Exact solutions of general states of harmonic oscillator in 1 and 2 dimensions: Student's supplement

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Abstract

The purpose of this paper is two-fold. First, we would like to write down algebraic expression for the wave function of general excited state of harmonic oscillator which doesn't include derivative signs (this is to be contrasted with typical physics textbook which only gets rid of derivative signs for first few excited states, while leaving derivatives in when it comes to Hermite polynomial for general n). Secondly, we would like to write similar expression for two dimensional case as well. In the process of tackling two dimensions, we will highlight the interplay between Cartesian and polar coordinates in 2D in the context of an oscillator. All of the above mentioned results have probably been derived by others but unfortunately they are not easily available. The purpose of this paper is to make it easier for both students and general public to look up said results and their derivations, should the need arise. We also attempt to illustrate different angles from which one could look at the problem and this way encourage students to think more deeply about the material.

1. Introduction

Typical textbook presentation of harmonic oscillator (see, for example [1] and [2]) proceeds as follows: first creation/annihilation operators, their properties, and energy spectrum are described, then by use of creation operators the wave functions for first four or five excited states are given, and finally a general excited state is expressed in terms of Hermite polynomial where the latter is defined in terms of n-th derivative of $e^{-x^2/2}$. However, there is one thing the textbooks skip. On the one hand, they give algebraic expression for Hermite polynomial *without* derivative sign for *first few states*; on the other hand, they give an expression *with* derivative sign *for general state*; but they *never* give an expression *without* derivative sign *for general state*. There is no good reason for such omission since said expression is known (see [3]). But even if one does check out [3], one won't see the derivation of such expression, only the result. One of the purposes of this paper is to provide the derivation of the above expression. I am not claiming to do anything new: after all, in order for such an expression to be found in [3], it must have been derived *somehow* by someone else; and besides, once the reader sees that expression in [3], they should be able to carry out the induction on their own. My task is to simply make such derivation more easily available for both students and public, which could potentially save them a lot of time.

Apart from that, my other purpose is to expand the treatment of harmonic oscillator to two dimensions. On the first glance, one can argue that it is a "trivial" issue of separation of variables. What makes it a lot less "trivial", however, is that a typical state obtained by separation of variables is not rotationally symmetric. The reason for this is that any given "Cartesian" state is a linear combination of "polar" states of different angular momenta. Each polar state "by itself" is rotationally symmetric, but their linear combination is not, due to the fact that difference in their angular momenta implies difference in phase factor they pick up with rotation. Such an issue has been acknowledged in [4], and the differential equation for Harmonic oscillator in polar coordinates was given. However, solving the polar differential equation given in [4] is quite difficult. In this paper we pick a different approach. We take the solutions for n-th excited state in x multiplied by ground state in y, and then replace x with $(re^{i\theta} + re^{-i\theta})/2$ (since in y direction we have a ground state, we don't have to worry about it other than extra factor $e^{-m\omega y^2/2}$) ; after replacing x^k with a binomial expansion of the above, we can readily extract terms proportional to $e^{iL\theta}$ which would thus "isolate" the state with fixed angular momentum L.

The above way of doing it raises some interesting questions. In particular, one still has raising/lowering operators described in polar coordinates (see Eq 41- 45, 54- 57, 67 and 73) which would allow us to generate any state with fixed energy and angular momentum by acting on a vacuum state *in polar coordinates* with those operators enough times. This procedure is clearly very different from the one described in the previous paragraph; so will these two procedures really give the same answer? As a physicist, I know the answer is yes; but a mathematician would want to verify it (such verification is left to the reader – see Exercise 1). This is just one of several other examples where we can either save a lot of work by "trusting physics" or work a lot harder and carry various mathematical proofs that different "physics" approaches would in fact give us the same answer. As a matter of fact, from the physicist's point of view, this paper could have been reduced to Sections 2-4 and 6-8. On the other hand, Sections 5, 9 and 10, which make paper twice longer than it could have been, are basically verifications that different results "match" the way they are "supposed" to. And, in fact, there are even more verifications that should be done – some of which are listed as exercises in Section 11.

Since different approaches differ in difficulty, I have picked the easiest possible approach in obtaining "original" answer; but then I was "forced" to face more difficulties in "verification" parts where I had to bring in "alternative" ways of getting the answer (which would be more difficult than the hand-picked way I started with). What this means is that, on the one hand, the student can "get" what is going on based on the "easy" chapters and then "challenge themselves" on the difficult ones. This combination might allow a student to tackle "challenges" themselves without outside help, thus gain deeper understanding of material. In particular, the student will get a lot of opportunities to think about interplay between polar and Cartesian coordinates as well as the properties of angular momentum.

2. Harmonic oscillator in 1D: The basics

The Hamiltonian of harmonic oscillator is given by

$$H = \frac{m\omega^2 x^2}{2} - \frac{1}{2m}\frac{d^2}{dx^2} \tag{1}$$

We will define raising and lowering operator as

$$a^\dagger = \sqrt{\frac{m\omega}{2}}x - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx}\ ,\ a = \sqrt{\frac{m\omega}{2}}x + \frac{1}{\sqrt{2m\omega}}\frac{d}{dx} \tag{2}$$

which produces commutation relations

$$[a, a^\dagger] = 1 \tag{3}$$

and allows us to rewrite Hamiltonian as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) \tag{4}$$

Thus, the eigenstates of H are the same as the ones of $a^\dagger a$, with eigenvalues "shifted" by $\omega/2$, which corresponds to "vacuum energy". The ground state $|\psi_0\rangle$ would satisfy

$$H|\psi_0\rangle = \frac{\omega}{2}|\psi_0\rangle \tag{5}$$

if it satisfies $a^\dagger a|\psi_0\rangle = 0$, or, equivalently,

$$a|\psi_0\rangle = 0 \tag{6}$$

By using Eq 2 for a, the solution of the latter takes the form

$$\psi_0(x) = N_0 e^{-\frac{m\omega x^2}{2}} \tag{7}$$

In order to determine the value of N_0, we note that

$$1 = \int \left(N_0 e^{-m\omega x^2/2}\right)^2 dx = N_0^2 \int e^{-m\omega x^2} dx \tag{8}$$

Thus,

$$N_0 = \left(\int e^{-m\omega x^2} dx\right)^{-1/2} \tag{9}$$

In order to evaluate the above integral, we note that

$$\left(\int_{-\infty}^{\infty} e^{-\frac{cx^2}{2}} dx\right)^2 = \int e^{-c(x^2+y^2)/2} dx dy = \int_0^\infty e^{-cr^2/2} 2\pi r dr = \frac{2\pi}{a} \tag{10}$$

Here, $r = \sqrt{x^2+y^2}$ is the distance from origin in (x,y)-plane, and $2\pi r$ comes from the radius of the circle. The above calculation implies that

$$\int_{-\infty}^{\infty} e^{-\frac{cx^2}{2}} dx = \sqrt{\frac{2\pi}{c}} \tag{11}$$

and, therefore,

$$N_0 = \left(\int e^{-m\omega x^2} dx \right)^{-1/2} = \left(\frac{m\omega}{\pi} \right)^{1/4} \tag{12}$$

leading to

$$\psi_0(x) = \left(\frac{m\omega}{\pi} \right)^{1/4} e^{-m\omega x^2/2} \tag{13}$$

The fact that $a\psi_0 = 0$ implies that $a^\dagger a \psi_0 = 0$. Now, let us define $|\psi_n\rangle$ according to

$$|\psi_n\rangle = \frac{N_n}{N_0} (a^\dagger)^n |\psi_0\rangle \tag{14}$$

where N_n is selected in such a way that

$$\langle \psi_n | \psi_n \rangle = 1 \tag{15}$$

Let us prove by induction that

$$(a^\dagger a)|\psi_n\rangle = n|\psi_n\rangle \tag{16}$$

First of all, the above is true for $n = 0$. Now, suppose it holds for some other n. Eq 14 implies that

$$|\psi_{n+1}\rangle = \frac{N_{n+1}}{N_n} a^\dagger |\psi_n\rangle \tag{17}$$

Therefore,

$$(a^\dagger a)|\psi_{n+1}\rangle = \frac{N_{n+1}}{N_n}(a^\dagger a)(a^\dagger|\psi_n\rangle) = \frac{N_{n+1}}{N_n} a^\dagger (a^\dagger a + [a, a^\dagger])|\psi_n\rangle = \tag{18}$$

$$= \frac{N_{n+1}}{N_n} a^\dagger (n+1)|\psi_n\rangle = (n+1) a^\dagger \frac{N_{n+1}}{N_n} |\psi_n\rangle = (n+1)|n+1\rangle \tag{19}$$

Thus, by induction, Eq 16 holds. Substitution of Eq 16 into Eq 4 further implies that

$$H|\psi_n\rangle = \left(n + \frac{1}{2} \right) \omega |\psi_n\rangle \tag{20}$$

Now, if we apply Eq 16 to Eq 17, we obtain

$$1 = \langle \psi_{n+1} | \psi_{n+1} \rangle = \left(\frac{N_{n+1}}{N_n} \right)^2 \langle \psi_n | a a^\dagger | \psi_n \rangle = \left(\frac{N_{n+1}}{N_n} \right)^2 \langle \psi_n | ([a, a^\dagger] + a^\dagger a) | \psi_n \rangle =$$

$$= \left(\frac{N_{n+1}}{N_n} \right)^2 \langle \psi_n | (1 + a^\dagger a) | \psi_n \rangle = \left(\frac{N_{n+1}}{N_n} \right)^2 (n+1) \tag{21}$$

Therefore,

$$\frac{N_{n+1}}{N_n} = \frac{1}{\sqrt{n+1}} \tag{22}$$

which, by induction, implies that

$$N_n = \frac{N_0}{\sqrt{n!}} = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} \tag{23}$$

By substituting Eq 13 and Eq 2 , we obtain

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi}\right)^{1/4}\left(\sqrt{\frac{m\omega}{2}}x - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx}\right)^n e^{-\frac{m\omega x^2}{2}} \tag{24}$$

Now, let us introduce an operator $\hat{x}$ as distinct entity from variable x, defined in the following way:

$$\hat{x}1 = x \ , \ \frac{d}{dx}\hat{x} = 1 + \hat{x}\frac{d}{dx} \ , \ \frac{d}{dx}x = 1 \tag{25}$$

For reasons that will be clear shortly, we will rewrite Eq 24 in the following way,

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega\hat{x}^2}{2}}\left[e^{\frac{m\omega\hat{x}^2}{2}}\left(\sqrt{\frac{m\omega}{2}}\hat{x} - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx}\right)e^{-\frac{m\omega\hat{x}^2}{2}}\right]^n 1 \tag{26}$$

Now, we notice that

$$\frac{d}{dx}(e^{-cx^2/2}f(x)) = -cxe^{-cx^2/2}f(x) + e^{-cx^2/2}\frac{df}{dx} \tag{27}$$

which can be rewritten in terms of an *operator equation*,

$$\frac{d}{dx}e^{-c\hat{x}^2/2} = -c\hat{x}e^{-c\hat{x}^2/2} + e^{-c\hat{x}^2/2}\frac{d}{dx} \tag{28}$$

and, therefore,

$$e^{c\hat{x}^2/2}\frac{d}{dx}e^{-c\hat{x}^2/2} = -c\hat{x} + \frac{d}{dx} \tag{29}$$

At the same time, it is easy to see that

$$e^{c\hat{x}^2/2}\hat{x}e^{-c\hat{x}^2/2} = \hat{x} \tag{30}$$

Equations 29 and 30 imply that

$$e^{m\omega\hat{x}^2/2}\left(\sqrt{\frac{m\omega}{2}}x - \frac{1}{2m\omega}\frac{d}{dx}\right)e^{-m\omega x^2/2} = \sqrt{2m\omega}\ x - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx} \tag{31}$$

Therefore, Eq 26 can be rewritten as

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega\hat{x}^2}{2}}\left(\sqrt{2m\omega}\ \hat{x} - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx}\right)^n 1 \tag{32}$$

3. Harmonic oscillator in 2D: The basics

Let us now discuss operators that take us from one state to the other. In two dimensions, we have raizing and lowering operators corresponding to both axis:

$$a_x^\dagger = \sqrt{\frac{m\omega}{2}}x - \frac{1}{\sqrt{2m\omega}}\frac{d}{dx} \ , \ a_x = \sqrt{\frac{m\omega}{2}}x + \frac{1}{\sqrt{2m\omega}}\frac{d}{dx} \tag{33}$$

$$a_y^\dagger = \sqrt{\frac{m\omega}{2}}y - \frac{1}{\sqrt{2m\omega}}\frac{d}{dy} \ , \ a_y = \sqrt{\frac{m\omega}{2}}y + \frac{1}{\sqrt{2m\omega}}\frac{d}{dy} \tag{34}$$

Thus, simple argument based on separation of variables tells us that

$$\psi_{n_1 n_2}(x,y) = \psi_{n_1}(x)\psi_{n_2}(y) \tag{35}$$

Since both $\hat{x}$ and d/dx simultaneously commute with both $\hat{y}$ and d/dy, we obtain

$$\psi_{n_1 n_2}(x,y) = \sqrt{\frac{m\omega}{\pi n_1! n_2!}} e^{-\frac{m\omega(\hat{x}^2+\hat{y}^2)}{2}} \times \tag{36}$$

$$\times \left(\sqrt{2m\omega}\ \hat{x} - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial x}\right)^{n_1} \left(\sqrt{2m\omega}\ \hat{y} - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial y}\right)^{n_2} 1$$

and the energy of the above state is

$$E_{n_1 n_2} = \omega\left(n_1 + \frac{1}{2}\right) + \omega\left(n_2 + \frac{1}{2}\right) = \omega(n_1 + n_2 + 1) \tag{37}$$

It is, however, more natural to use the linear combinations of these operators given by

$$a_{++} = \frac{a_x^\dagger + i a_y^\dagger}{\sqrt{2}} \ , \ a_{+-} = \frac{a_x^\dagger - i a_y^\dagger}{\sqrt{2}} \tag{38}$$

$$a_{-+} = \frac{a_x + i a_y}{\sqrt{2}} \ , \ a_{--} = \frac{a_x - i a_y}{\sqrt{2}} \tag{39}$$

and, therefore

$$a_{++}^\dagger = a_{--} \ , \ a_{+-}^\dagger = a_{-+} \ , \ a_{-+}^\dagger = a_{+-} \ , \ a_{--}^\dagger = a_{++} \tag{40}$$

One can also show that they satisfy the following commutation relations:

$$[a_{-+}, a_{+-}] = [a_{--}, a_{++}] = 1 \tag{41}$$

$$[a_{+-}, a_{-+}] = [a_{++}, a_{--}] = -1 \tag{42}$$

$$[a_{++}, a_{-+}] = [a_{-+}, a_{++}] = [a_{+-}, a_{--}] = [a_{--}, a_{+-}] = 0 \tag{43}$$

$$[a_{++}, a_{+-}] = [a_{+-}, a_{++}] = [a_{-+}, a_{--}] = [a_{--}, a_{-+}] = 0 \tag{44}$$

$$[a_{++}, a_{++}] = [a_{+-}, a_{+-}] = [a_{-+}, a_{-+}] = [a_{--}, a_{--}] = 0 \tag{45}$$

Some of what we included is trivial (such as last line which basically says operators commute with themselves or second line which follows from first line). The reason we included "trivial" results is so that we have total of 16 commutation relations thus making sure we haven't missed anything. Finally, one can also show that

$$a_x = \frac{a_{-+} + a_{--}}{\sqrt{2}} \ , \ a_y = \frac{i(a_{--} - a_{-+})}{\sqrt{2}} \tag{46}$$

$$a_x^\dagger = \frac{a_{++} + a_{+-}}{\sqrt{2}} \ , \ a_y^\dagger = \frac{i(a_{+-} - a_{++})}{\sqrt{2}} \tag{47}$$

In order to express them in terms of derivatives in polar coordinates, we recall that

$$x = r\cos\theta \ , \ y = r\sin\theta \tag{48}$$

$$\frac{\partial}{\partial x} = -\frac{\sin\theta}{r} \ , \ \frac{\partial}{\partial y} = \frac{\cos\theta}{r} \tag{49}$$

and, therefore,

$$a_x^\dagger = \sqrt{\frac{m\omega}{2}}x - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial x} = \left(\sqrt{\frac{m\omega}{2}}r - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\cos\theta + \frac{1}{\sqrt{2m\omega}}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{50}$$

$$a_x = \sqrt{\frac{m\omega}{2}}x + \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial x} = \left(\sqrt{\frac{m\omega}{2}}r + \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\cos\theta - \frac{1}{\sqrt{2m\omega}}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{51}$$

$$a_y^\dagger = \sqrt{\frac{m\omega}{2}}y - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial y} = \left(\sqrt{\frac{m\omega}{2}}r - \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\cos\theta - \frac{1}{\sqrt{2m\omega}}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{52}$$

$$a_y = \sqrt{\frac{m\omega}{2}}y + \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial y} = \left(\sqrt{\frac{m\omega}{2}}r + \frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\cos\theta + \frac{1}{\sqrt{2m\omega}}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{53}$$

Thus,

$$a_{++} = \frac{a_x^\dagger + ia_y^\dagger}{\sqrt{2}} = \frac{e^{i\theta}}{2}\left(r\sqrt{m\omega} - \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial r} - \frac{i}{r\sqrt{m\omega}}\frac{\partial}{\partial\theta}\right) \tag{54}$$

$$a_{+-} = \frac{a_x^\dagger - ia_y^\dagger}{\sqrt{2}} = \frac{e^{-i\theta}}{2}\left(r\sqrt{m\omega} - \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial r} + \frac{i}{r\sqrt{m\omega}}\frac{\partial}{\partial\theta}\right) \tag{55}$$

$$a_{-+} = \frac{a_x + ia_y}{\sqrt{2}} = \frac{e^{i\theta}}{2}\left(r\sqrt{m\omega} + \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial r} + \frac{i}{r\sqrt{m\omega}}\frac{\partial}{\partial\theta}\right) \tag{56}$$

$$a_{--} = \frac{a_x - ia_y}{\sqrt{2}} = \frac{e^{-i\theta}}{2}\left(r\sqrt{m\omega} + \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial r} - \frac{i}{r\sqrt{m\omega}}\frac{\partial}{\partial\theta}\right) \tag{57}$$

