{\psi}{\zeta};\ \rho_0 \equiv \frac{\epsilon\kappa^2\zeta}{a^2};\ \rho_e^* \equiv \frac{\rho_e}{\rho_0} \quad (1)$$

Where, λ_D is the 'Debye length', ϵ is the permittivity of the fluid.

A. Cylindrical geometry of annular cross-section

Here the domain is bounded by two concentric right cylinders, see Fig. (1); the 'inner' and 'outer' radii are R_i and R_o, normalized as: $r_i \equiv R_i/a; r_o \equiv R_o/a$; the annular ends are also walls.

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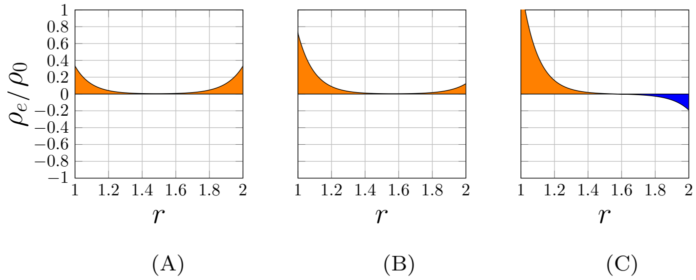

FIG. 2. (Color online) Charge density distribution within a fluid, bounded by two concentric cylinders. $\kappa = 10.0$; $q_0 = 0.1$. (a) $v = 0$; excess charges accumulate near boundaries. (b) $v = 0.6$; an applied voltage redistributes charges. (c) $v = 1.5$; strong voltage segregates negative charges even if $q_0 > 0$. Compare this figure with that in Ref. [13]

We start with a formula that we derived in Ref. [14] (within its Supplementary Material).

$$\therefore \rho_e^* = -\psi^* \tag{2}$$

We will define some useful quantities, which will be useful later. The net charge present in the total domain i.e Q_{TOT} is given by $\iiint \rho_e \mathrm{d}V$, where $\mathrm{d}V = R\,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}z$. If ρ_e depends only upon R, we have,

$$Q_{TOT} = \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_{R_i}^{R_o} \rho_e\,R\,\mathrm{d}R = 2\pi L \int_{R_i}^{R_o} \rho_e\,R\,\mathrm{d}R \tag{3}$$

We define a quantity Q_{1c}, then using Eq. (1) we write its non-dimensional form q_{1c}; the subscript '$1c$' means an 1-D problem in cylindrical geometry.

$$Q_{1c} \equiv \frac{Q_{TOT}}{2\pi L} = \int_{R_i}^{R_o} \rho_e\,R\,\mathrm{d}R = \rho_0 a^2 \int_{r_i}^{r_o} \rho_e^*\,r\,\mathrm{d}r \tag{4}$$

$$q_{1c} \equiv \frac{Q_{1c}}{(\rho_0 a^2)} = \int_{r_i}^{r_o} \rho_e^*\,r\,\mathrm{d}r \tag{5}$$

Q_{1c} is the net charge present in the fluid within an angular sector of unit radian, per unit axial length. Q_{1c} has dimension 'charge per length' e.g. $Coulomb \cdot meter^{-1}$, unlike in the 1-D problem for a rectangular domain, where it had dimension $Coulomb \cdot meter^{-2}$ (see Ref. [15]).

Now, ψ and ρ_e are also related by Poisson's equation in electrostatics (PES), which is given by,

$$\nabla^2 \psi = -\frac{\rho_e}{\epsilon} \tag{6}$$

For a cylindrical geometry, ∇^2 is given by,

$$\nabla^2 \equiv \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{7}$$

In the special case, where ψ varies only in the'radial' direction, the PES reduces to,

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}\psi}{\mathrm{d}R}\right) = -\frac{\rho_e}{\epsilon} \tag{8}$$

using Eq. (1) we first make PES non-dimensional:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right)\left(\frac{\zeta}{a^2}\right) = -\frac{\rho_0}{\epsilon}\rho_e^* = -\left(\frac{\epsilon\kappa^2\zeta}{a^2\epsilon}\right)\rho_e^*$$
$$\Rightarrow \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right) = -\kappa^2\rho_e^* \tag{9}$$

Using Eq. (2) in Eq. (9) we get non-dimensional PBE in 1-D cylindrical (radial) coordinates:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right) = \kappa^2\psi^* \tag{10}$$