We can now take linear combinations of the above expressions to obtain the following results:

$$re^{i\theta} = \frac{a_{++} + a_{-+}}{\sqrt{m\omega}} \tag{58}$$

$$re^{-i\theta} = \frac{a_{+-} + a_{--}}{\sqrt{m\omega}} \tag{59}$$

$$e^{i\theta}\frac{\partial}{\partial r} = \sqrt{m\omega}(a_{-+} - a_{++}) \tag{60}$$

$$e^{-i\theta}\frac{\partial}{\partial r} = \sqrt{m\omega}(a_{--} - a_{+-}) \tag{61}$$

The angular momentum is given by

$$L = xp_y - yp_x \tag{62}$$

which, in polar coordinates becomes

$$L = -i\frac{\partial}{\partial \theta} \tag{63}$$

By substituting

$$x = \frac{a_x + a_x^\dagger}{\sqrt{2m\omega}}\ ,\ p_x = \frac{-i(a_x - a_x^\dagger)}{\sqrt{2m\omega}} \tag{64}$$

$$y = \frac{a_y + a_y^\dagger}{\sqrt{2m\omega}}\ ,\ p_y = \frac{-i(a_y - a_y^\dagger)}{\sqrt{2m\omega}} \tag{65}$$

into Eq 62 we obtain

$$L = -i(a_x^\dagger a_y - a_y^\dagger a_x) \tag{66}$$

and by using Eq 46 and Eq 47, the above evaluates to

$$L = \frac{a_{++}a_{--} - a_{+-}a_{-+}}{2} \tag{67}$$

which, in combination with Eq 63 implies that

$$\frac{\partial}{\partial \theta} = \frac{i(a_{++}a_{--} - a_{+-}a_{-+})}{2} \tag{68}$$

Now, we can obtain the action of the operators given in Eq 56, 54, 57 and 55 in two different ways. On the one hand, we can look at the right hand sides, and compare them to Eq 63. This tells us that operators in Eq 56 and Eq 54 raise angular momentum by a unit, while the operators in Eq 57 and 55 lower it by a unit. On the other hand, we can look at the left hand sides of Eq ref128a, 54, 57 and 55 and, in each case, take their commutator with angular momentum, as described in Eq 67:

$$[L, a_{++}] = a_{++}\ ,\ [L, a_{+-}] = -a_{+-}\ ,\ [L, a_{-+}] = a_{-+}\ ,\ [L, a_{--}] = -a_{--} \tag{69}$$

From this we know that operators that add to angular momentum are a_{++} and a_{-+}, while operators that subtract from angular momentum are a_{+-} and a_{--}. On the other hand, by looking at Eq 38 and 39, it is evident that the operators that add to energy are a_{++} and a_{+-} while the operators that subtract from energy are a_{-+} and a_{--}. The way to remember what we just found out is that the "first sign" indicates the addition/subtraction of "unit" to/from energy, while the "second sign" indicates the addition/subtraction of "unit" to/from angular momentum. This is the reason behind our choice of notation. In terms of action on states, our results imply

$$a_{-+}|\psi_{nL}\rangle = A_{nL}|\psi_{n-1,L+1}\rangle\ ,\ a_{--}|\psi_{nL}\rangle = B_{nL}|\psi_{n-1,L-1}\rangle \tag{70}$$

$$a_{++}|\psi_{nL}\rangle = C_{nL}|\psi_{n+1,L+1}\rangle \ , \ a_{+-}|\psi_{nL}\rangle = D_{nL}|\psi_{n+1,L-1}\rangle \tag{71}$$

where the coefficients A_{nL}, B_{nL}, C_{nL} and D_{nL} will be computed later. We will also define *number operator* as follows:

$$\hat{n}_x = a_x^\dagger a_x \ , \ \hat{n}_y = a_y^\dagger a_y \ , \ \hat{n} = \hat{n}_x + \hat{n}_y \tag{72}$$

The operator $\hat{n}$ can be re-expressed in polar coordinates as

$$\hat{n} = a_{++}a_{-+} + a_{+-}a_{--} \tag{73}$$

Now, from the fact that $n = n_x + n_y$ where $n_x \geq 0$ and $n_y \geq 0$, it is obvious that $n = 0$ if and only if $n_x = n_y = 0$. Therefore, $n = 0$ state is unique; namely, $\psi(x,y) = \psi_0(x)\psi_0(y)$ where ψ_0 is 1-dimensional ground state. From symmetry, we know that $\psi_0(x)\psi_0(y)$ happens to have $L = 0$. In other words, we have ψ_{00} but we do *not* have ψ_{0L} for $L \neq 0$.

Now, suppose we were to have ψ_{1L}. If we were to *simultaneously* get zero by acting on it with $(a_x + ia_y)/\sqrt{2}$ and $(a_x - a_y)/\sqrt{2}$, then the linear combination of these two zeros would tell us that a_x and a_y would each send it to zero, thus making it a vacuum state. Since we know that ψ_{1L} is *not* a vacuum, we have shown by contradiction that *either* $(a_x + ia_y)/\sqrt{2}$ or $(a_x - ia_y)/\sqrt{2}$ would send it to something non-zero. *But* we know that both of them would send it to $n = 0$ state, and the *only* $n = 0$ state that is non-zero is the one that happens to have $L = 0$. Thus, the only two ways for us to accomplish it is to either have ψ_{11}, which would be sent to ψ_{00} via $(a_x - ia_y)/\sqrt{2}$, or to have $\psi_{1,-1}$, which would be sent to ψ_{00} via $(a_x + ia_y)/\sqrt{2}$.

Now suppose we have ψ_{2L}. By the same argument as before, either $(a_x + ia_y)/\sqrt{2}$ or $(a_x - ia_y)/\sqrt{2}$ should send it to some non-zero entity. But each of these two operators would send it to $n = 1$ state and, as we have just shown, the only $n = 1$ states are ψ_{11} and $\psi_{1,-1}$. If we are to arrive there from ψ_{2L} then either $L + 1 \in \{1, -1\}$ (in which case we arrive there through $(a_x + ia_y)/\sqrt{2}$) or $L - 1 \in \{1, -1\}$ (in which case we arrive there through $(a_x - ia_y)/\sqrt{2}$). The statement "either $L + 1 \in \{1, -1\}$ or $L - 1 \in \{1, -1\}$" is equivalent to the statement "either $L \in \{0, -2\}$ or $L \in \{2, 0\}$" which, in turn, is equivalent to the statement $L \in \{-2, 0, 2\}$.

We can now try to generalize it: for any given n, the allowed values of L are $\{-n, -n + 2, \cdots, n - 2, n\}$. In other words, they satisfy two conditions: first of all, $-n \leq L \leq n$ and, secondly, $n - L$ is even. Now, suppose it is true for a given n, and let us look at $n + 1$. As before, $\psi_{n+1,L}$ should be sent to non-zero entity by either $(a_x + ia_y)/\sqrt{2}$ or by $(a_x - ia_y)/\sqrt{2}$. In the former case, $L + 1 \in \{-n, -n + 2, \cdots, n - 2, n\}$ and in the latter case $L - 1 \in \{-n, -n + 2, \cdots, n - 2, n\}$. This means that, in the former case, $L \in \{-n-1, -n+1, \cdots, n-3, n-1\}$ and in the latter case $Lin\{-n+1, -n+3, \cdots, n-1, n+1\}$. Thus, if we combine both cases, we obtain $L \in \{-n - 1, -n + 1, \cdots, n - 1, n + 1\}$. Thus, the induction step works.

Let us now calculate A_{nL}, B_{nL}, C_{nL} and D_{nL}.

$$A_{nL}^2 = \langle\psi_{nL}|a_{+-}a_{-+}|\psi_{nL}\rangle \ , \ B_{nL} = |\psi_{nL}|a_{++}a_{--}|\psi_{nL}\rangle \tag{74}$$

$$C^2_{nL} = |\psi_{nL}|a_{--}a_{++}|\psi_{nL}\rangle \ , \ D^2_{nL} = \langle\psi_{nL}|a_{-+}a_{+-}|\psi_{nL}\rangle \tag{75}$$

By looking at Eq 66 it is easy to see that

$$a_{+-}a_{-+} = \frac{\hat{n} - \hat{L}}{2} \tag{76}$$

$$a_{++}a_{--} = \frac{\hat{n} + \hat{L}}{2} \tag{77}$$

By using Eq 41 and 42, we obtain

$$a_{--}a_{++} = 1 + \frac{\hat{n} + \hat{L}}{2} \tag{78}$$

$$a_{-+}a_{+-} = 1 + \frac{\hat{n} - \hat{L}}{2} \tag{79}$$

Therefore,

$$A^2_{nL} = \frac{n-L}{2} \ , \ B^2_{nL} = \frac{n+L}{2} \ , \ C^2_{nL} = 1 + \frac{n+L}{2} \ , \ D^2_{nL} = 1 + \frac{n-L}{2} \tag{80}$$

Let us now check consistency between equation 70 and 71. We can relabel n and L in such a way that any given equation only has n, L, $n-1$ and $L-1$ and does *not* have either $n+1$ or $L+1$:

$$a_{-+}|\psi_{n,L-1}\rangle = A_{n,L-1}|\psi_{n-1,L}\rangle \ , \ a_{--}|\psi_{nL}\rangle = B_{nL}|\psi_{n-1,L-1}\rangle \tag{81}$$

$$a_{++}|\psi_{n-1,L-1}\rangle = C_{n-1,L-1}|\psi_{nL}\rangle \ , \ a_{+-}|\psi_{n-1,L}\rangle = D_{n-1,L}|\psi_{n,L-1}\rangle \tag{82}$$

Therefore,

$$|\psi_{nL}\rangle = \frac{1}{C_{n-1,L-1}} a_{++}|\psi_{n-1,L-1}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}} a_{++}a_{--}|\psi_{nL}\rangle \tag{83}$$

$$|\psi_{n-1,L-1}\rangle = \frac{1}{B_{nL}} \frac{a_x - ia_y}{\sqrt{2}} |\psi_{nL}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}} a_{--}a_{++}|\psi_{n-1,L-1}\rangle \tag{84}$$

$$|\psi_{n-1,L}\rangle = \frac{1}{A_{n,L-1}} a_{-+}|\psi_{n,L-1}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}} a_{-+}a_{+-}|\psi_{n-1,L}\rangle \tag{85}$$

$$|\psi_{n,L-1}\rangle = \frac{1}{D_{n-1,L}} a_{+-}|\psi_{n-1,L}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}} a_{+-}a_{-+}|\psi_{n,L-1}\rangle \tag{86}$$

By using Eq 76, 77, 78 and 79, we obtain

$$|\psi_{nL}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}} \frac{\hat{n} + \hat{L}}{2} |\psi_{nL}\rangle \tag{87}$$

$$|\psi_{n-1,L-1}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}} \left(1 + \frac{\hat{n} + \hat{L}}{2}\right) |\psi_{n-1,L-1}\rangle \tag{88}$$

$$|\psi_{n-1,L}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}} \left(1 + \frac{\hat{n} - \hat{L}}{2}\right) |\psi_{n-1,L}\rangle \tag{89}$$

$$|\psi_{n,L-1}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}}\frac{\hat{n}-\hat{L}}{2}|\psi_{n,L-1}\rangle \tag{90}$$

Now, we have to be careful when converting operators into scalars. For example,

$$\hat{n}|\psi_{n-1,L}\rangle = (n-1)|\psi_{n-1,L}\rangle \neq n|\psi_{n-1,L}\rangle \tag{91}$$

It is crucial to keep track of where we have a hat and where we don't. Keeping this in mind, the above four equations become

$$|\psi_{nL}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}}\frac{n+L}{2}|\psi_{nL}\rangle \tag{92}$$

$$|\psi_{n-1,L-1}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}}\left(1+\frac{(n-1)+(L-1)}{2}\right)|\psi_{n-1,L-1}\rangle \tag{93}$$

$$|\psi_{n-1,L}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}}\left(1+\frac{(n-1)-L}{2}\right)|\psi_{n-1,L}\rangle \tag{94}$$

$$|\psi_{n,L-1}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}}\frac{n-(L-1)}{2}|\psi_{n,L-1}\rangle \tag{95}$$

which evaluate to

$$|\psi_{nL}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}}\frac{n+L}{2}|\psi_{nL}\rangle \tag{96}$$

$$|\psi_{n-1,L-1}\rangle = \frac{1}{B_{nL}C_{n-1,L-1}}\frac{n+L}{2}|\psi_{n-1,L-1}\rangle \tag{97}$$

$$|\psi_{n-1,L}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}}\frac{n-L+1}{2}|\psi_{n-1,L}\rangle \tag{98}$$

$$|\psi_{n,L-1}\rangle = \frac{1}{A_{n,L-1}D_{n-1,L}}\frac{n-L+1}{2}|\psi_{n,L-1}\rangle \tag{99}$$

Thus, the consistency requires that

$$B_{nL}C_{n-1,L-1} = \frac{n+L}{2}\ ,\ A_{n,L-1}D_{n-1,L} = \frac{n-L+1}{2} \tag{100}$$

Now, if we compare it to the definitions of A, B, C and D given in Eq 80, we will find that the above conditions indeed hold:

$$B_{nL}C_{n-1,L-1} = \sqrt{\frac{n+L}{2}}\sqrt{1+\frac{(n-1)+(L-1)}{2}} = \frac{n+L}{2} \tag{101}$$

$$A_{n,L-1}D_{n-1,L} = \sqrt{\frac{n-(L-1)}{2}}\sqrt{1+\frac{(n-1)-L}{2}} = \frac{n-L+1}{2} \tag{102}$$

4. a^n and $(a^\dagger)^n$ in 1D: Hand waving calculation

So far we have shown how to make a "single step" by using various operators that take us from one step to the other. One can easily repeat said "single step" a few times and thus make two steps, three steps, and so forth. But generalizing it to n steps is not as simple. The goal of this section is to carry out combinatoric arguments that would allow us to do that. In particular, we would like to compute the expression of the form $(kx + l\partial_x)^n$, where

$$k = \sqrt{2m\omega}\ ,\ l = -\frac{1}{\sqrt{2m\omega}} \tag{103}$$

Let us start with an example of $n = 4$. In order to see what happens combinatorically, let us try "more general" expression, $(kx_{a_1} + l\partial_{a_1}) \cdots (kx_{a_4} + l\partial_{a_4})$ and then substitute a "special case" of $a_1 = a_2 = a_3 = a_4 = 1$. We will use the commutation relation

$$[\partial_{a_i}, x_{a_j}] = \delta^{a_i}_{a_j} \tag{104}$$

in order to "push" all of x-s to the left and all of ∂-s to the right. After some straightforward, yet tedious, calculation, one finds that

$$\begin{gathered}(k_{a_1}x_{a_1} + l_{a_1}\partial_{a_1})(k_{a_2}x_{a_2} + l_{a_2}\partial_{a_2})(k_{a_3}x_{a_3} + l_{a_3}\partial_{a_3})(k_{a_4}x_{a_4} + l_{a_4}\partial_{a_4}) = \\
= k_{a_1}k_{a_2}k_{a_3}k_{a_4}x_{a_1}x_{a_2}x_{a_3}x_{a_4} + l_{a_1}k_{a_2}k_{a_3}k_{a_4}x_{a_2}x_{a_3}x_{a_4}\partial_{a_1} + k_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_1}x_{a_3}x_{a_4}\partial_{a_2} + \\
+ k_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_1}x_{a_2}x_{a_4}\partial_{a_3} + k_{a_1}k_{a_2}k_{a_3}l_{a_4}x_{a_1}x_{a_2}x_{a_3}\partial_{a_4} + k_{a_1}k_{a_2}l_{a_3}l_{a_4}x_{a_1}x_{a_2}\partial_{a_3}\partial_{a_4} + \\
+ k_{a_1}l_{a_2}k_{a_3}l_{a_4}x_{a_1}x_{a_3}\partial_{a_2}\partial_{a_4} + k_{a_1}l_{a_2}l_{a_3}k_{a_4}x_{a_1}x_{a_4}\partial_{a_2}\partial_{a_3} + l_{a_1}k_{a_2}k_{a_3}l_{a_4}x_{a_2}x_{a_3}\partial_{a_1}\partial_{a_4} + \\
+ l_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_2}x_{a_4}\partial_{a_1}\partial_{a_3} + l_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_3}x_{a_4}\partial_{a_1}\partial_{a_2} + k_{a_1}l_{a_2}l_{a_3}l_{a_4}x_{a_1}\partial_{a_2}\partial_{a_3}\partial_{a_4} + \\
+ l_{a_1}k_{a_2}l_{a_3}l_{a_4}x_{a_2}\partial_{a_1}\partial_{a_3}\partial_{a_4} + l_{a_1}l_{a_2}k_{a_3}l_{a_4}x_{a_3}\partial_{a_1}\partial_{a_2}\partial_{a_4} + l_{a_1}l_{a_2}l_{a_3}k_{a_4}x_{a_4}\partial_{a_1}\partial_{a_2}\partial_{a_3} + \\
+ l_{a_1}l_{a_2}l_{a_3}l_{a_4}\partial_{a_1}\partial_{a_2}\partial_{a_3}\partial_{a_4} + \delta^{a_1}_{a_2}l_{a_1}k_{a_2}k_{a_3}k_{a_4}x_{a_3}x_{a_4} + \delta^{a_1}_{a_2}l_{a_1}k_{a_2}k_{a_3}l_{a_4}x_{a_3}\partial_{a_4} + \\
+ \delta^{a_1}_{a_2}l_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_4}\partial_{a_3} + \delta^{a_1}_{a_2}l_{a_1}k_{a_2}l_{a_3}l_{a_4}\partial_{a_3}\partial_{a_4} + \delta^{a_1}_{a_3}l_{a_1}k_{a_2}k_{a_3}k_{a_4}x_{a_2}x_{a_4} + \\
+ \delta^{a_1}_{a_3}l_{a_1}k_{a_2}k_{a_3}l_{a_4}x_{a_2}\partial_{a_4} + \delta^{a_1}_{a_3}l_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_4}\partial_{a_2} + \delta^{a_1}_{a_3}l_{a_1}l_{a_2}k_{a_3}l_{a_4}\partial_{a_2}\partial_{a_4} + \\
+ \delta^{a_1}_{a_4}l_{a_1}k_{a_2}k_{a_3}k_{a_4}x_{a_2}x_{a_3} + \delta^{a_1}_{a_4}l_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_2}\partial_{a_3} + \delta^{a_1}_{a_4}l_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_3}\partial_{a_2} + \\
+ \delta^{a_1}_{a_4}l_{a_1}l_{a_2}l_{a_3}k_{a_4}\partial_{a_2}\partial_{a_3} + \delta^{a_2}_{a_3}k_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_1}x_{a_4} + \delta^{a_2}_{a_3}k_{a_1}l_{a_2}k_{a_3}l_{a_4}x_{a_1}\partial_{a_4} + \\
+ \delta^{a_2}_{a_3}l_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_4}\partial_{a_1} + \delta^{a_2}_{a_3}l_{a_1}l_{a_2}k_{a_3}l_{a_4}\partial_{a_1}\partial_{a_4} + \delta^{a_2}_{a_4}k_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_1}x_{a_3} + \\
+ \delta^{a_2}_{a_4}k_{a_1}l_{a_2}l_{a_3}k_{a_4}x_{a_1}\partial_{a_3} + \delta^{a_2}_{a_4}l_{a_1}l_{a_2}k_{a_3}k_{a_4}x_{a_3}\partial_{a_1} + \delta^{a_2}_{a_4}l_{a_1}l_{a_2}l_{a_3}k_{a_4}\partial_{a_1}\partial_{a_3} + \\
+ \delta^{a_3}_{a_4}k_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_1}x_{a_2} + \delta^{a_3}_{a_4}k_{a_1}l_{a_2}l_{a_3}k_{a_4}x_{a_1}\partial_{a_2} + \delta^{a_3}_{a_4}l_{a_1}k_{a_2}l_{a_3}k_{a_4}x_{a_2}\partial_{a_1} + \\
+ \delta^{a_3}_{a_4}l_{a_1}l_{a_2}l_{a_3}k_{a_4}\partial_{a_1}\partial_{a_2} + \delta^{a_1}_{a_2}\delta^{a_3}_{a_4}l_{a_1}k_{a_2}l_{a_3}k_{a_4} + \delta^{a_1}_{a_3}\delta^{a_2}_{a_4}l_{a_1}l_{a_2}k_{a_3}k_{a_4} + \delta^{a_1}_{a_4}\delta^{a_2}_{a_3}l_{a_1}l_{a_2}k_{a_3}k_{a_4}\end{gathered} \tag{105}$$

If we now set

$$a_1 = a_2 = a_3 = a_4 = 1 \tag{106}$$

the above evaluates to

$$(kx_1 + l\partial_1)^4 = k_1^4x_1^4 + 4k_1^3l_1x_1^3\partial_1 + 6k_1^2l_1^2x_1^2\partial_1^2 + 4k_1l_1^3x_1\partial_1^3 + l_1^4\partial_1^4 +$$

$$+6k_1^3l_1x^2 + 12k_1^2l_1^2x_1\partial_1 + 6k_1l_1^3\partial_1^2 + 3k_1^2l_1^2 \tag{107}$$

In order to be able to generalize the above from $n = 4$ to general n, we would have to first look closely at $n = 4$ case and see exactly how the above coefficients were produced. First of all, it is clear that k comes for a ride with x while l comes for a ride with ∂, so for simplicity we will drop k-s and l-s. Now, let us look at the numerical coefficients of each term:

1. There is only one way of producing x_1^4, namely, $x_{a_1}x_{a_2}x_{a_3}x_{a_4}$, which is why it comes with coefficient of 1.

2. In order to produce $x_1^3\partial_1$ we can "select" $i \in \{1, 2, 3, 4\}$ and then write $x_{a_1} \cdots x_{a_{i-1}}x_{a_{i+1}} \cdots x_{a_4}\partial_{a_i}$. Since there are four choices of $i \in \{1, 2, 3, 4\}$, the term $x_1^3\partial_1$ comes with coefficient 4.

3. In order to produce $x_1^2\partial_1^2$ we have to select $\{i_1, i_2\} \in \{1, 2, 3, 4\}$ such that $i_1 < i_2$. This will "dictate" the values of i_3 and i_4 if we demand that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$ and $i_3 < i_4$. Then we use $x_{i_1}x_{i_2}\partial_{i_3}\partial_{i_4}$ to produce $x_1^2\partial_1^2$. There are 6 choices of $i_1 < i_2$, and for each such choice there is a unique choice of $i_3 < i_4$. Thus, $x_1^2\partial_1^2$ comes with the coefficient 6.

4. In order to produce $x_1\partial_1^3$ we can "select" $i \in \{1, 2, 3, 4\}$ and then write $x_{a_i}\partial_{a_1} \cdots \partial_{a_{i-1}}\partial_{a_{i+1}} \cdots \partial_{a_4}$. Since there are four choices of $i \in \{1, 2, 3, 4\}$, the term $x_1^3\partial_1$ comes with coefficient 4.

5. There is only one way of producing ∂_1^4, namely, $\partial_{a_1}\partial_{a_2}\partial_{a_3}\partial_{a_4}$, which is why it comes with coefficient of 1.

6. In order to produce x_1^2 we need to start off from three x-s, one ∂, and then "contract" one of the x-s with the ∂ by replacing them with $\delta^{a_i}_{a_j}$, thus leaving two un-contracted x-s. For example, if we decided to contract ∂_{a_1} with x_{a_2}, we would obtain $\delta^{a_1}_{a_2}x_{a_3}x_{a_4}$. Now, in order to compute the number of all possible ways of doing that, we can utilize the following algorithm. First, we will select $i_1 < i_2$ and "make up our mind" that we want to "contract" $\partial_{a_{i_1}}$ with $x_{a_{i_2}}$. After that, we have unique way of specifying $i_3 < i_4$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$ and including un-contracted $x_{a_{i_3}}$ and $x_{a_{i_4}}$, thus producing $\delta^{a_{i_1}}_{a_{i_2}}x_{a_{i_3}}x_{a_{i_4}}$. Finally, when we set $a_1 = a_2 = a_3 = a_4$, the coefficient of $\delta^{a_{i_1}}a_{i_2}$ will be replaced by 1, thus producing x_1^2. Since there are 6 ways of specifying $i_1 < i_2$ and for each such way there is a unique way of specifying $i_3 < i_4$, we see that x_1^2 comes with coefficient of 6.

7. In order to produce $x_1\partial_1$ we need to start off from two x-s, two ∂-s, and then "contract" one of the x-s with one of ∂-s by replacing them with $\delta^{a_i}_{a_j}$, thus leaving one un-contracted x and one un-contracted ∂. For example, if we decided to contract ∂_{a_1} with x_{a_2}, we would obtain either $\delta^{a_1}_{a_2}x_{a_3}\partial_{a_4}$ or $\delta^{a_1}_{a_2}x_{a_4}\partial_{a_3}$. Now, in order to compute the number of all possible ways of doing that, we can utilize the following algorithm. First, we will select $i_1 < i_2$ and "make up our mind" that we want to "contract" $\partial_{a_{i_1}}$ with $x_{a_{i_2}}$ (in other words, we already know that $\delta^{a_{i_1}}_{a_{i_2}}$ will be one of the factors). After that, we have unique way of specifying $i_3 < i_4$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Then we have *two* choices:either include $x_{a_{i_3}}\partial_{a_{i_4}}$ or include $x_{a_{i_4}}\partial_{a_{i_3}}$. The former would result in $\delta^{a_{i_1}}_{a_{i_2}}x_{a_{i_3}}\partial_{a_{i_4}}$ and the latter with $\delta^{a_{i_1}}_{a_{i_2}}x_{a_{i_4}}\partial_{a_{i_3}}$. Finally, once we set $a_1 = a_2 = a_3 = a_4$, all of the coefficients of $\delta^{a_i}_{a_j}$ will be replaced by 1, thus we obtain $x_1\partial_1$. Since we have 6 choices of $i_1 < i_2$, 1 choice of $i_3 < i_4$ and 2 choices between $x_{a_{i_3}}\partial_{a_{i_4}}$ and $x_{a_{i_4}}\partial_{a_{i_3}}$, the total number of choices is $6 \times 1 \times 2 = 12$, hence $x_1\partial_1$ comes with coefficient 12.

8. In order to produce ∂_1^2 we need to start off from three ∂-s, one x, and then "contract" one of the ∂-s with the x by replacing them with $\delta^{a_i}_{a_j}$, thus leaving two un-contracted ∂-s. For example, if we decided to contract ∂_{a_1} with x_{a_2}, we would obtain $\delta^{a_1}_{a_2}\partial_{a_3}\partial_{a_4}$. Now, in order to compute the number of all possible ways of doing that, we can utilize the following algorithm. First, we will select $i_1 < i_2$ and "make up our mind" that we want to "contract" $\partial_{a_{i_1}}$ with $x_{a_{i_2}}$. After that, we have unique way of specifying $i_3 < i_4$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$ and including un-contracted $\partial_{a_{i_3}}$ and $\partial_{a_{i_4}}$, thus producing $\delta^{a_{i_1}}_{a_{i_2}}\partial_{a_{i_3}}\partial_{a_{i_4}}$. Finally, when we set $a_1 = a_2 = a_3 = a_4$, the coefficient of $\delta^{a_{i_1}}a_{i_2}$ will be replaced by 1, thus producing ∂_1^2. Since there are 6 ways of specifying $i_1 < i_2$ and for each such way there is a unique way of specifying $i_3 < i_4$, we see that ∂_1^2 comes with coefficient of 6.

9.In order to produce a scalar, we need to contract all four indexes. We first specify $i_1 < i_2$. Then we have a unique way of specifying $i_3 < i_4$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Then we contract i_1 with i_2 and i_3 with i_4, producing $\delta^{a_{i_1}}_{a_{i_2}}\delta^{a_{i_3}}_{a_{i_4}}$. Then, by setting $a_1 = a_2 = a_3 = a_4$ this produces 1. Now, since there are 6 ways of selecting $i_1 < i_2$ and for each of those there is 1 way of selecting $i_3 < i_4$, on the first glance one might wrongly say that coefficient is $6 \times 1 = 6$. The reason the actual coefficient is 3 is that the situation $(i_1, i_2, i_3, i_4) = (1, 2, 3, 4)$ is equivalent to the situation $(i_1, i_2, i_3, i_4) = (3, 4, 1, 2)$ since both would produce $\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}$. In other words, we have 2 contractions and, therefore, 2! ways of permuting said contractions. Thus, we have to divide our result by 2!, which would produce $6 \times 1/2! = 3$. The latter, in fact, coincides with the actual coefficient we have obtained.

Let us now use the above examples to generalize from $(x_1 + \partial_1)^4$ to $(x_1 + \partial_1)^n$. Suppose we are interested in the coefficient next to $x^A\partial^B$. If $A + B < n$, we need to "get rid of" all the rest of x-s and ∂-s by means of contractions. This means that $n - A - B$ should be even, and also that the number of x-s and ∂-s we are getting rid of should be the same. In other words, we start out with the number of x-s being $A + C$, the number of ∂-s being $B + C$ and then introducing C contractions in order to bring down the number of x-s from $A + C$ to A, and number of ∂-s from $B + C$ to B. This means that

$$A + B + 2C = n \tag{108}$$

Now, the crucial thing is to count the number of ways we can accomplish the above. By looking at examples earlier, it is apparent that "a way" amounts to our specification which of the indexes correspond to x, which to ∂, and which are contracted with which. Going back to $n = 4$, if I say that index 1 is contracted with index 3, index 2 corresponds to x and index 4 corresponds to ∂, then we can immediately read off that we have started with $\partial_{a_1}x_{a_2}x_{a_3}\partial_{a_4}$ and produced $\delta^{a_1}_{a_3}x_{a_2}\partial_{a_4}$. The way we know that we had $\partial_{a_1}\cdots x_{a_3}\cdots$ as opposed to $x_{a_1}\cdots\partial_{a_3}\cdots$ is that we know from our earlier examples that ∂ and x contract only if ∂ has smaller index than x does. This means that we do not have to specify how we fill the contractions. As long as we say what contractions *are* by listing pairs of numbers, there will only be *one* way of filling them with x-s and ∂-s. On the other hand, among non-contracted indexes x-s and ∂-s can be distributed in any way we choose. Thus, while continuing to insist that 1 is contracted with 3, we could also say that 2 is occupied by ∂ and 4 by x. Then we would produce $\delta^{a_1}_{a_3}x_4\partial_2$. Notably, ∂ would still stay at the right of x: after all, our entire goal is to push all the ∂-s to the right. But, at the same time, it would have smaller index. On the other hand, we *do* have a constraint speficying the *total number* of non-contracted x-s and the *total number* of non-contracted ∂-s: in case of this example these are $A = 1$ and $B = 1$, respectively. So, going back to the general case, our strategy involves the following steps:

1. Select the choice of C contractions
2. Distribute A x-s and B ∂-s among $n - 2C$ of non-contracted cells

Let us specify each new contraction one by one. So, the number of choices of first contraction is $n(n-1)/2$, the number of choices of second contraction is $(n-2)(n-3)/2$, and when we reach the last contraction, we would have $(n+2-2C)(n+1-2C)/2$ options. Furthermore, by looking at part 9, we see that "first" specifying $\delta^{a_1}_{a_2}$ and "after that" specifying $\delta^{a_3}_{a_4}$ is the same as doing it in reverse order. Thus, we have to divide that product by $C!$. Therefore,

$$\sharp\{\text{ways to contract}\} = \frac{1}{C!}\frac{n(n-1)}{2}\cdots\frac{(n+2-2C)(n+1-2C)}{2} = \frac{n!}{2^C C!(n-2C)!} \tag{109}$$

Now, once we have established which elements are contracted, we have to decide which of the non-contracted elements will be occupied by x and which by ∂. At this point, we have $n - 2C$ non-contracted "cells" left, and we have A x-s and B ∂-s to distribute among them. It is easy to see that the number of ways of doing so is

$$\sharp\{\text{non} - \text{contracted choices}\} = \frac{(n-2C)!}{A!B!} \tag{110}$$

Therefore, the total number of ways of producing $x^A\partial^B$ is

$$\sharp\{\text{complete prescriptions}\} = \frac{n!}{2^C C!(n-2C)!}\frac{(n-2C)!}{A!B!} = \frac{n!}{2^C A!B!C!} \tag{111}$$

Now, when we contract ∂ with x, we still retain l that came with ∂ and k that came with x. Thus, despite the fact that the number of non-contracted x-s is A, the number of k-s is $A + C$. Likewise, despite the fact that the number of non-contracted ∂-s is B, the number of l-s is $B + C$. This, together with Eq 108, tells us that

$$\left(kx + l\frac{d}{dx}\right)^n = \sum_{A+B+2C=n} \frac{n!k^{A+C}l^{B+C}}{2^C A!B!C!} x^A \left(\frac{d}{dx}\right)^B \tag{112}$$

Now, Eq 108 allows us to remove B by replacing it with

$$B = n - 2C - A \tag{113}$$

This, however, immediately tells us that

$$A < n - 2C \tag{114}$$

which, in turn, tells us that $C \leq n/2$. Since C is integer, this means that

$$C \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{115}$$

Thus, we obtain

$$\left(kx + l\frac{d}{dx}\right)^n = n! \sum_{C=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{A=0}^{n-2C} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx}\right)^{n-2C-A} \tag{116}$$

where we have pulled out $n!$ outside of the sum, as constant factor.

5. a^n and $(a^\dagger)^n$ in 1D: Proof by induction

So far we have made an "intuitive" argument in favor of Eq 116. Now, in order to know for sure that our "intuition" didn't let us down, we now have to verify Eq 116 by induction. Let us rewrite Eq 116 as

$$\left(kx + l\frac{d}{dx}\right)^n = n! \sum_{(A,C)\in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx}\right)^{n-2C-A} \tag{117}$$

where

$$S_n = \left\{ (A,C) \middle| A \in \mathbb{N}, C \in \mathbb{N}, 0 \leq C \leq \left\lfloor \frac{n}{2} \right\rfloor, 0 \leq A \leq n-2C \right\} \tag{118}$$

Since the combinatorial arguments we were using are quite hand waving, it is instructive to prove our result by induction. In other words, we will act with $kx + ld/dx$ on the right hand side and show that the result will be the same, up to replacing n with $n+1$. Let us first act with d/dx. It is easy to see that

$$\frac{d}{dx}\frac{x^A}{A!} = \frac{x^{A-1}}{(A-1)!} + \frac{x^A}{A!}\frac{d}{dx}\ ,\ A \geq 1 \tag{119}$$

However, since $(-1)!$ is undefined, the above only holds for $A \geq 1$. On the other hand, for $A = 0$ one trivially has

$$\frac{d}{dx}\frac{x^0}{0!} = \frac{d}{dx} = \frac{x^0}{0!}\frac{d}{dx} \tag{120}$$

Therefore,

$$\begin{aligned} &\frac{d}{dx} \sum_{(A,C)\in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx}\right)^{n-2C-A} = \\ &= \sum_{A\geq 1;(A,C)\in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! (A-1)! (n-2C-A)!} x^{A-1} \left(\frac{d}{dx}\right)^{n-2C-A} + \\ &+ \sum_{(A,C)\in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx}\right)^{n+1-2C-A} \end{aligned} \tag{121}$$

Now, as far as the first step is concerned, we can re-label A by introducing

$$A' = A - 1 \tag{122}$$

Now, we are hoping to claim that the induction step works. Thus, we are trying to replace n with $n+1$. In order to do it, we re-label C as well via

$$C' = C + 1 \tag{123}$$

which results in

$$x^{A-1}\left(\frac{d}{dx}\right)^{n-2C-A} = x^{A'}\left(\frac{d}{dx}\right)^{n+1-2C'-A'} \tag{124}$$

At the same time, the coefficients still have A and C, so it would take a little bit of algebra to show that we would, indeed, obtain the desired result. Before we do that, however, let us address a bit more serious issue: making sure that we will end up summing over $(A', C') \in S_{n+1}$. We can re-express the condition given in the sum of the first term on the right hand side in the following way:

$$\left\{\begin{array}{l} A \geq 1 \\ (A, C) \in S_n \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} A \geq 1 \\ 0 \leq C \leq \lfloor \frac{n}{2} \rfloor \\ 0 \leq A \leq n - 2C \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n}{2} \rfloor \end{array}\right\} \tag{125}$$