Its general solution (see Ref. [16]), with arbitrary constants A and B, is given by,

$$\psi^* = AI_0(\kappa r) + BK_0(\kappa r) \tag{11}$$

Where, I_0 and K_0 are modified Bessel functions of order 0. We need two conditions to fix A and B. We get one condition by integrating PES i.e. Eq. (9) and using Eq. (5),

$$\int_{r_i}^{r_o}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right)\right] r\,\mathrm{d}r = -\kappa^2\int_{r_i}^{r_o}\rho_e^*\,r\,\mathrm{d}r$$
$$\Rightarrow \left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right)\Big|_{r=r_o} - \left(r\frac{\mathrm{d}\psi^*}{\mathrm{d}r}\right)\Big|_{r=r_i} = -q_{1c}\kappa^2 \tag{12}$$

We assume the potential difference (scaled with ζ) between outer and inner curved boundaries i.e. 'v' to be known,

$$\psi^*(r_o) - \psi^*(r_i) = v \tag{13}$$

We solve PBE i.e. Eq. (10) using two conditions given by Eq. (12) and Eq. (13). We use the following formulae, see Ref. [18, 19]:

$$\frac{\mathrm{d}I_0(\xi)}{\mathrm{d}\xi} = I_1(\xi)\ ;\ \frac{\mathrm{d}K_0(\xi)}{\mathrm{d}\xi} = -K_1(\xi) \tag{14}$$

From Eq. (11) we get,

$$\frac{\mathrm{d}\psi^*}{\mathrm{d}r} = A\frac{\mathrm{d}I_0(\kappa r)}{\mathrm{d}r} + B\frac{\mathrm{d}K_0(\kappa r)}{\mathrm{d}r}$$
$$= \kappa\left[AI_1(\kappa r) - BK_1(\kappa r)\right] \tag{15}$$

From Eq. (12) and Eq. (15) we get,

$$r_o\kappa\left[AI_1(\kappa r_o)-BK_1(\kappa r_o)\right]-r_i\kappa\left[AI_1(\kappa r_i)-BK_1(\kappa r_i)\right]=-q_{1c}\kappa^2$$
$$\Rightarrow\left[r_oI_1(\kappa r_o)-r_iI_1(\kappa r_i)\right]A-\left[r_oK_1(\kappa r_o)-r_iK_1(\kappa r_i)\right]B=-q_{1c}\kappa \tag{16}$$

From Eq. (11) and Eq. (13) we get,

$$\left[AI_0(\kappa r_o)+BK_0(\kappa r_o)\right]-\left[AI_0(\kappa r_i)+BK_0(\kappa r_i)\right]=v$$
$$\Rightarrow\left[I_0(\kappa r_o)-I_0(\kappa r_i)\right]A+\left[K_0(\kappa r_o)-K_0(\kappa r_i)\right]B=v \tag{17}$$

We write Eq. (16) and Eq. (17) together in a compact, matrix form:

$$\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\end{pmatrix} \tag{18}$$

Where,

$$C_{11}\equiv\left[r_oI_1(\kappa r_o)-r_iI_1(\kappa r_i)\right] \tag{19}$$
$$C_{12}\equiv(-1)\times\left[r_oK_1(\kappa r_o)-r_iK_1(\kappa r_i)\right] \tag{20}$$
$$C_{21}\equiv\left[I_0(\kappa r_o)-I_0(\kappa r_i)\right] \tag{21}$$
$$C_{22}\equiv\left[K_0(\kappa r_o)-K_0(\kappa r_i)\right] \tag{22}$$
$$d_1\equiv-q_{1c}\kappa \tag{23}$$
$$d_2\equiv v \tag{24}$$

The determinant Δ of the above 2×2 matrix is given by,

$$\Delta\equiv C_{11}\cdot C_{22}-C_{12}\cdot C_{21} \tag{25}$$

Finally we write A and B in terms of known quantities,

$$A=(d_1\cdot C_{22}-d_2\cdot C_{12})/\Delta \tag{26}$$
$$B=(-d_1\cdot C_{21}+d_2\cdot C_{11})/\Delta \tag{27}$$