Apart from that, we also see that

$$1 \leq A \leq n - 2C \Longrightarrow 1 \leq n - 2c \Longrightarrow C \leq \frac{n-1}{2} \Longrightarrow C \leq \left\lfloor \frac{n-1}{2} \right\rfloor \tag{126}$$

This shows that

$$\left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n}{2} \rfloor \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n-1}{2} \rfloor \end{array}\right\} \tag{127}$$

At the same time, the fact that

$$0 \leq C \leq \left\lfloor \frac{n-1}{2} \right\rfloor \Longrightarrow 0 \leq C \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{128}$$

shows that

$$\left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n-1}{2} \rfloor \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n}{2} \rfloor \end{array}\right\} \tag{129}$$

Therefore, 127 and 129 together imply that

$$\left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n}{2} \rfloor \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n-1}{2} \rfloor \end{array}\right\} \tag{130}$$

This, together with Eq 125 implies that

$$\left\{\begin{array}{l} A \geq 1 \\ (A, C) \in S_n \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} 1 \leq A \leq n - 2C \\ 0 \leq C \leq \lfloor \frac{n-1}{2} \rfloor \end{array}\right\} \tag{131}$$

Now, we know from Eq 122 and 123 that

$$A' = A - 1\ ,\ C' = C + 1 \tag{132}$$

Therefore,

$$1 \leq A \leq n-2C \Longleftrightarrow 0 \leq A' \leq n-1-2C \Longleftrightarrow 0 \leq A' \leq n+1-2C' \tag{133}$$

where we have used Eq 122 during the first step and Eq 123 during the second step. Furthermore, by using

$$\left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-1}{2} + 1 \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor \tag{134}$$

we have

$$0 \leq C \leq \left\lfloor \frac{n-1}{2} \right\rfloor \Longleftrightarrow 1 \leq C' \leq \left\lfloor \frac{n+1}{2} \right\rfloor \tag{135}$$

Now, let us keep A and C on the left hand side of Eq 131 while, at the same time, use Eq 133 and 135 in order to convert everything to A' and C' on right hand side. Thus, we obtain

$$\left\{ \begin{array}{l} A \geq 1 \\ (A, C) \in S_n \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} 0 \leq A' \leq n+1-2C' \\ 1 \leq C' \leq \lfloor \frac{n+1}{2} \rfloor \end{array} \right\} \tag{136}$$

By comparing it to the definition of S_{n+1} (which can be obtained by replacing n with $n+1$ in Eq 118), we can rewrite Eq 136 as

$$\left\{ \begin{array}{l} A \geq 1 \\ (A, C) \in S_n \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} C' \geq 1 \\ (A', C') \in S_{n+1} \end{array} \right\} \tag{137}$$

So, in order to obtain the sum over S_{n+1}, we have to make sure that summands are equal to 0 for $C = 0$, which would allow us to include $C = 0$ terms in the sum without any consequences. To remind the reader, all of the above manipulations were pertaining to *first* term on the right hand side of Eq 121. That term has $1/C!$ in it, which is the same as $1/(C'-1)!$. So we can re-express it as $C'/C'!$, and then note that it is equal to zero for $C' = 0$, which would allow us to include $C' = 0$ term. More precisely, what we will do is the following:

$$\sum_{A \geq 1; (A,C) \in S_n} \frac{\cdots}{C!} = \sum_{A \geq 1; (A,C) \in S_n} \frac{(C+1)(\cdots)}{(C+1)!} = \sum_{C' \geq 1 \; (A',C') \in S_{n+1}} \frac{C'(\cdots)}{C'!} = \sum_{(A',C') \in S_{n+1}} \frac{C'(\cdots)}{C'!} \tag{138}$$

The first step holds for both zero and non-zero C. The second step is merely conversion C into C' as well as re-writing the condition of the sum by means of Eq 137. The third step amounts to dropping $C' \geq 1$ from the condition under the sum since we note that the terms corresponding to $C' = 0$ at the third expression are zero anyway. This will lead to the last expression which is the sum over S_{n+1} as desired. Once this is established, the conversion of the summands of the first term on the right hand side of Eq 137 boils down to simple substitution which, after some trivial algebra, produces

$$\sum_{A \geq 1; (A,C) \in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C!(A-1)!(n-2C-A)!} x^{A-1} \left(\frac{d}{dx} \right)^{n-2C-A} =$$

$$= \sum_{(A',C') \in S_{n+1}} \frac{2C'}{l} \frac{k^{A'+C'} l^{n+1-A'-C'}}{2^{C'} C'! A'! (n+1-2C'-A')!} x^{A'} \left(\frac{d}{dx} \right)^{n+1-2C'-A'} \tag{139}$$

Let us now move to the second term on the right hand side of Eq 137. This time, the powers of x and d/dx match the desired ones for $n+1$. So there is no need to replace A or C with anything else. However, we would like to change $(n-2C-A)!$ in denominator into $(n+1-2C-A)!$. This we will do by using

$$\frac{1}{(n-2C-A)!} = \frac{n+1-2C-A}{(n+1-2C-A)!} \tag{140}$$

Now, if we show that the right hand side of Eq 140 is equal to zero for $(A,C) \in S_{n+1} \setminus S_n$, then, keeping in mind that $S_n \subset S_{n+1}$, we would be able to replace the sum over S_n with the one over S_{n+1}, as long as we replace left hand side of Eq 140 with right hand side whenever it occurs. Now, in order for (A,C) to be an element of $S_{n+1} \setminus S_n$ we have to assume that it is an element of S_{n+1} and then find a way of "disqualifying" it from S_n. From Eq 118, we see that there are two ways of doing "disqualifying" part: either $C > \lfloor n/2 \rfloor$ or $A > n-2C$. We will break it into two separate cases.

Case 1: $(A,C) \in S_{n+1}$ but $C > \lfloor n/2 \rfloor$.

Since $(A,C) \in S_{n+1}$, we know that $C \leq \lfloor (n+1)/2 \rfloor$. Thus, the assumption that $C > \lfloor n/2 \rfloor$ implies that $\lfloor n/2 \rfloor < \lfloor (n+1)/2 \rfloor$ and, therefore, n is odd. Thus, $n+1$ is even and $C = (n+1)/2$ holds exactly. This means that $n+1-2C = 0$. Now, the fact that $(A,C) \in S_{n+1}$ implies that $A \leq n+1-2C$. This, together with $n+1-2C = 0$ implies that $A = 0$ holds exactly. Therefore, $(n+1-2C-A)/(n+1-2C-A)! = (0-0)/(0-0)! = 0/1 = 0$.

Case 2: $(A,C) \in S_{n+1}$ but $A > n-2C$.

Since $(A,C) \in S_{n+1}$, we know that $A \leq n+1-2C$. In combination with assumption $A > n-2C$ this implies exact equality $A = n+1-2C$. Therefore, $n+1-2C-A = 0$ and, therefore, $(n+1-2C-A)(n+1-2C-A)! = 0/0! = 0/1 = 0$.

The combination of above two cases implies that

$$\forall (A,C) \in S_{n+1} \setminus S_n \left(\frac{n+1-2C-A}{(n+1-2C-A)!} = 0 \right) \tag{141}$$

This means that if we substitute Eq 140 into the last term on the right hand side of Eq 121,we can freely replace S_n with S_{n+1} under the sum, which lead to

$$\sum_{(A,C)\in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx} \right)^{n+1-A-C} = \tag{142}$$

$$= \sum_{(A,C)\in S_{n+1}} \frac{n+1-2C-A}{l} \frac{k^{A+C} l^{n+1-A-C}}{2^C C! A! (n+1-2C-A)!} x^A \left(\frac{d}{dx} \right)^{n+1-2C-A}$$

We will now change the notation in Eq 139: we will "drop" the "prime" signs while, at the same time, treating A and C the same way we were treating A' and C': in other words, $(A',C') \in S_{n+1}$ will be replaced by $(A,C) \in S_{n+1}$, $2C'/l$ will be replaced by $2C/l$, and so forth. By substitutting Eq 142, together with the "relabeled" version of Eq 139, on the right hand side of Eq 121 we obtain

$$\frac{d}{dx}\sum_{(A,C)\in S_n}\frac{k^{A+C}l^{n-A-C}}{2^C C!A!(n-2C-A)!}x^A\left(\frac{d}{dx}\right)^{n-2C-A}= \tag{143}$$

$$=\sum_{(A,C)\in S_{n+1}}\frac{n+1-A}{l}\frac{k^{A+C}l^{n+1-A-C}}{2^C C!A!(n+1-2C-A)!}x^A\left(\frac{d}{dx}\right)^{n+1-2C-A}$$

So far we have found the outcome of the product of d/dx with the sum in question. Since our final goal is to compute the action of $kx+ld/dx$, our next step is to evaluate the product of x with that sum,

$$x\sum_{(A,C)\in S_n}\frac{k^{A+C}l^{n-A-C}}{2^C C!A!(n-2C-A)!}x^A\left(\frac{d}{dx}\right)^{n-2C-A}=$$

$$=\sum_{(A,C)\in S_n}\frac{k^{A+C}l^{n-A-C}}{2^C C!A!(n-2C-A)!}x^A\left(\frac{d}{dx}\right)^{n+1-2C-A} \tag{144}$$

By defining

$$A''=A+1 \tag{145}$$

this becomes

$$x\sum_{(A,C)\in S_n}\frac{k^{A+C}l^{n-A-C}}{2^C C!A!(n-2C-A)!}x^A\left(\frac{d}{dx}\right)^{n-2C-A}=$$

$$=\sum_{(A''-1,C)\in S_n}\frac{A''}{k}\frac{k^{A''+C}l^{n+1-A''-C}}{2^C C!A''!(n+1-2C-A'')!}x^{A''}\left(\frac{d}{dx}\right)^{n+1-2C-A''} \tag{146}$$

Once again, our goal is to replace $(A''-1,C)\in S_n$ with $(A'',C)\in S_{n+1}$. First, we note that

$$(A''-1,C)\in S_n \Longleftrightarrow \left\{\begin{array}{l} 0\le A''-1\le n-2C \\ 0\le C\le \lfloor\frac{n}{2}\rfloor \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} 1\le A''\le n+1-2C \\ 0\le C\le \lfloor n/2\rfloor \end{array}\right\} \tag{147}$$

and

$$(A'',C)\in S_{n+1} \Longleftrightarrow \left\{\begin{array}{l} 0\le A''\le n+1-2C \\ 0\le C\le \lfloor\frac{n+1}{2}\rfloor \end{array}\right\} \tag{148}$$

which tells us that

$$(A''-1,C)\in S_n \Longrightarrow (A'',C)\in S_{n+1} \tag{149}$$

Therefore, our only concern is a situation when $(A'',C)\in S_{n+1}$ holds but $(A''-1,C)\in S_n$ does not. We would like to show that in this case the summand is zero. There are only two ways of "disqualifying" $(A''-1,C)$ from S_n: either $A''-1$ should be outside the $\{0,\cdots,n\}$ range, or else C should be greater than $\lfloor n/2\rfloor$. We will consider these two cases separately.

Case 3: $(A'',C)\in S_{n+1}$ but $A''-1$ is outside of $\{0,\cdots,n\}$ range.

Since $(A'', C) \in S_{n+1}$, we know that $A'' \in \{0, \cdots, n+1\}$. Thus, we know that $A''-1 \leq n$. Therefore, the only way for $A''-1$ *not to be* part of $\{0, \cdots, n\}$ is to have $A''=0$ and $A''-1=-1$. This, however, will imply that $A''/A''! = 0/0! = 0$. Since the summand includes multiplication by $A''/A''!$, this means that it is zero.

Case 4: $(A'', C) \in S_{n+1}$ but $C > \lfloor n/2 \rfloor$.

The fact that $(A'', C) \in S_{n+1}$ implies that $C \leq \lfloor (n+1)/2 \rfloor$. This, in combination with the assumption $C > \lfloor n/2 \rfloor$ implies that $C = \lfloor (n+1)/2 \rfloor$ holds exactly. This also implies that $\lfloor (n+1)/2 \rfloor > \lfloor n/2 \rfloor$, which is equivalent to saying that n is odd. The fact that n is odd implies that $\lfloor (n+1)/2 \rfloor = (n+1)/2$ and, therefore, $C = (n+1)/2$. Now, the fact that $(A'', C) \in S_{n+1}$ implies that $0 \leq A'' \leq n+1-2C$. By substituting $C = (n+1)/2$, we obtain $0 \leq A'' \leq 0$ which means that $A''=0$. Therefore, $A''/A''! = 0/0! = 0$. So the fact that summand includes $A''/A''!$ implies that summand is zero.

Thus, the combination of Case 3 and Case 4 tells us that, whenever $(A'', C) \in S_{n+1}$ holds while $(A''-1, C) \in S_n$ does not, the corresponding summand is equal to zero, as long as the summands are expressed in the form that includes $A''/A''!$. Besides, we have shown earlier that if $(A''-1, C) \in S_n$ holds, then $(A'', C) \in S_{n+1}$ holds as well. These two statements together imply that the sum over $(A''-1, C) \in S_n$ can be replaced with the sum over $(A'', C) \in S_{n+1}$, provided that summands contain $A''/A''!$. Since the right hand side of Eq 146, indeed, contains $A''/A''!$, we can rewrite Eq 146 as

$$x \sum_{(A,C) \in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx} \right)^{n-2C-A} =$$

$$= \sum_{(A'',C) \in S_{n+1}} \frac{A''}{k} \frac{k^{A''+C} l^{n+1-A''-C}}{2^C C! A''! (n+1-2C-A'')!} x^{A''} \left(\frac{d}{dx} \right)^{n+1-2C-A''} \tag{150}$$

We will now re-label the above equation by simply dropping the "double prime" sign while treating A the same exact way we were treating A. Thus, $(A'', C) \in S_{n+1}$ will be replaced with $(A, C) \in S_{n+1}$, $k^{A''+C}$ will be replaced with k^{A+C}, and so forth. If we combine the "relabeled" version of Eq 150 with the Eq 143, we obtain

$$\left(kx + l\frac{d}{dx} \right) \sum_{(A,C) \in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx} \right)^{n-2C-A} =$$

$$= (n+1) \sum_{(A,C) \in S_{n+1}} \frac{k^{A+C} l^{n+1-A-C}}{2^C C! A! (n+1-A-C)!} x^A \left(\frac{d}{dx} \right)^{n+1-2C-A} \tag{151}$$

Finally, if we remember to include $n!$ outside the sum in Eq 117, we obtain

$$\left(kx + l\frac{d}{dx} \right) n! \sum_{(A,C) \in S_n} \frac{k^{A+C} l^{n-A-C}}{2^C C! A! (n-2C-A)!} x^A \left(\frac{d}{dx} \right)^{n-2C-A} =$$

$$= (n+1)! \sum_{(A,C) \in S_{n+1}} \frac{k^{A+C} l^{n+1-A-C}}{2^C C! A! (n+1-A-C)!} x^A \left(\frac{d}{dx} \right)^{n+1-2C-A} \tag{152}$$

which ultimately proves that induction step holds. It is trivial to show that Eq 117 holds for $n = 0$. So, by induction, it holds for all n.

6. General excited state in 1D

Now that we have proven the relevant operator equation, obtaining wave function is straightforward. We substitute Eq 116 into Eq 32. Since Eq 116 will act on 1, all the terms with non-zero power of d/dx will drop out. This means that the only terms that remain are the ones for which $n - 2C - A = 0$. Therefore, we can replace A with $n - 2C$. Thus, we obtain

$$\left(k\hat{x} + l\frac{d}{dx}\right)^n 1 = n! \sum_{C=0}^{\lfloor \frac{n}{2} \rfloor} \frac{k^{n-C} l^C}{2^C C!(n-2C)!} x^{n-2C} \tag{153}$$

If we now substitute it into Eq 32, we obtain

$$\psi_n(x) = \left[\frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega x^2/2}\right] \left[n! \sum_{C=0}^{\lfloor n/2 \rfloor} \frac{(\sqrt{2m\omega})^{n-C}\left(-\frac{1}{\sqrt{2m\omega}}\right)^C}{2^C C!(n-2C)!} x^{n-2C}\right] \tag{154}$$

which, after simple cancellation and combining of multiples, evaluates to

$$\psi_n(x) = \sqrt{n!}\left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega x^2/2} \sum_{C=0}^{\lfloor n/2 \rfloor} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C}}{2^C C!(n-2C)!} x^{n-2C} \tag{155}$$

Interestingly, the above wave function is already normalized; after all, the coefficient $n!(m\omega/\pi)^{1/4}$ comes from the normalization of ψ_0, and $n!$ in Eq 154 comes from the factors needed to preserve normalization in transition from ψ_0 to ψ_n. We will explicitly check that it is the case in Section 9.