We plug in the expressions of A and B given by Eq. (26) and Eq. (27) in Eq. (11) and rearrange terms; then we use Eq. (23), Eq. (24) ; finally we use Eq. (2) i.e. $\rho_e^*=-\psi^*$,

$$\begin{aligned}\psi^*&=\frac{(d_1\cdot C_{22}-d_2\cdot C_{12})}{\Delta}I_0(\kappa r)+\frac{(-d_1\cdot C_{21}+d_2\cdot C_{11})}{\Delta}K_0(\kappa r)\\ &=\frac{1}{\Delta}\left[d_1\left\{C_{22}I_0(\kappa r)-C_{21}K_0(\kappa r)\right\}-d_2\left\{C_{12}I_0(\kappa r)-C_{11}K_0(\kappa r)\right\}\right]\\ &=\frac{(-1)}{\Delta}\left[q_{1c}\kappa\left\{C_{22}I_0(\kappa r)-C_{21}K_0(\kappa r)\right\}+v\left\{C_{12}I_0(\kappa r)-C_{11}K_0(\kappa r)\right\}\right]\end{aligned} \tag{28}$$

$$\boxed{\therefore\,\rho_e^*=\frac{1}{\Delta}\left[q_{1c}\kappa\left\{C_{22}I_0(\kappa r)-C_{21}K_0(\kappa r)\right\}+v\left\{C_{12}I_0(\kappa r)-C_{11}K_0(\kappa r)\right\}\right]} \tag{29}$$

Using Eq. (29) we plot ρ_e^* vs r in Fig. (2) for the range $[1\le r\le 2]$ i.e. we took $a=R_i$ and $R_o=2R_i$, so that $r_i=1$, and $r_o=2$. The explanation of the plots goes along the same line as the rectangular geometry (Ref. [13]); if there is a net amount of charge in the domain, the charges accumulate near the two boundaries. An applied voltage redistributes the charges; positive and negative charges move towards the walls of lower and higher potentials respectively. A strong voltage can segregate two types of charges.

B. Cylindrical geometry of circular cross-section

Here we derive the formula of ρ_e for a right cylindrical domain of 'circular' cross-section, unlike the 'annular' one. The general solution to the PBE is given by Eq. (11). The function $K_0(\kappa r)$ blows up as $r\to 0$. Hence, in order to prevent ψ^* from blowing up as $r\to 0$, we must set $B=0$. We get,

$$\psi^*=AI_0(\kappa r) \tag{30}$$

For the special case $r_i=0$, we write ${}^0q_{1c}$ instead of q_{1c}. Using Eq. (2) in Eq. (5), with $r_i=0$, we get,

$$\int_0^{r_o}\psi^*\,r\,\mathrm{d}r=-{}^0q_{1c} \tag{31}$$

Using Eq. (30) we get,

$$\int_0^{r_o} \psi^* \, r \, \mathrm{d}r = A \int_0^{r_o} I_0(\kappa r) \mathrm{d}r$$
$$= A \left. \frac{r I_1(\kappa r)}{\kappa} \right|_0^{r_o}$$
$$= A \frac{r_o I_1(\kappa r_o)}{\kappa} = -{}^0q_{1c}$$
$$\Rightarrow A = -\frac{{}^0q_{1c}\kappa}{r_0 I_1(\kappa r_o)} \quad (32)$$

$$\Rightarrow \psi^* = -\frac{{}^0q_{1c}\kappa}{r_0 I_1(\kappa r_o)} I_0(\kappa r) \quad (33)$$

Using Eq. (2) we finally have the formula for ρ_e^*,

$$\boxed{\rho_e^* = \frac{{}^0q_{1c}\kappa}{r_0 I_1(\kappa r_o)} I_0(\kappa r)} \quad (34)$$

III. SEMI INFINITE RECTANGULAR GEOMETRY

Let's consider a plane, which, in its one side, bounds an infinite ocean of ionic solution. Here ρ_e varies only in the direction normal to the plane (x-direction); we set the origin on the plane so that x varies from 0 to ∞. The general solution is the same as given in Ref. [13] (see the supplementary materials in that reference), however, the arbitrary constants of integration are fixed with different conditions. The non-dimensional PBE is $\mathrm{d}^2\psi^*/\mathrm{d}\eta^2 = \kappa^2\psi^*$; its general solution is given by:

$$\psi^* = A \exp(\kappa\eta) + B \exp(-\kappa\eta) \quad (35)$$

We must set 'A' to zero to prevent the solution to blow up when $\eta \to \infty$. Hence, using Eq. (2) we get

$$\rho_e^* = -B \exp(-\kappa\eta) \quad (36)$$
$$\Rightarrow \int_0^{\infty} \rho_e^* \mathrm{d}\eta = -B \int_0^{\infty} \exp(-\kappa\eta) \mathrm{d}\eta$$
$$= -B \left(\left. \frac{\exp(-\kappa\eta)}{-\kappa} \right|_0^{\infty} \right)$$
$$= -B \left(\frac{0-1}{-\kappa} \right) = -\frac{B}{\kappa} \quad (37)$$

Let's define the following quantities (the subscript '$1r$' means 1-D and rectangular; left-superscript '∞' means infinite domain):

$${}^{\infty}Q_{1r} \equiv \int_0^{\infty} \rho_e \, \mathrm{d}x \quad (38)$$
$${}^{\infty}q_{1r} \equiv \int_0^{\infty} \rho_e^* \, \mathrm{d}\eta = \frac{1}{\rho_0 a} \int_0^{\infty} \rho_e \, \mathrm{d}x = \frac{{}^{\infty}Q_{1r}}{(\rho_0 a)} \quad (39)$$

Please note the difference between Eq. (5) and Eq. (39); the dimensions of Q_{1c} and ${}^{\infty}Q_{1r}$ are different in two cases. We get,

$$B = -({}^{\infty}q_{1r}\kappa) \quad (40)$$

Finally we write the formula of ψ^* and ρ_e^*:

$$\psi^* = -({}^{\infty}q_{1r}\kappa) \exp(-\kappa\eta) \quad (41)$$

$$\boxed{\rho_e^* = ({}^{\infty}q_{1r}\kappa) \exp(-\kappa\eta)} \quad (42)$$

$$\boxed{\rho_e = \left(\frac{{}^{\infty}Q_{1r}\kappa}{a} \right) \exp(-\kappa x/a)} \quad (43)$$

Let's write the formula for the potential at boundary:

$$\psi_0^* \equiv \psi^*|_{\eta=0} = -{}^{\infty}q_{1r}\kappa \quad (44)$$

IV. FINITE RECTANGULAR GEOMETRY: AN ALTERNATIVE DERIVATION

For a finite, rectangular geometry we derived the formula of ρ_e in Ref. [13, 14], where ρ_e varies between two boundaries at $x = \pm a$ i.e. $\eta = \pm 1$; a minute correction of the formula can be found in Ref. [15]. Here we derive the same formula in a different way. We use a different symbol Q_{1r} instead of Q_0. For the scaled potential difference (V/ζ) between boundaries we use the symbol v instead of δ.

$$Q_{1r} \equiv \int_{-a}^{+a} \rho_e dx \quad (45)$$

From Eq. (38) and Eq. (45) we see that ${}^{\infty}Q_{1r}$ and Q_{1r} differs only in the limits of integration. Integrating Eq. (2) we get,

$$\int_{-1}^{+1} \psi^* d\eta = -\int_{-1}^{+1} \rho_e^* d\eta = -\int_{-a}^{+a} \frac{\rho_e}{\rho_0} d\left(\frac{x}{a}\right)$$
$$= -\frac{1}{(\rho_0 a)} \int_{-a}^{+a} \rho_e dx = -\frac{Q_{1r}}{(\rho_0 a)}$$
$$\Rightarrow \int_{-1}^{+1} \psi^* d\eta = -q_{1r} \quad (46)$$

$$\text{Where,} \quad q_{1r} \equiv \frac{Q_{1r}}{(\rho_0 a)} \quad (47)$$

We use Eq. (46) as one condition to solve PBE; the other condition is given by,

$$\psi^*(+1) - \psi^*(-1) = v \quad (48)$$

Integrating the general solution to the PBE i.e. Eq. (35) and then using Eq. (46) we get,

$$\int_{-1}^{+1} \psi^* d\eta = A \left[\frac{\exp(+\kappa) - \exp(-\kappa)}{\kappa} \right] + B \left[\frac{\exp(-\kappa) - \exp(\kappa)}{-\kappa} \right]$$
$$= [A + B] \frac{2\sinh(\kappa)}{\kappa} = -q_{1r} \quad (49)$$