7. Wave functions in 2D, without normalization

Let us now turn to two dimensional oscillator. In order to find the wave function, we consider the following trick: we will imagine an oscillator that is in n-th excited state with respect to x and ground state with respect to y,

$$\psi(x, y) = \psi_n(x)\psi_0(y) \tag{156}$$

where ψ_0 is given by Eq 13 and ψ_n is given by Eq 155. We then introduce polar coordinates,

$$x = r\cos\theta \ , \ y = r\sin\theta \tag{157}$$

and do some algebra that would lead to the expression of the form

$$\psi_n(r\cos\theta)\psi_0(r\sin\theta) = \sum \psi_{nL}(r)e^{iL\theta} \tag{158}$$

and then $\psi_{nL}(r)e^{iL\theta}$ will automatically become a solution with total excitation n and angular momentum L, up to some normalization. Let us now do it more explicitly. By substituting Eq 13 and Eq 155 into Eq 156, we obtain

$$\psi_{n0}(x,y) = \sqrt{\frac{m\omega}{\pi}} e^{-m\omega(x^2+y^2)/2} \sum_{C=0}^{\lfloor n/2 \rfloor} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C}}{2^C C!(n-2C)!} x^{n-2C} \tag{159}$$

which, after substitution of 157 becomes

$$\psi_{n0}(r,\theta) = \sqrt{\frac{m\omega}{\pi}} e^{-m\omega r^2/2} \sum_{C=0}^{\lfloor n/2 \rfloor} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C}}{2^C C!(n-2C)!} r^{n-2C} (\cos\theta)^{n-2C} \tag{160}$$

Now, by using

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{161}$$

and then expanding $(\cos\theta)^{n-2C}$ in terms of its $e^{i\theta}$ and $e^{-i\theta}$ components, the above becomes

$$\psi(r,\theta) = \sqrt{\frac{m\omega}{\pi}} e^{-m\omega r^2/2} \sum_{C=0}^{\lfloor n/2 \rfloor} \left(\frac{(-1)^C (2m\omega)^{\frac{n}{2}-C}}{2^C C!(n-2C)!} r^{n-2C} \sum_{D=0}^{n-2C} \binom{n-2C}{D} \frac{e^{i\theta(2C+2D-n)}}{2^{n-2C}} \right) \tag{162}$$

Now, we will set

$$L = 2C + 2D - n \tag{163}$$

and replace the sum over C and D with the sum over L and C (thus, we will also rearrange the order of the sums). We will also do some simple combining of factors, leading to

$$\psi(r,\theta) = \sqrt{\frac{m\omega}{\pi}} e^{-m\omega r^2/2} \sum_{L\in\{-n,-n+2,\cdots,n-2,n\}} \sum_{C=0}^{\min\left(\frac{n-L}{2},\frac{n+L}{2}\right)} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C} r^{n-2C} e^{iL\theta}}{2^{n-C} C! \left(\frac{n+L}{2}-C\right)! \left(\frac{n-L}{2}-C\right)!} \tag{164}$$

Because of the factor $e^{iL\theta}$, the Eq 63 tells us that L represents angular momentum. Thus, we can now produce a state with fixed angular momentum L by "extracting" L-th term from the above sum. Now, we recall from previous discussion that, for any given n, the only "allowed" angular momenta are $L \in \{-n, -n+2, \cdots, n-2, n\}$; in other words, $n-L$ is even. Thus, L-s, as described in Eq 163 "cover" all possible angular momenta; and, similarly, the upper bound of the sum of Eq 164 is integer.

It should be noted that the above described state is *not* normalized (as a matter of fact, since $\psi(r,\theta)$ is norm 1, any of the "ingredient" states should have norm less than 1). Thus, we introduce normalization constant N_{nL}, into which we will also absorb other constant factors; thus,

$$\psi_{nL}(r,\theta) = N_{nL} e^{-m\omega r^2/2} \sum_{C=0}^{\min\left(\frac{n-L}{2},\frac{n+L}{2}\right)} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C} r^{n-2C} e^{iL\theta}}{2^{n-C} C! \left(\frac{n+L}{2}-C\right)! \left(\frac{n-L}{2}-C\right)!} \tag{165}$$

Clearly, the letter n in ψ_{nL} no longer represents the excitation along x-axis; after all, the above is eigenstate of L, and L does not commute with excitation along any particular axis. However, L *does* commute with Hamiltonian, and the latter is given by

$$\hat{H} = \left(\hat{n}_x + \frac{1}{2}\right) + \left(\hat{n}_y + \frac{1}{2}\right) = \hat{n}_x + \hat{n}_y + 1 \tag{166}$$

Thus, we interpret the above n as

$$\hat{n} = \hat{n}_x + \hat{n}_y \ , \ \text{where } \hat{n}_x = \hat{a}_x^\dagger \hat{a}_x \ , \ \hat{n}_y = \hat{a}_y^\dagger \hat{a}_y \tag{167}$$

After all, the state we have started with, which corresponds to $n_y = 0$ can, indeed, be described as $n = n_x$; all the other states we produce have the same n but uncertain n_x-s which, in turn, are being compensated by uncertainty in equal and opposite deviation of n_y.

8. Normalization of two-dimensional oscillator

Let us now compute the normalization constant. Doing so by brute force is increasingly complicated. Therefore, in this section, we will introduce some tricks that would allow us to avoid brute force calculation altogether. But, for those of you that are curious, we will perform the brute force calculation in Section 10 and demonstrate that the results match.

Noticing $e^{-m\omega r^2/2}$ in

$$\psi_{nL}(r,\theta) = N_{nL} e^{-m\omega r^2/2} \sum_{C=0}^{\min\left(\frac{n-L}{2},\frac{n+L}{2}\right)} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C} r^{n-2C} e^{iL\theta}}{2^{n-C} C! \left(\frac{n+L}{2}-C\right)! \left(\frac{n-L}{2}-C\right)!} \tag{168}$$

it is easy to see that, by cancellation of r- terms,

$$\left(\sqrt{\frac{m\omega}{2}} r + \frac{1}{\sqrt{2m\omega}} \frac{\partial}{\partial r}\right) \left(f(r,\theta) e^{-m\omega r^2/2}\right) = \frac{e^{-m\omega r^2/2}}{\sqrt{2m\omega}} \frac{\partial f}{\partial r} \tag{169}$$

Therefore,

$$\left(\sqrt{\frac{m\omega}{2}} r + \frac{1}{\sqrt{2m\omega}} \frac{\partial}{\partial r}\right) \psi_{nL} = \tag{170}$$

$$= \frac{N_{nL}}{\sqrt{2m\omega}} e^{-\frac{m\omega r^2}{2} + iL\theta} \sum_{c=0}^{\min\left(\frac{n+L}{2},\frac{n-L}{2}\right)} \frac{(-1)^c (2m\omega)^{\frac{n}{2}-c}}{2^{n-c}} \frac{(n-2c) r^{n-2c-1}}{c! \left(\frac{n+L}{2}-c\right)! \left(\frac{n-L}{2}-c\right)!}$$

and, after moving outside factor of $\sqrt{2m\omega}$ into the sum, we obtain,

$$\left(\sqrt{\frac{m\omega}{2}} r + \frac{1}{\sqrt{2m\omega}} \frac{\partial}{\partial r}\right) \psi_{nL} = \tag{171}$$

$$= N_{nL} e^{-\frac{m\omega r^2}{2} + iL\theta} \sum_{c=0}^{\min\left(\frac{n+L}{2},\frac{n-L}{2}\right)} \frac{(-1)^c (2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c}} \frac{(n-2c) r^{n-2c-1}}{c! \left(\frac{n+L}{2}-c\right)! \left(\frac{n-L}{2}-c\right)!}$$

We then re-express $n-2c$ in the numerator as

$$n-2c=\left(\frac{n-L}{2}-c\right)+\left(\frac{n+L}{2}-c\right) \tag{172}$$

and use that to split the expression under the sum into two parts:

$$\left(\sqrt{\frac{m\omega}{2}}r+\frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\psi_{nL}=$$

$$=N_{nL}e^{-\frac{m\omega r^2}{2}+iL\theta}\sum_{c=0}^{\min\left(\frac{n+L}{2},\frac{n-L}{2}\right)}\left[\frac{(-1)^c(2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c}}\times\right. \tag{173}$$
$$\left.\times\left(\frac{\left(\frac{n-L}{2}-c\right)r^{n-2c-1}}{c!\left(\frac{n+L}{2}-c\right)!\left(\frac{n-L}{2}-c\right)!}+\frac{\left(\frac{n+L}{2}-c\right)r^{n-2c-1}}{c!\left(\frac{n+L}{2}-c\right)!\left(\frac{n-L}{2}-c\right)!}\right)\right]$$

Now we note that

$$\frac{k}{k!}=\begin{cases}(k-1)! & \text{if } k\geq 1\\ 0 & \text{if } k=0\end{cases} \tag{174}$$

and, therefore,

$$\sum_{k=0}^{m}\frac{\mu_k k}{k!}=\sum_{k=1}^{m}\frac{\mu_k}{(k-1)!} \tag{175}$$

where $k=0$ term is dropped because it is equal to zero, which is why lower bound of the sum changes from $k=0$ to $k=1$. Similarly, if we have $(m-k)/(m-k)!$, then the lower bound would remain the same, while the upper bound would change from $m=k$ to $m=k-1$:

$$\sum_{k=0}^{m}\frac{\mu_k(m-k)}{(m-k)!}=\sum_{k=0}^{m-1}\frac{\mu_k}{(m-k-1)!} \tag{176}$$

Keeping these things in mind, we can rewrite Eq 173 in the following way:

$$\left(\sqrt{\frac{m\omega}{2}}r+\frac{1}{\sqrt{2m\omega}}\frac{\partial}{\partial r}\right)\psi_{nL}=$$

$$=N_{nL}e^{-\frac{m\omega r^2}{2}+iL\theta}\left(\sum_{c=0}^{\min\left(\frac{n+L}{2},\frac{n-L}{2}-1\right)}\frac{(-1)^c(2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c}}\frac{r^{n-1-2c}}{c!\left(\frac{n+L}{2}-c\right)\left(\frac{n-L}{2}-c-1\right)!}+\right. \tag{177}$$
$$\left.+\sum_{c=0}^{\min\left(\frac{n+L}{2}-1,\frac{n-L}{2}\right)}\frac{(-1)^c(2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c}}\frac{r^{n-1-2c}}{c!\left(\frac{n+L}{2}-1-c\right)!\left(\frac{n-L}{2}-c\right)!}\right)$$

We will now rewrite it as

$$\frac{1}{N_{nL}}\left(r\sqrt{2m\omega}+\sqrt{\frac{2}{m\omega}}\frac{\partial}{\partial r}\right)\psi_{nL}=$$

$$= e^{-i\theta} e^{-\frac{m\omega r^2}{2} + i\theta(L+1)} \sum_{c=0}^{\min\left(\frac{(n-1)+(L+1)}{2}, \frac{(n-1)-(L+1)}{2}\right)} \frac{(-1)^c (2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c-1}} \frac{r^{n-1-2c}}{c!\left(\frac{(n-1)+(L+1)}{2} - c\right)!\left(\frac{(n-1)-(L+1)}{2} - c\right)!} +$$

$$+ e^{i\theta} e^{-\frac{m\omega r^2}{2} + i\theta(L-1)} \sum_{c=0}^{\min\left(\frac{(n-1)+(L-1)}{2}, \frac{(n-1)-(L-1)}{2}\right)} \frac{(-1)^c (2m\omega)^{\frac{n-1}{2}-c}}{2^{n-c-1}} \frac{r^{n-1-2c}}{c!\left(\frac{(n-1)+(L-1)}{2} - c\right)!\left(\frac{(n-1)-(L-1)}{2} - c\right)!}$$

where we have multiplied both sides by $2/N_{nL}$. By using Eq 168 we can rewrite the above as

$$\frac{1}{nN_{nL}} \left(r\sqrt{2m\omega} + \sqrt{\frac{2}{m\omega}} \frac{\partial}{\partial r} \right) \psi_{nL} = \frac{e^{-i\theta} \psi_{n-1,L+1}}{N_{n-1,L+1}} + \frac{e^{i\theta} \psi_{n-1,L-1}}{N_{n-1,L-1}} \tag{178}$$

Let us now compute it in a different way. Linear combination of Eq 56 and 57 tells us that

$$a_{-+} e^{-i\theta} + a_{--} e^{i\theta} = r\sqrt{m\omega} + \frac{1}{\sqrt{m\omega}} \frac{\partial}{\partial r} \tag{179}$$

By acting with this on $|\psi_{nL}\rangle$ and using Eq 70, we obtain

$$\left(r\sqrt{m\omega} + \frac{1}{\sqrt{m\omega}} \frac{\partial}{\partial r} \right) |\psi_{nL}\rangle = e^{-i\theta} A_{nL} |\psi_{n-1,L+1}\rangle + e^{i\theta} B_{nL} |\psi_{n-1,L-1}\rangle \tag{180}$$

By comparing Eq 178 and 180, we obtain

$$\frac{1}{N_{n-1,L+1}} = \frac{\sqrt{2}}{nN_{nL}} A_{nL} \ , \ \frac{1}{N_{n-1,L-1}} = \frac{\sqrt{2}}{nN_{nL}} B_{nL} \tag{181}$$

By dividing these two equations one by the other, we obtain

$$\frac{N_{n-1,L-1}}{N_{n-1,L+1}} = \frac{A_{nL}}{B_{nL}} \tag{182}$$

and by substituting the values of A_{nL} and B_{nL} given in Eq 80, this becomes

$$\frac{N_{n-1,L-1}}{N_{n-1,L+1}} = \sqrt{\frac{n-L}{n+L}} \tag{183}$$

By relabeling n and L, we can rewrite it as

$$\frac{N_{n,L-2}}{N_{nL}} = \sqrt{\frac{n-L+2}{n+L}} \tag{184}$$

Therefore,

$$\frac{N_{n,L-4}}{N_{nL}} = \frac{N_{n,L-4}}{N_{n,L-2}} \frac{N_{n,L-2}}{N_{nL}} = \sqrt{\frac{n-L+4}{N+L-2}} \sqrt{\frac{n-L+2}{N+L}} = \sqrt{\frac{(N-L+4)(N-L+2)}{(N+L-2)(N+L)}} \tag{185}$$

By induction, one can show that

$$\frac{N_{n,L-2M}}{N_{nL}} = \sqrt{\frac{\prod_{j=1}^{M} (n-L+2j)}{\prod_{i=1}^{M} (n+L+2-2i)}} \tag{186}$$

Now, we have shown earlier that, for any given n, the allowed values of L are $L \in \{-n, -n+2, \cdots, n-2, n\}$; thus, $n-L$ and $n+L$ are both even. Therefore, we can rewrite Eq 168 as

$$\frac{N_{n,L-2M}}{N_{nL}} = \sqrt{\frac{\prod_{j=1}^{M}\left(\frac{n-L}{2}+j\right)}{\prod_{i=1}^{M}\left(\frac{n+L}{2}+1-i\right)}} = \sqrt{\frac{\frac{\left(\frac{n-L}{2}+M\right)!}{\left(\frac{n-L}{2}\right)!}}{\frac{\left(\frac{n+L}{2}\right)!}{\left(\frac{n+L}{2}-M\right)!}}} = \sqrt{\frac{\left(\frac{n-L}{2}+M\right)!\left(\frac{n+L}{2}-M\right)!}{\left(\frac{n-L}{2}\right)!\left(\frac{n+L}{2}\right)!}} \tag{187}$$

Now, we will first compute N_{nn} and then use the above ratio to produce N_{nL}. By substituting $L = n$ into Eq 165, we obtain

$$\psi_{nn}(r,\theta) = N_{nn}e^{-m\omega r^2/2}\sum_{C=0}^{0}\frac{(-1)^C(2m\omega)^{\frac{n}{2}-C}r^{n-2C}e^{in\theta}}{2^{n-C}C!(n-C)!(0-C)!} =$$

$$= N_{nn}e^{-m\omega r^2/2}\frac{(2m\omega)^{n/2}r^n e^{in\theta}}{2^n n!} = \frac{N_{nn}}{n!}\left(\frac{m\omega}{2}\right)^{n/2} r^n e^{\frac{-m\omega r^2}{2}+in\theta} \tag{188}$$

Therefore,

$$\langle\psi_{nn}|\psi_{nn}\rangle = \left(\frac{N_{nL}}{n!}\right)^2\left(\frac{m\omega}{2}\right)^n\int d^2r\; r^{2n}e^{-m\omega r^2} \tag{189}$$

By using

$$d^2r = 2\pi r dr \tag{190}$$

this becomes

$$\langle\psi_{nn}|\psi_{nn}\rangle = 2\pi\left(\frac{N_{nn}}{n!}\right)^2\left(\frac{m\omega}{2}\right)^n\int_0^\infty r^{2n+1}e^{-m\omega r^2}dr \tag{191}$$

Let us now evaluate the above integral. A single induction step takes the form

$$\int_0^\infty r^{2l+1}e^{-r^2/2}dr = \int_0^\infty r^{2l}e^{-r^2/2}d\frac{r^2}{2} = -\int_0^\infty r^{2k}de^{-r^2/2} = \tag{192}$$

$$= -\left(r^{2l}e^{-r^2/2}\Big|_0^\infty - \int_0^\infty e^{-r^2/2}dr^{2l}\right) = \int_0^\infty e^{-r^2/2}(2l)r^{2l-1}dr = 2l\int_0^\infty e^{-r^2/2}r^{2l-1}dr$$

Now, if we do this repeatedly k times, we get

$$\int_0^\infty r^{2k+1}e^{-r^2/2}dr = 2k\int_0^\infty e^{-r^2/2}r^{2k-1}dr = (2k)(2k-2)\int_0^\infty e^{-r^2/2}r^{2k-3}dr-$$

$$= \cdots = (2k)(2k-2)\cdots(4)(2)\int_0^\infty e^{-r^2/2}\;rdr = 2^k(k)(k-1)\cdots(2)(1)\int_0^\infty e^{-r^2/2}\;rdr =$$

$$= 2^k k!\int_0^\infty e^{-r^2/2}\;rdr = 2^k k!\int_0^\infty e^{-r^2/2}d(r^2/2) = -2^k k! e^{-r^2/2}\Big|_0^\infty = 2^k k! \tag{193}$$

and, therefore, by using new variable

$$s = r\sqrt{2m\omega} \tag{194}$$

we obtain

$$\int_0^\infty r^{2k+1} e^{-m\omega r^2} dr = \int_0^\infty \left(\frac{s}{\sqrt{2m\omega}} \right)^{2k+1} e^{-s^2/2}\, d\left(\frac{s}{\sqrt{2m\omega}} \right) =$$

$$= \frac{1}{(2m\omega)^{k+1}} \int_0^\infty s^{2k+1} e^{-s^2/2} ds = \frac{2^k k!}{(2m\omega)^{k+1}} = \frac{k!}{2\ (m\omega)^{n+1}} \tag{195}$$

Now, if we substitute Eq 195 into Eq 189, we obtain

$$\langle \psi_{nn} | \psi_{nn} \rangle = 2\pi \left(\frac{N_{nn}}{n!} \right)^2 \left(\frac{m\omega}{2} \right)^n \frac{n!}{2\ (m\omega)^{n+1}} = \frac{N_{nn}^2 \pi}{2^n n! m\omega} \tag{196}$$

Thus, the normalization implies that

$$\langle \psi_{nn} | \psi_{nn} \rangle = 1 \Longrightarrow N_{nn} = \sqrt{\frac{2^n n! m\omega}{\pi}} \tag{197}$$

Now, by substituting $L = n$ into Eq 187 we obtain

$$\frac{N_{n,n-2M}}{N_{nn}} = \sqrt{\frac{M!(n-M)!}{n!}} \tag{198}$$

and if we now re-label L so that $L = n - 2M$, then Eq 198 becomes

$$\frac{N_{nL}}{N_{nn}} = \sqrt{\frac{\left(\frac{n-L}{2}\right)! \left(\frac{n+L}{2}\right)!}{n!}} \tag{199}$$

Thus, Eq 197 in combination with Eq 199 imply that

$$N_{nL} = \sqrt{\frac{\left(\frac{n-L}{2}\right)! \left(\frac{n+L}{2}\right)!}{n!}} \sqrt{\frac{2^n n! m\omega}{\pi}} \tag{200}$$

Therefore, Eq 168 becomes

$$\psi_{nL}(r,\theta) = \sqrt{\frac{\left(\frac{n-L}{2}\right)! \left(\frac{n+L}{2}\right)!}{n!}} \sqrt{\frac{2^n n! m\omega}{\pi}} e^{-m\omega r^2/2} \times$$

$$\times \sum_{C=0}^{\min\left(\frac{n-L}{2}, \frac{n+L}{2}\right)} \frac{(-1)^C (2m\omega)^{\frac{n}{2}-C} r^{n-2C} e^{iL\theta}}{2^{n-C} C! \left(\frac{n+L}{2} - C\right)! \left(\frac{n-L}{2} - C\right)!} \tag{201}$$

If we didn't know where the above equation came from, the fact that it is norm 1 is far from obvious; but if one plugs in various specific numbers one can see that the norm "happens" to be 1 "each time". The same is true on both accounts for the 1D case worked out in Section 6. In the two sections that follow we will perform proofs by induction that this is indeed the case for all numbers rather than just some type of coincidence, and we will make sure to carry out such proofs without any appeal to "physics".

9. Verification of orthonormality for 1D

Keeping with our habit of doing 1D before 2D, let us go back and check that the 1D wave function we obtained in Section 6 is properly normalized. And then in Section 10 we will do the same for the 2D case we just obtained.

Naively, the inner product $\langle \psi_p | \psi_q \rangle$ (where $|\psi_n\rangle$ is defined by Eq 155) is given by

$$\langle \psi_p | \psi_q \rangle = \sqrt{p!q!}\sqrt{\frac{m\omega}{\pi}} \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \left(\frac{(-1)^{c_1+c_2}(2m\omega)^{\frac{p+q}{2}-c_1-c_2}}{2^{c_1+c_2} c_1! c_2! (p-2c_1)!(q-2c_2)!} \int x^{p+q-2c_1-2c_2} e^{-m\omega x^2} dx \right) \tag{202}$$

Let us now evaluate the integral. First of all, we note that if $p+q$ is odd, then the above expression produces a sum of integrals of odd functions which is zero; thus,

$$p+q \text{ is odd} \implies \langle \psi_p | \psi_q \rangle = 0 \tag{203}$$

Therefore, our only concern is the situation where $p+q$ is even. The induction step takes the following form:

$$\int_{-\infty}^{\infty} e^{-x^2/2} x^l dx = \int_{-\infty}^{\infty} e^{-x^2/2} x^{l-1} d\frac{x^2}{2} = -\int_{-\infty}^{\infty} x^{l-1} de^{-x^2/2} =$$
$$= -\left(x^{l-1} e^{-x^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2/2} dx^{l-1} \right) = \int_{-\infty}^{\infty} e^{-x^2/2} dx^{l-1} = \tag{204}$$
$$= \int_{-\infty}^{\infty} e^{-x^2/2} (l-1) x^{l-2} dx = (l-1) \int_{-\infty}^{\infty} e^{-x^2/2} x^{l-2} dx$$

By repeating this step k times we obtain

$$\int_{-\infty}^{\infty} e^{-x^2/2} x^{2k} dx = (2k-1) \int_{-\infty}^{\infty} e^{-x^2/2} x^{2k-2} dx = (2k-1)(2k-3) \int_{-\infty}^{\infty} e^{-x^2/2} x^{2k-4} dx =$$
$$= \cdots = (2k-1)(2k-3)\cdots(3)(1) \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}(2k-1)(2k-3)\cdots(3)(1) =$$
$$= \sqrt{2\pi} \prod_{l \leq 2k \text{ , } l \text{ is odd}} l = \sqrt{2\pi} \frac{\prod_{l' \leq 2k} l'}{\prod_{l'' \leq 2k \text{ , } l'' \text{ is even}} l''} = \tag{205}$$
$$= \sqrt{2\pi} \frac{\prod_{l' \leq 2k} l'}{\prod_{l''' \leq k} (2l''')} = \sqrt{2\pi} \frac{\prod_{l' \leq 2k} l'}{2^k \prod_{l''' \leq k} l'''} = \frac{(2k)! \sqrt{2\pi}}{2^k k!}$$

Now, by setting

$$x' = x\sqrt{2m\omega} \tag{206}$$

we obtain

$$\int e^{-m\omega x^2} x^{2k} dx = \int e^{-x'^2/2} \left(\frac{x'}{\sqrt{2m\omega}} \right)^{2k} d\frac{x'}{\sqrt{2m\omega}} =$$

$$= \frac{1}{(2m\omega)^{k+\frac{1}{2}}} \int e^{-x'^2/2} x'^{2k} dx' \frac{1}{(2m\omega)^{k+\frac{1}{2}}} \frac{(2k)!\sqrt{2\pi}}{2^k k!} = \frac{(2k)!\sqrt{\pi}}{2^{2k} k! (m\omega)^{k+\frac{1}{2}}} \tag{207}$$

By setting $k = n - c_1 - c_2$ we obtain .

$$\int e^{-m\omega x^2} x^{2n-2c_1-2c_2} dx = \frac{(2n-2c_1-2c_2)!\sqrt{\pi}}{2^{2n-2c_1-2c_2}(n-c_1-c_2)!(m\omega)^{n-c_1-c_2+\frac{1}{2}}} \tag{208}$$

Now, the above can be substituted into Eq 202 only if $p+q$ is even, which is fine with us given that we already know the answer for odd functions is zero (see Eq 203). Thus,

$$p+q \text{ is even} \Longrightarrow \langle \psi_p | \psi_q \rangle = \sqrt{p!q!}\sqrt{\frac{m\omega}{\pi}} \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_1=0}^{\lfloor q/2 \rfloor} \left(\frac{(-1)^{c_1+c_2}(2m\omega)^{\frac{p+q}{2}-c_1-c_2}}{2^{c_1+c_2}c_1!c_2!(p-2c_1)!(q-2c_2)!} \times \right.$$

$$\left. \times \frac{(p+q-2c_1-2c_2)!\sqrt{\pi}}{2^{p+q-2c_1-2c_2}(\frac{p+q}{2}-c_1-c_2)!(m\omega)^{\frac{p+q}{2}-c_1-c_2+\frac{1}{2}}} \right) \tag{209}$$

which, after combining of factors and cancellations, evaluates to

$$p+q \text{ is even} \Longrightarrow \tag{210}$$

$$\Longrightarrow \langle \psi_p | \psi_q \rangle = \frac{\sqrt{p!q!}}{2^{(p+q)/2}} \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \left(\frac{(-1)^{c_1+c_2}}{c_1!c_2!(p-2c_1)!(q-2c_2)!} \frac{(p+q-2c_1-2c_2)!}{(\frac{p+q}{2}-c_1-c_2)!} \right)$$

But, as we have earlier remarked, orthonormality condition has to hold; in other words,

$$\langle \psi_p | \psi_q \rangle = \delta^p_q \tag{211}$$

Comparison of Eq 210 to Eq 211 tells us that

$$p+q \text{ is even} \Longrightarrow \tag{212}$$

$$\Longrightarrow \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_1=0}^{\lfloor q/2 \rfloor} \left(\frac{(-1)^{c_1+c_2}}{c_1!c_2!(p-2c_1)!(q-2c_2)!} \frac{(p+q-2c_1-2c_2)!}{(\frac{p+q}{2}-c_1-c_2)!} \right) = \frac{2^{(p+q)/2}}{\sqrt{p!q!}} \delta^p_q$$

Now, note the following:

$$p = q \Longrightarrow \frac{2^{(p+q)/2}}{\sqrt{p!q!}} \delta^p_q = \frac{2^p}{p!} = \frac{2^q}{q!} \tag{213}$$

$$p \neq q \Longrightarrow \frac{2^{(p+q)/2}}{\sqrt{p!q!}} \delta^p_q = 0 \tag{214}$$

Therefore, for general p and q,

$$\frac{2^{(p+q)/2}}{\sqrt{p!q!}} \delta^p_q = \frac{2^p}{p!} \delta^p_q = \frac{2^q}{q!} \delta^p_q \tag{215}$$

which means that we can rewrite Eq 212 as

$$p+q \text{ is even} \Longrightarrow \tag{216}$$

$$\implies \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_1=0}^{\lfloor q/2 \rfloor} \left(\frac{(-1)^{c_1+c_2}}{c_1!c_2!(p-2c_1)!(q-2c_2)!} \frac{(p+q-2c_1-2c_2)!}{(\frac{p+q}{2}-c_1-c_2)!} \right) = \frac{2^p}{p!}\delta^p_q = \frac{2^q}{q!}\delta^p_q$$

In other words, we have used "physics" in order to prove "math". The first reaction to this is that something must be wrong with our "physics" *unless* we can prove that same "math" by using "math alone". Let us do the latter. Thus, our task is to prove Eq 212 by induction, without resorting to either physics *or* calculus. Let us define

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies X_{pqr} = \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r-c_1-c_2)!} \tag{217}$$

Therefore,

$$r = \frac{p+q}{2} \implies X_{pqr} = \text{desired expression} \tag{218}$$

or, in other words,

$$X_{p,q,(p+q)/2} = \frac{2^p}{p!}\delta^p_q = \frac{2^q}{q!}\delta^p_q \tag{219}$$

but we find it more convenient to define X for general (p,q,r) first. The reason for this is that it is a lot more difficult to figure out how to "go" from $(p,q,(p+q)/2)$ to $(p+1,q+1,1+(p+q)/2)$ than it is to figure out how to do "simpler" steps, such as "going" from (p,q,r) to $(p+1,q,r)$. The Eq 217 is only well defined if

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \tag{220}$$

because in order for $(r-c_1-c_2)!$ to be well defined, we need to have $r \geq c_1+c_2$, while the sum includes the term where $c_1 = \lfloor p/2 \rfloor$ and $c_2 = \lfloor q/2 \rfloor$. This restriction is okay with us, since our "target" $r = (p+q)/2$ falls into the above domain. We don't need any other restrictions for other factorials to hold. The fact that upper bounds for c_1 and c_2 are $\lfloor p/2 \rfloor$ and $\lfloor q/2 \rfloor$ automatically implies that $p-2c_1 \geq 0$ and $q-2c_2 \geq 0$ and, therefore, $p+q-2c_1-2c_2 \geq 0$.

In order to be able to use induction, we have to relate X_{pqr} to the "neighboring" values of X where p, q or r are being altered by either 0 or 1. In order to obtain such an expression, we will multiply X_{pqr} by a "unit" expressed in the following way:

$$1 = \frac{p-2c_1}{p+q-2c_1-2c_2} + \frac{q-2c_2}{p+q-2c_1-2c_2} \tag{221}$$

Thus, we obtain

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies$$

$$\implies X_{pqr} = \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r-c_1-c_2)!} \; \frac{p-2c_1}{p+q-2c_1-2c_2} +$$

$$+ \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r-c_1-c_2)!} \; \frac{q-2c_2}{p+q-2c_1-2c_2} \tag{222}$$

Now, the $p-2c_1$ in the numerator implies that a given term will be sent to 0 if $p-2c_1=0$; this means that, if it is convenient to us, we can change the upper bound of the sum, provided that it would only amount to throwing away said zero. If p is odd then zeros won't occur so the upper bound of the sum would have to remain the same; namely, $\lfloor p/2 \rfloor$. On the other hand, if p is even, then the above zero would occur at $c_1 = \lfloor p/2 \rfloor = p/2$, thus we can change upper bound of the sum to $p/2-1$. We then note that, in both cases, the "new" upper bound happens to coincide with $\lfloor (p-1)/2 \rfloor$. Thus, we use the latter for general p:

$$\text{First Term} \implies 0 \leq c_1 \leq \left\lfloor \frac{p-1}{2} \right\rfloor \tag{223}$$

Similarly, we also change upper bound for summation over c_2 to $\lfloor (q-1)/2 \rfloor$:

$$\text{Second Term} \implies 0 \leq c_2 \leq \left\lfloor \frac{q-1}{2} \right\rfloor \tag{224}$$

Now, since $p-2c_1$ occurs only in first term while $q-2c_2$ only in second term, the upper bound for summation over q in the first term and over p in the second term would remain the same:

$$\text{First Term} \implies 0 \leq c_2 \leq \left\lfloor \frac{q}{2} \right\rfloor \tag{225}$$

$$\text{Second Term} \implies 0 \leq c_1 \leq \left\lfloor \frac{p}{2} \right\rfloor \tag{226}$$

In light of new upper bounds, we can use the following in order to re-express the expression under the sum:

$$\text{first term} \implies p-2c_1 \geq 1 \implies \frac{p-2c_1}{(p-2c_1)!} = \frac{1}{(p-1-2c_1)!} \tag{227}$$

$$\text{second term} \implies q-2c_2 \geq 1 \implies \frac{q-2c_2}{(q-2c_2)!} = \frac{1}{(q-1-2c_2)!} \tag{228}$$

$$\text{first term} \implies p-2c_1 \geq 1 \implies p+q-2c_1-2c_2 \geq 1 \implies$$
$$\implies \frac{(p+q-2c_1-2c_2)!}{p+q-2c_1-2c_2} = (p+q-2c_1-2c_2-1)! \tag{229}$$

$$\text{second term} \implies q-2c_2 \geq 1 \implies p+q-2c_1-2c_2 \geq 1 \implies$$
$$\implies \frac{(p+q-2c_1-2c_2)!}{p+q-2c_1-2c_2} = (p+q-2c_1-2c_2-1)! \tag{230}$$

The "≥ 1" part was crucial because that is what allowed us to avoid "$(-1)!$" when doing $k!/k = (k-1)!$ By taking into account everything we said so far, our new expression becomes

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies$$

$$\implies X_{pqr} = \sum_{c_1=0}^{\lfloor (p-1)/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}((p-1)+q-2c_1-2c_2)!}{c_1!c_2!((p-1)-2c_1)!(q-2c_2)!(r-c_1-c_2)!} +$$

$$+\sum_{c_1=0}^{\lfloor p/2\rfloor}\sum_{c_2=0}^{\lfloor (q-1)/2\rfloor}\frac{(-1)^{c_1+c_2}(p+(q-1)-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!((q-1)-2c_2)!(r-c_1-c_2)!} \tag{231}$$

By comparing the right hand side to the definition of X_{pqr} as given in Eq 217, we can rewrite the above as

$$r \geq \left\lfloor\frac{p}{2}\right\rfloor + \left\lfloor\frac{q}{2}\right\rfloor \Longrightarrow X_{pqr} = X_{p-1,q,r} + X_{p,q-1,r} \tag{232}$$

This will be one thing that will help us with induction. But we notice that the above expression leaves r unaltered. So let us derive some other expression that would alter r. Recalling that r satisfies the condition given in Eq 220, we can re-express Eq 217 in the following way:

$$r \geq \left\lfloor\frac{p}{2}\right\rfloor + \left\lfloor\frac{q}{2}\right\rfloor \Longrightarrow \tag{233}$$

$$\Longrightarrow X_{pqr} = \sum_{c_1=0}^{\lfloor p/2\rfloor}\sum_{c_2=0}^{\lfloor q/2\rfloor}\frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!}(r+1-c_1-c_2) \tag{234}$$

which we then rewrite as

$$r \geq \left\lfloor\frac{p}{2}\right\rfloor + \left\lfloor\frac{q}{2}\right\rfloor \Longrightarrow$$

$$\begin{aligned}\Longrightarrow X_{pqr} = (r+1)\sum_{c_1=0}^{\lfloor p/2\rfloor}\sum_{c_2=0}^{\lfloor q/2\rfloor}\frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} - \\ -\sum_{c_1=0}^{\lfloor p/2\rfloor}\sum_{c_2=0}^{\lfloor q/2\rfloor}\frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!c_1}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} - \\ -\sum_{c_1=0}^{\lfloor p/2\rfloor}\sum_{c_2=0}^{\lfloor q/2\rfloor}\frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!c_2}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!}\end{aligned} \tag{235}$$

Now, the second term will become 0 for $c_1 = 0$. Thus, we are free to replace the lower bound of the sum on the second term with $c_1 = 1$. Similarly, due to c_2 in the numerator of the third term, we replace the lower bound of that sum with $c_2 = 1$. However, since c_2 doesn't occur in the first term, the sum over c_2 on the first term continues to start from $c_2 = 0$; and, similarly, the sum over c_1 on the second term continues to start with $c_1 = 0$. Since neither c_1 nor c_2 is present in the numerator of the first term, no bounds are changed there. Thus,

$$\text{First Term} \Longrightarrow 0 \leq c_1 \leq \left\lfloor\frac{p}{2}\right\rfloor,\ 0 \leq c_2 \leq \left\lfloor\frac{q}{2}\right\rfloor \tag{236}$$

$$\text{Second Term} \Longrightarrow 1 \leq c_1 \leq \left\lfloor\frac{p}{2}\right\rfloor,\ 0 \leq c_2 \leq \left\lfloor\frac{q}{2}\right\rfloor \tag{237}$$

$$\text{Third Term} \Longrightarrow 0 \leq c_1 \leq \left\lfloor\frac{p}{2}\right\rfloor,\ 1 \leq c_2 \leq \left\lfloor\frac{q}{2}\right\rfloor \tag{238}$$

We can now use $c_1 \geq 1$ and $c_2 \geq 1$, wherever thats the case, in order to replace $c_1!/c_1$ and $c_2!/c_2$ with $1/(c_1-1)!$ and $1/(c_2-1)!$ respectively:

$$\text{Second Term} \implies c_1 \geq 1 \implies \frac{c_1}{c_1!} = \frac{1}{(c_1-1)!} \tag{239}$$

$$\text{Third Term} \implies c_2 \geq 1 \implies \frac{c_2}{c_2!} = \frac{1}{(c_2-1)!} \tag{240}$$