Using Eq. (35) and Eq. (48) we get,

$$\begin{aligned}[A\exp(\kappa)+B\exp(-\kappa)]-[A\exp(-\kappa)+B\exp(\kappa)]=v\\ \Rightarrow A[\exp(\kappa)-\exp(-\kappa)]-B[\exp(\kappa)-\exp(-\kappa)]=v\\ \Rightarrow [A-B]\,2\sinh(\kappa)=v\end{aligned} \tag{50}$$

From Eq. (49) and Eq. (50) we get,

$$A+B=-\frac{q_{1r}\kappa}{2\sinh(\kappa)} \tag{51}$$

$$A-B=\frac{v}{2\sinh(\kappa)} \tag{52}$$

From the above two equations we solve for A and B,

$$A=\frac{[v-q_{1r}\kappa]}{4\sinh(\kappa)};\quad B=-\frac{[v+q_{1r}\kappa]}{4\sinh(\kappa)} \tag{53}$$

Plugging in these expressions for A and B in Eq. (35), and rearranging terms we get the required expression for ψ^*; then, using Eq. (2) we get the formula of ρ^*; then we return to the dimensional variables:

$$\psi^*=\frac{1}{2\sinh(\kappa)}\left[v\cdot\sinh(\kappa\eta)-(q_{1r}\kappa)\cdot\cosh(\kappa\eta)\right] \tag{54}$$

$$\boxed{\rho_e^*=\frac{1}{2\sinh(\kappa)}\left[(q_{1r}\kappa)\cdot\cosh(\kappa\eta)-v\cdot\sinh(\kappa\eta)\right]} \tag{55}$$

$$\boxed{\rho_e=\frac{\rho_0}{2\sinh(\kappa)}\left[\kappa\left(\frac{Q_{1r}}{\rho_0 a}\right)\cosh\left(\frac{\kappa x}{a}\right)-\left(\frac{V}{\zeta}\right)\sinh\left(\frac{\kappa x}{a}\right)\right]} \tag{56}$$

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[1] P. Debye and E. Huckel, Physikalische Zeitschrift **24**, 185 (1923).
[2] R. B. Schoch, J. Han, and P. Renaud, Rev. Mod. Phys. **80**, 839 (2008).
[3] S. Ghosal, Annu. Rev. Fluid Mech. **38**, 309 (2006).
[4] M. Horanyi, Annu. Rev. Astron. Astrophys. **34**, 383418 (1996).
[5] Y. Liu, S. Q. Liu, and K. Xu, Phys. of Plasmas **19**, 073702 (2012).
[6] T. W. Chevalier, U. S. Inan, and T. F. Bell, Radio Sci. **45**, RS1010 (2010).
[7] D. Lis, E. H. G. Backus, J. Hunger, S. H. Parekh, and M. Bonn, Science **344**, 1138 (2014).
[8] J. Lyklema, *Fundamentals of Interface and Colloid Science,*, Vol. 2 (Academic, London, 1991).
[9] J. P. H. Goedbloed and S. Poedts, *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas* (Cambridge University Press, 2004).
[10] L. Patacchini and I. H. Hutchinson, Plasma Phys. Control. Fusion **53**, 025005 (2011).
[11] A. N. Rao and D. W. Grainger, Biomater. Sci. **2**, 436 (2014).
[12] F. Fogolari, A. Brigo, and H. Molinari, J. Mol. Recognit. **15**, 377 (2002).
[13] R. Chakraborty, "Debye length cannot be interpreted as screening or shielding length," http://www.vixra.org/abs/1502.0051 (2015).
[14] R. Chakraborty, "Physical solution of poisson-boltzmann equation," http://www.vixra.org/abs/1408.0093 (2014).
[15] R. Chakraborty, "Debye length cannot be interpreted as screening or shielding length," (2015), J. Phys. : Cond. Mat. (unpublished).
[16] K. Bohinc, A. Iglič, and T. Slivnik, Electrothniški vestnik, Electrotechnical Rev., Ljubljana, Slovenija **75**, 82 (2008).
[17] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 4th ed. (Academic Press, Inc., San Diego, USA, 1995).
[18] "BesselI," http://functions.wolfram.com/Bessel-TypeFunctions/BesselI/20/01/02/0004/ (2001), (Documentation of Wolfram Research).
[19] "BesselK," http://functions.wolfram.com/Bessel-TypeFunctions/BesselK/20/01/02/0004/ (2001), (Documentation of Wolfram Research).