Thus, we can rewrite X_{pqr} as

$$\begin{aligned} r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies & \\ \implies X_{pqr} = (r+1) \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} & \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} - \\ - \sum_{c_1=1}^{\lfloor p-/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} & \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{(c_1-1)!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} - \\ - \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=1}^{\lfloor q/2 \rfloor} & \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!(c_2-1)!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} \end{aligned} \tag{241}$$

Now, in the second term, we will re-label c_1 so as to replace the sum from $c_1 = 1$ to $\lfloor (p-1)/2 \rfloor$ with the sum from $c_1 = 0$ to $\lfloor (p-1)/2 \rfloor - 1$. Similarly, in the third term, we will re-label c_2 so as to replace the sum from $c_2 = 1$ to $\lfloor (q-1)/2 \rfloor$ with the sum from $c_2 = 0$ to $\lfloor (q-1)/2 \rfloor - 1$. On the other hand, c_2 on the second term and c_1 on the third term will remain the same:

$$\begin{aligned} r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies & \\ \implies X_{pqr} = (r+1) \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} & \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} - \\ - \sum_{c_1=1}^{\lfloor p/2 \rfloor - 1} \sum_{c_2=0}^{\lfloor q/2 \rfloor} & \frac{(-1)^{1+c_1+c_2}(p+q-2c_1-2-2c_2)!}{c_1!c_2!(p-2-2c_1)!(q-2c_2)!(r-c_1-c_2)!} - \\ - \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor - 1} & \frac{(-1)^{1+c_1+c_2}(p+q-2c_1-2c_2-2)!}{c_1!c_2!(p-2c_1)!(q-2-2c_2)!(r-c_1-c_2)!} \end{aligned} \tag{242}$$

Now, by noticing that

$$\left\lfloor \frac{p}{2} \right\rfloor - 1 = \left\lfloor \frac{p-2}{2} \right\rfloor, \quad \left\lfloor \frac{q}{2} \right\rfloor - 1 = \left\lfloor \frac{q-2}{2} \right\rfloor \tag{243}$$

and also that

$$-(-1)^{1+c_1+c_2} = +(-1)^{c_1+c_2} \tag{244}$$

we can rewrite the above as

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies$$

$$\implies X_{pqr} = (r+1) \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2)!}{c_1!c_2!(p-2c_1)!(q-2c_2)!(r+1-c_1-c_2)!} +$$

$$+ \sum_{c_1=1}^{\lfloor (p-2)/2 \rfloor} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2-2c_2)!}{c_1!c_2!(p-2-2c_1)!(q-2c_2)!(r-c_1-c_2)!} + \tag{245}$$

$$+ \sum_{c_1=0}^{\lfloor p/2 \rfloor} \sum_{c_2=0}^{\lfloor (q-2)/2 \rfloor} \frac{(-1)^{c_1+c_2}(p+q-2c_1-2c_2-2)!}{c_1!c_2!(p-2c_1)!(q-2-2c_2)!(r-c_1-c_2)!}$$

By comparing this to Eq 217, we can rewrite it as

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies X_{pqr} = (r+1)X_{p,q,r+1} + X_{p-2,q,r} + X_{p,q,r-2} \tag{246}$$

On the other hand, by repeated use of Eq 232, we have

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies X_{pqr} = X_{p-1,q,r} + X_{p,q-1,r} =$$

$$= (X_{p-2,q,r} + X_{p-1,q-1,r}) + (X_{p-1,q-1,r} + X_{p,q-2,r}) = X_{p-2,q,r} + 2X_{p-1,q-1,r} + X_{p,q-2,r} \tag{247}$$

By comparing right hand sides of Eq 246 and Eq 247, we have

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies$$

$$\implies (r+1)X_{p,q,r+1} + X_{p-2,q,r} + X_{p,q,r-2} = X_{p-2,q,r} + 2X_{p-1,q-1,r} + X_{p,q-2,r} \tag{248}$$

By canceling $X_{p-2,q,r}$ and $X_{p,q-2,r}$ on both sides, we obtain

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies (r+1)X_{p,q,r+1} = 2X_{p-1,q-1,r} \tag{249}$$

and, therefore,

$$r \geq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies X_{p,q,r+1} = \frac{2}{r+1} X_{p-1,q-1,r} \tag{250}$$

By re-labeling r this becomes

$$r \geq 1 + \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \implies X_{p,q,r} = \frac{2}{r} X_{p-1,q-1,r-1} \tag{251}$$

the extra 1 in the domain of r, which comes from re-labeling, leaves out $r = \lfloor p/2 \rfloor + \lfloor q/2 \rfloor$. Now, our only aim is to include $r = (p+q)/2$, where $p+q$ is even; or, in other words, either p and q are both even, or they are both odd. Now, we observe that

$$p \text{ and } q \text{ are even } \implies \frac{p+q}{2} = \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \tag{252}$$

$$p \text{ and } q \text{ are odd} \implies \frac{p+q}{2} = \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor + 1 \tag{253}$$

Thus, $(p, q, (p+q)/2)$ meet the pre-condition of Eq 251 if they are odd, but they fail to meet it if they are even. Therefore, for the case of odd p and q, we automatically know that

$$p \text{ and } q \text{ are odd} \implies X_{p,q,(p+q)/2} = X_{p-1,q,(p+q)/2} + X_{p,q-1,(p+q)/2} \tag{254}$$

but for the case of even p and q we need to do some extra work. We will re-express $X_{p,q,(p+q)/2}$ as

$$X_{p,q,(p+q)/2} = X_{p-1,q,(p+q)/2} + X_{p,q-1,(p+q)/2} \tag{255}$$

Since our only concern is the case of even p and q, we note that

$$p \text{ and } q \text{ are even} \implies \frac{p+q}{2} = 1 + \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor = 1 + \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q-1}{2} \right\rfloor \tag{256}$$

thus, as long as p and q are even, we know that both $(p-1, q, (p+q)/2)$ and $(p, q-1, (p+q)/2)$ fall into the domain of Eq 251, which allows us to re-express them as

$$p \text{ and } q \text{ are even} \implies \left\{ \begin{array}{l} X_{p-1,q,(p+q)/2} = \frac{4}{p+q} X_{p-2,q-1,(p+q)/2-1} \\ X_{p,q-1,(p+q)/2} = \frac{4}{p+q} X_{p-1,q-2,(p+q)/2-1} \end{array} \right\} \tag{257}$$

By substituting this into Eq 255, we obtain

$$p \text{ and } q \text{ are even} \implies X_{p,q,(p+q)/2} = \frac{4}{p+q} X_{p-2,q-1,(p+q)/2-1} + \frac{4}{p+q} X_{p-1,q-2,(p+q)/2-1} \tag{258}$$

But Eq 254, after re-labeling p and q, implies that

$$p \text{ and } q \text{ are even} \implies X_{p-1,q-1,(p+q)/2-1} = X_{p-2,q-1,(p+q)/2-1} + X_{p-1,q-2,(p+q)/2-1} \tag{259}$$

By comparing Eq 258 and Eq 259, we obtain

$$p \text{ and } q \text{ are even} \implies X_{p,q,(p+q)/2} = \frac{4}{p+q} X_{p-1,q-1,(p+q)/2-1} \tag{260}$$

Taking together Eq 254 and Eq 260 we see that

$$p + q \text{ is even} \implies X_{p,q,(p+q)/2} = \frac{4}{p+q} X_{p-1,q-1,(p+q)/2-1} \tag{261}$$

But, if $p + q$ is even, then $(p-1) + (q-1)$ is also even, and so is $(p-2) + (q-2)$, and so forth. Thus,

$$p + q \text{ is even} \implies X_{p,q,(p+q)/2} = \frac{4}{p+q} X_{p-1,q-1,(p+q)/2-1} =$$

$$= \frac{4}{p+q} \frac{4}{(p-1)+(q-1)} X_{p-2,q-2,(p+q)/2-2} = \cdots =$$

$$= \frac{4^{\min(p,q)} X_{p-\min(p,q),q-\min(p,q),(p+q)/2-\min(p,q)}}{\prod_{k=0}^{\min(p,q)-1} ((p-k)+(q-k))} =$$

$$= \frac{4^{\min(p,q)} X_{\max(0,p-q),\max(0,q-p),|q-p|/2}}{\prod_{k=0}^{\min(p,q)-1} \left(2\left(\frac{p+q}{2} - k\right)\right)} = \tag{262}$$

$$= \frac{4^{\min(p,q)} X_{\max(0,p-q),\max(0,q-p),|q-p|/2}}{2^{\min(p,q)} \prod_{k=0}^{\min(p,q)-1} \left(\frac{p+q}{2} - k\right)} =$$

$$= \frac{2^{\min(p,q)} X_{\max(0,p-q),\max(0,q-p),|q-p|/2}}{\prod_{k=0}^{\min(p,q)-1} \left(\frac{p+q}{2} - k\right)} =$$

$$= \frac{2^{\min(p,q)}((p+q)/2 - \min(p,q))! X_{\max(0,p-q),\max(0,q-p),|q-p|/2}}{((p+q)/2)!} =$$

$$= \frac{2^{\min(p,q)}|(q-p)/2|!}{((p+q)/2)!} X_{\max(0,p-q),\max(0,q-p),|q-p|/2}$$

we can rewrite the above result as

$$p+q \text{ is even } \Longrightarrow X_{p,q,(p+q)/2} = \begin{cases} \frac{2^p((q-p)/2)!}{((p+q)/2)!} X_{0,q-p,(q-p)/2} & \text{if } p < q \\ \frac{2^q((p-q)/2)!}{((p+q)/2)!} X_{p-q,0,(p-q)/2} & \text{if } p > q \\ \frac{2^p}{p!} X_{000} = \frac{2^q}{q!} X_{000} & \text{if } p = q \end{cases} \tag{263}$$

We will, therefore, compute X_{p0r}, X_{0qr} and X_{000} and then substitute them with appropriate relabeling of p and q into the above equation. By substituting $p = 0$ into Eq 217, we obtain

$$r \geq \left\lfloor \frac{q}{2} \right\rfloor \Longrightarrow X_{0qr} = \sum_{c_1=0}^{0} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_1+c_2}(0+q-2c_1-2c_2)!}{c_1!c_2!(0-2c_1)!(q-2c_2)!(r-c_1-c_2)!} =$$

$$= \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{0+c_2}(0+q-2*0-2c_2)!}{0!c_2!(0-2*0)!(q-2c_2)!(r-0-c_2)!} = \tag{264}$$

$$= \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_2}(q-2c_2)!}{c_2!(q-2c_2)!(r-c_2)!} = \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_2}}{c_2!(r-c_2)!}$$

Therefore, if we assume $r = \lfloor q/2 \rfloor$, we obtain

$$X_{0,q,\lfloor q/2 \rfloor} = \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_2}}{c_2!(\lfloor q/2 \rfloor - c_2)!} = \frac{1}{\lfloor q/2 \rfloor!} \sum_{c_2=0}^{\lfloor q/2 \rfloor} \binom{\lfloor q/2 \rfloor}{c_2} (-1)^{c_2}(+1)^{\lfloor q/2 \rfloor - c_2} =$$

$$= \frac{(-1+1)^{\lfloor q/2 \rfloor}}{\lfloor q/2 \rfloor!} = \frac{0^{\lfloor q/2 \rfloor}}{\lfloor q/2 \rfloor!} \tag{265}$$

Now we have to be careful. The expression $0^n = 0$ holds only for $n \geq 1$; as far as 0^0 is concerned, its value is a "philosophical" question. Thus, the only thing we have found out is that

$$\left\lfloor \frac{q}{2} \right\rfloor \geq 1 \Longrightarrow X_{0,q,\lfloor q/2 \rfloor} = 0 \tag{266}$$

On the other hand, for the case of $\lfloor q/2 \rfloor = 0$ we have

$$\left\lfloor \frac{q}{2} \right\rfloor = 0 \Longrightarrow X_{0,q,\lfloor q/2 \rfloor} = \sum_{c_2=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{c_2}}{c_2!(\lfloor q/2 \rfloor - c_2)!} = \sum_{c_2=0}^{0} \frac{(-1)^{c_2}}{c_2!(0-c_2)!} = \frac{(-1)^0}{0!(0-0)!} = 1 \quad (267)$$

In other words, for general q,

$$X_{0,q,\lfloor q/2 \rfloor} = \delta_0^q \quad (268)$$

and, similarly,

$$X_{p,0,\lfloor q/2 \rfloor} = \delta_0^p \quad (269)$$

If we use these results for Eq 263, we obtain

$$p+q \text{ is even } \Longrightarrow X_{p,q,(p+q)/2} = \begin{cases} 0 & \text{if } p<q \\ 0 & \text{if } p>q \\ \frac{2^p}{p!} = \frac{2^q}{q!} & \text{if } p=q \end{cases} \quad (270)$$

which can be rewritten as

$$p+q \text{ is even } \Longrightarrow X_{p,q,(p+q)/2} = \frac{2^p}{p!}\delta_q^p = \frac{2^q}{q!}\delta_q^p \quad (271)$$

which agrees with Eq 216. Now, the reason we wanted Eq 216 to hold is that, from our earlier discussion,

$$\text{Eq 216 holds } \Longrightarrow \langle \psi_p | \psi_q \rangle = \delta_q^p \text{ for even } p+q \quad (272)$$

On the other hand, inspection of Eq 202 tells us that

$$\int (\text{odd function}) = 0 \Longrightarrow \langle \psi_p | \psi_q \rangle = 0 \text{ for odd } p+q \quad (273)$$

as has also been observed in Eq 203. Thus, by combining the above two statements, we have

$$\forall p, q(\langle \psi_p | \psi_q \rangle = 0) \quad (274)$$

as desired.

10. Verification of orthonormality for 2D

Let us now turn to 2D case and verify the orthonormality of Eq 201. If we take inner product between two states given in Eq 168, we find

$$\langle \psi_{pL} | \psi_{qL} \rangle = N_{pL} N_{qL} \sum_{c_1=0}^{\min\left(\frac{p-L}{2}, \frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2}, \frac{q+L}{2}\right)} \left[\left(\int d^3r \; e^{-m\omega r^2} r^{p+q-2c_1-2c_2} \right) \times \right.$$

$$\left. \times \left(\frac{(-1)^{c_1+c_2} (2m\omega)^{\frac{p+q}{2}-c_1-c_2}}{2^{p+q-c_1-c_2} c_1! c_2! \left(\frac{p+L}{2}-c_1\right)! \left(\frac{p-L}{2}-c_1\right)! \left(\frac{q+L}{2}-c_2\right)! \left(\frac{q-L}{2}-c_2\right)!} \right) \right] \quad (275)$$

by using

$$d^2 r = 2\pi r dr \tag{276}$$

this becomes

$$\langle \psi_{pL} | \psi_{qL} \rangle = 2\pi N_{pL} N_{qL} \sum_{c_1=0}^{\min\left(\frac{p-L}{2}, \frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2}, \frac{q+L}{2}\right)} \Bigg[\left(\int_0^\infty dr \; e^{-m\omega r^2} r^{p+q-2c_1-2c_2+1} \right) \times$$

$$\times \left(\frac{(-1)^{c_1+c_2} (2m\omega)^{\frac{p+q}{2}-c_1-c_2}}{2^{p+q-c_1-c_2} c_1! c_2! \left(\frac{p+L}{2}-c_1\right)! \left(\frac{p-L}{2}-c_1\right)! \left(\frac{q+L}{2}-c_2\right)! \left(\frac{q-L}{2}-c_2\right)!} \right) \Bigg] \tag{277}$$

Let us assume that $p+q$ is even. By replacing n with $(p+q)/2 - c_1 - c_2$ in Eq 195 we obtain

$$p+q \text{ is even} \implies \int_0^\infty r^{p+q-2c_1-2c_2+1} e^{-m\omega r^2} dr = \frac{\left(\frac{p+q}{2} - c_1 - c_2\right)!}{2\,(m\omega)^{\frac{p+q}{2}-c_1-c_2+1}} \tag{278}$$

By plugging in Eq 278 into Eq 277, we obtain

$$p+q \text{ is even} \implies \langle \psi_{pL} | \psi_{qL} \rangle = 2\pi N_{pL} \sum_{c_1=0}^{\min\left(\frac{p-L}{2}, \frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2}, \frac{q+L}{2}\right)} \Bigg[\left(\frac{\left(\frac{p+q}{2} - c_1 - c_2\right)!}{2\,(m\omega)^{\frac{p+q}{2}-c_1-c_2+1}} \right) \times$$

$$\times \left(\frac{(-1)^{c_1+c_2} (2m\omega)^{\frac{p+q}{2}-c_1-c_2}}{2^{p+q-c_1-c_2} c_1! c_2! \left(\frac{p+L}{2}-c_1\right)! \left(\frac{p-L}{2}-c_1\right)! \left(\frac{q+L}{2}-c_2\right)! \left(\frac{q-L}{2}-c_2\right)!} \right) \Bigg] \tag{279}$$

which, after some combining of factors and some cancellations, becomes

$$p+q \text{ is even} \implies \langle \psi_{pL} | \psi_{qL} \rangle = \tag{280}$$

$$= \frac{\pi N_{pL} N_{qL}}{2^{(p+q)/2} m\omega} \sum_{c_1=0}^{\min\left(\frac{p-L}{2}, \frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2}, \frac{q+L}{2}\right)} \frac{(-1)^{c_1+c_2} \left(\frac{p+q}{2} - c_1 - c_2\right)!}{c_1! c_2! \left(\frac{p+L}{2}-c_1\right)! \left(\frac{p-L}{2}-c_1\right)! \left(\frac{q+L}{2}-c_2\right)! \left(\frac{q-L}{2}-c_2\right)!}$$

By substituting Eq 200 for N_{pL} and N_{qL}, we obtain

$$p+q \text{ is even} \implies \langle \psi_{pL} | \psi_{qL} \rangle = \frac{\pi \sqrt{\frac{\left(\frac{p-L}{2}\right)! \left(\frac{p+L}{2}\right)!}{p!}} \sqrt{\frac{2^p p! m\omega}{\pi}} \sqrt{\frac{\left(\frac{q-L}{2}\right)! \left(\frac{q+L}{2}\right)!}{n!}} \sqrt{\frac{2^q q! m\omega}{\pi}}}{2^{(p+q)/2} m\omega} \times$$

$$\times \sum_{c_1=0}^{\min\left(\frac{p-L}{2}, \frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2}, \frac{q+L}{2}\right)} \frac{(-1)^{c_1+c_2} \left(\frac{p+q}{2} - c_1 - c_2\right)!}{c_1! c_2! \left(\frac{p+L}{2}-c_1\right)! \left(\frac{p-L}{2}-c_1\right)! \left(\frac{q+L}{2}-c_2\right)! \left(\frac{q-L}{2}-c_2\right)!} \tag{281}$$

which evaluates to

$$p+q \text{ is even} \implies \langle \psi_{pL} | \psi_{qL} \rangle = \sqrt{\left(\frac{p-L}{2}\right)! \left(\frac{p+L}{2}\right)! \left(\frac{q-L}{2}\right)! \left(\frac{q+L}{2}\right)!} \times$$

$$\times \sum_{c_1=0,c_2=0}^{\min\left(\frac{n-L}{2},\frac{n+L}{2}\right)} \frac{(-1)^{c_1+c_2}\left(\frac{p+q}{2}-c_1-c_2\right)!}{c_1!c_2!\left(\frac{p+L}{2}-c_1\right)!\left(\frac{p-L}{2}-c_1\right)!\left(\frac{q+L}{2}-c_2\right)!\left(\frac{q-L}{2}-c_2\right)!} \tag{282}$$

The normalization

$$\langle \psi_{pL} | \psi_{qL} \rangle = \delta^p_q \tag{283}$$

tells us that

$$p+q \text{ is even } \Longrightarrow$$

$$\Longrightarrow \sum_{c_1=0}^{\min\left(\frac{p-L}{2},\frac{p+L}{2}\right)} \sum_{c_2=0}^{\min\left(\frac{q-L}{2},\frac{q+L}{2}\right)} \frac{(-1)^{c_1+c_2}\left(\frac{p+q}{2}-c_1-c_2\right)!}{c_1!c_2!\left(\frac{p+L}{2}-c_1\right)!\left(\frac{p-L}{2}-c_1\right)!\left(\frac{q+L}{2}-c_2\right)!\left(\frac{q-L}{2}-c_2\right)!} =$$

$$= \frac{\delta^p_q}{\sqrt{\left(\frac{p-L}{2}\right)!\left(\frac{p+L}{2}\right)!\left(\frac{q-L}{2}\right)!\left(\frac{q+L}{2}\right)!}} \tag{284}$$

Notably, we were using "physics" to prove the above, which is strange, since outcome of said "physics" is pure math. Let us, therefore, see whether or not we can derive said "math" without reference to "physics. Let us define

$$Y_{P_1P_2Q_1Q_2R} = \sum_{c=0}^{\min(P_1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!}{c!d!(P_1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!} \tag{285}$$

In this language, what we have just found out is

$$\text{Physics} \Longrightarrow Y_{\frac{p-L}{2},\frac{p+L}{2},\frac{q-L}{2},\frac{q+L}{2},\frac{p+q}{2}} = \frac{\delta^p_q}{\sqrt{\left(\frac{p-L}{2}\right)!\left(\frac{p+L}{2}\right)!\left(\frac{q-L}{2}\right)!\left(\frac{q+L}{2}\right)!}} \text{ if } p+q \text{ is even} \tag{286}$$

Our goal is to prove the above result by using pure math.

Theorem 1

$$P_1 Y_{P_1P_2Q_1Q_2R} = Y_{P_1-1,P_2,Q_1,Q_2,R} - Y_{P_1-1,P_2-1,Q_1,Q_2,R-1} \tag{287}$$

Proof: We can multiply each of the summand of Eq 285 by

$$P_1 = (P_1 - c) + c \tag{288}$$

and then splitting it into two terms: one containing $P_1 - c$ and the other c:

$$P_1 Y_{P_1P_2Q_1Q_2R} = \sum_{c=0}^{\min(P_1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!(P_1-c)}{c!d!(P_1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!} +$$

$$+ \sum_{c=0}^{\min(P_1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!c}{c!d!(P_1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!} \tag{289}$$

Now we observe that

$$\frac{0}{0!} = \frac{0}{1} = 0 \tag{290}$$

Therefore, the summand in the first term is zero for $c = P_1$. This means that, instead of summing c over $\{0, \cdots, \min(P_1, P_2)\}$ we can sum it over $\{0, \cdots, \min(P_1, P_2)\} \setminus \{P_1\}$:

$$\sum_{c=0}^{\min(P_1,P_2)} \frac{P_1 - c}{(P_1 - c)!}(\cdots) = \sum_{c \in \{0,\cdots,\min(P_1,P_2)\} \setminus \{P_1\}} \frac{P_1 - c}{(P_1 - c)!}(\cdots) \tag{291}$$

Now, we notice that

$$P_1 \leq P_2 \Longrightarrow \{0, \cdots, \min(P_1, P_2)\} \setminus \{P_1\} = \{0, \cdots, P_1\} \setminus \{P_1\} =$$

$$= \{0, \cdots, P_1 - 1\} = \{0, \cdots, \min(P_1 - 1, P_2)\} \tag{292}$$

$$P_2 \leq P_1 - 1 \Longrightarrow \{0, \cdots, \min(P_1, P_2)\} \setminus \{P_1\} = \{0, \cdots, P_2\} \setminus \{P_1\} =$$

$$= \{0, \cdots, P_2\} = \{0, \cdots, \min(P_1 - 1, P_2)\} \tag{293}$$

Now, we know that either $P_1 \leq P_2$ or $P_2 < P_1$ must hold. Furthermore, we know that $P_2 < P_1$ is equivalent to $P_2 \leq P_1 - 1$. Thus, we know that either $P_1 \leq P_2$ or $P_2 \leq P_1 - 1$ must hold:

$$\forall P_1 \forall P_2 ((P_1 \leq P_2) \vee (P_2 \leq P_1 - 1)) \tag{294}$$

This, in combination with Eq 292 and 293 tells us that

$$\forall P_1 \forall P_2 (\{0, \cdots, \min(P_1, P_2)\} \setminus \{P_1\} = \{0, \cdots, \min(P_1 - 1, P_2)\} \tag{295}$$

Thus, Eq 291 becomes

$$\sum_{c=0}^{\min(P_1,P_2)} \frac{P_1 - c}{(P_1 - c)!}(\cdots) = \sum_{c=0}^{\min(P_1-1,P_2)} \frac{P_1 - c}{(P_1 - c)!}(\cdots) \tag{296}$$

We then observe that

$$c \leq \min(P_1 - 1, P_2) \Longrightarrow c \leq p_1 - 1 \Longrightarrow P_1 - c \geq 1 \Longrightarrow \frac{P_1 - c}{(P_1 - c)!} = \frac{1}{(P_1 - 1 - c)!} \tag{297}$$

Thus we can rewrite Eq 296 as

$$\sum_{c=0}^{\min(P_1,P_2)} \frac{P_1 - c}{(P_1 - c)!}(\cdots) = \sum_{c=0}^{\min(P_1-1,P_2)} \frac{1}{(P_1 - 1 - c)!}(\cdots) \tag{298}$$

On the other hand, if we apply

$$\frac{0}{0!} = \frac{0}{1} = 0 \tag{299}$$

to the second term on the right hand side of Eq 289, then we will exclude $c = 0$ term:

$$\sum_{c=0}^{\min(P_1,P_2)} \frac{c}{c!}(\cdots) = \sum_{c \in \{0,\cdots,\min(P_1,P_2)\} \setminus \{0\}} \frac{c}{c!}(\cdots) = \sum_{c=1}^{\min(P_1,P_2)} \frac{c}{c!}(\cdots) \tag{300}$$

By noticing that

$$c \geq 1 \Longrightarrow \frac{c}{c!} = \frac{1}{(c-1)!} \tag{301}$$

we can rewrite Eq 300 as

$$\sum_{c=0}^{\min(P_1,P_2)} \frac{c}{c!}(\cdots) = \sum_{c=1}^{\min(P_1,P_2)} \frac{1}{(c-1)!}(\cdots) \tag{302}$$

By applying Eq 298 and 302 to Eq 289, we obtain

$$P_1 Y_{P_1P_2Q_1Q_2R} = \sum_{c=0}^{\min(P_1-1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!}{c!d!(P_1-1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!}+$$

$$+ \sum_{c=1}^{\min(P_1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!}{(c-1)!d!(P_1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!} \tag{303}$$

Now if we set

$$c' = c - 1 \tag{304}$$

then

$$(-1)^{c+d} = -(-1)^{c'+d}\ ,\ R-c-d = (R-1)-c'-d \tag{305}$$

$$P_1 - c = (P_1-1)-c'\ ,\ P_2 - c = (P_2-1)-c' \tag{306}$$

and, furthermore,

$$c \in \{1, \cdots, \min(P_1,P_2)\} \Longrightarrow c' \in \{0, \min(P_1-1,P_2-1)\} \tag{307}$$

By applying Eq 304, 305, 306 and 307 to the last sum in Eq 303, we obtain

$$P_1 Y_{P_1P_2Q_1Q_2R} = \sum_{c=0}^{\min(P_1-1,P_2)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c+d}(R-c-d)!}{c!d!(P_1-1-c)!(P_2-c)!(Q_1-d)!(Q_2-d)!}-$$

$$- \sum_{c'=0}^{\min(P_1-1,P_2-1)} \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c'+d}((R-1)-c'-d)!}{c'!d!((P_1-1)-c')!((P_2-1)-c')!(Q_1-d)!(Q_2-d)!} \tag{308}$$

By comparing the right hand side to Eq 285, we can rewrite it as

$$P_1 Y_{P_1P_2Q_1Q_2R} = Y_{P_1-1,P_2,Q_1,Q_2,R} - Y_{P_1-1,P_2-1,Q_1,Q_2,R} \tag{309}$$

which coincides with the claim of the Theorem. **QED.**

Theorem 2

$$Y_{0PQ_1Q_2R} = \frac{1}{P!} Y_{00Q_1Q_2R} \tag{310}$$

Proof From Eq 285 we know that

$$Y_{0PQ_1Q_2R} = \sum_{c_1=0}^{0} \sum_{c_2=0}^{\min(Q_1,Q_2)} \frac{(-1)^{c_1+c_2}(R-c_1-c_2)!}{c_1!c_2!(0-c_1)!(P-c_1)!(Q_1-c_1)!(Q_2-c_2)!} \tag{311}$$

This means that we can drop the summation over c_1 and replace it with

$$c_1 = 0 \tag{312}$$

leading to

$$Y_{0PQ_1Q_2R} = \sum_{c_2=0}^{\min(Q_1,Q_2)} \frac{(-1)^{0+c_2}(R-0-c_2)!}{0!c_2!(0-0)!(P-0)!(Q_1-d)!(Q_2-d)!} =$$

$$= \sum_{c_2=0}^{\min(Q_1,Q_2)} \frac{(-1)^{0+c_2}(R-0-c_2)!}{c_2!P!(Q_1-d)!(Q_2-d)!} = \frac{1}{P!} \sum_{c_2=0}^{\min(Q_1,Q_2)} \frac{(-1)^{0+c_2}(R-0-c_2)!}{c_2!P!(Q_1-d)!(Q_2-d)!} \tag{313}$$

By plugging in $P = 0$, we obtain

$$Y_{00Q_1Q_2R} = \sum_{c_2=0}^{\min(Q_1,Q_2)} \frac{(-1)^{0+c_2}(R-0-c_2)!}{c_2!P!(Q_1-d)!(Q_2-d)!} \tag{314}$$

And by comparing Eq 313 with Eq 314 we have

$$Y_{0PQ_1Q_2R} = \frac{1}{P!} Y_{00Q_1Q_2R} \tag{315}$$

Lemma 1

$$Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = \delta_0^{\min(Q_1,Q_2)} \tag{316}$$

Proof By inspecting Eq 285 for $P_1 = P_2 = 0$ we see that c is being summed from 0 to 0. In other words, we can "get rid" of sum over c and, instead, replace all of c-s with 0. Furthermore, again by noticing $P_1 = P_2 = 0$, we also know that $P_1 - c = 0 - 0 = 0$ and $P_2 - c = 0 - 0 = 0$. Thus, we can replace $c!(P_1 - c)!(P_2 - c)!$ with 1. Therefore, the summation becomes

$$Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^d(\max(Q_1,Q_2)-d)!}{d!(Q_1-d)!(Q_2-d)!} \tag{317}$$

Now, it is easy to see that

$$(Q_1-d)!(Q_2-d)! = (\min(Q_1,Q_2)-d)!(\max(Q_1,Q_2)-d)! \tag{318}$$

By doing this substitution, we obtain

$$Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^d(\max(Q_1,Q_2)-d)!}{d!(\min(Q_1,Q_2)-d)!(\max(Q_1,Q_2)-d)!} \tag{319}$$

Thus, $(\max(Q_1,Q_2)-d)!$ on numerator and denominator cancels out, leading to

$$Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = \sum_{d=0}^{\min(Q_1,Q_2)} \frac{(-1)^d}{d!(\min(Q_1,Q_2)-d)!} \tag{320}$$

The above matches binomial expression of $(-1+1)^{\min(Q_1,Q_2)}$. We have to be careful though since $0^n = 0$ holds only for $n \geq 1$ while the value of 0^0 is "controversial". Thus we do binomial formula in former case and brute force summing in latter case:

$$\min(Q_1,Q_2) \geq 1 \Longrightarrow Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = (-1+1)^{\min(Q_1,Q_2)} = 0 \tag{321}$$

$$\min(Q_1,Q_2) = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,\max(Q_1,Q_2)} = \sum_{d=0}^{0} \frac{(-1)^d}{d!(0-d)!} = \frac{(-1)^0}{0!(0-0)!} = 1 \tag{322}$$

These two results imply

$$Y_{00Q_1Q_2\max(Q_1,Q_2)} = \delta_0^{\min(Q_1,Q_2)} \tag{323}$$

as claimed. **QED**

Lemma 2

$$\left\{ \begin{array}{l} k \geq 0 \\ Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1,Q_2,k+\max(Q_1,Q_2)} = 0 \tag{324}$$

Proof Let us define set S as follows:

$$S = \{k \geq 0 | \forall Q_1 \geq k+1 \; \forall Q_2 \geq k+1 \; (Y_{0,0,Q_1,Q_2,k+\max(Q_1,Q_2)} = 0)\} \tag{325}$$

Now, by using Eq 309 with Q_1 playing the role of P_1, we see that

$$\left\{ \begin{array}{l} k \in S \\ Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1,Q_2,k+\max(Q_1,Q_2)} = 0 \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} - Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} = 0 \Longrightarrow$$
$$\Longrightarrow Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} \tag{326}$$

We therefore observe that

$$\left\{ \begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} k \in S \\ Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} \tag{327}$$

and, on the other hand,

$$\left\{ \begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} k \in S \\ Q_1 - 1 \geq k+1 \\ Q_2 - 1 \geq k+1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} = 0 \tag{328}$$

Pulling those two results together we have

$$\left\{ \begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} \\ Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1-1,Q_2-1)} = 0 \end{array} \right\} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = 0 \tag{329}$$

Now, we will utilize Eq 309 with Q_2 playing the role of P_1, thus making the following argument:

$$\left\{\begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array}\right\} \Longrightarrow Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = 0 \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1,Q_2)} - Y_{0,0,Q_1-2,Q_2-1,k+\max(Q_1-1,Q_2-1)} = 0 \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1-1,Q_2-1,k+\max(Q_1,Q_2)} = Y_{0,0,Q_1-2,Q_2-1,k+\max(Q_1-1,Q_2-1)} \tag{330}$$

Now if we re-label Q_1 and Q_2 by replacing them with Q_1+1 and Q_2+1, respectively, we obtain

$$\left\{\begin{array}{l} k \in S \\ Q_1+1 \geq k+2 \\ Q_2+1 \geq k+2 \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,(Q_1+1)-1,(Q_2+1)-1,k+\max(Q_1+1,Q_2+1)} = Y_{0,0,(Q_1+1)-2,(Q_2+1)-1,k+\max((Q_1+1)-1,(Q_2+1)-1)} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1,Q_2,(k+1)+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} \tag{331}$$

And, therefore,

$$\left\{\begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} k \in S \\ Q_1+1 \geq k+2 \\ Q_2+1 \geq k+2 \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1,Q_2,(k+1)+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} \tag{332}$$

Now, if we combine Eq 329 and Eq 332, we get

$$\left\{\begin{array}{l} k \in S \\ Q_1 \geq k+2 \\ Q_2 \geq k+2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} = 0 \\ Y_{0,0,Q_1,Q_2,(k+1)+\max(Q_1,Q_2)} = Y_{0,0,Q_1-1,Q_2,k+\max(Q_1,Q_2)} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow Y_{0,0,Q_1,Q_2,(k+1)+\max(Q_1,Q_2)} = 0 \tag{333}$$

Now, we can rewrite it as

$$k \in S \Longrightarrow [\forall Q_1 \geq k+2 \; \forall Q_2 \in k+2 \; (Y_{0,0,Q_1,Q_2,(k+1)+\max(Q_1,Q_2)} = 0)] \tag{334}$$

By comparing it to the definition of S given in Eq 325, we can rewrite the above as

$$k \in S \Longrightarrow k+1 \in S \tag{335}$$

Furthermore, we know from previous Lemma that

$$0 \in S \tag{336}$$

Thus, by induction,

$$S = \mathbb{N} \tag{337}$$

This, in combination with Eq 325, implies

$$\forall k \in \mathbb{N}\ \forall Q_1 \geq k+1\ \forall Q_2 \geq k+1\ (Y_{0,0,Q_1,Q_2,k+\max(Q_1,Q_2)} = 0) \tag{338}$$

as claimed. **QED**

Lemma 3

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 - 1 \Longrightarrow Y_{0,0,Q_1,Q_2,R} = 0 \tag{339}$$

Proof Let us define

$$k = R - \max(Q_1, Q_2) \tag{340}$$

Thus,

$$\max(Q_1, Q_2) \leq R \Longleftrightarrow R - \max(Q_1, Q_2) \geq 0 \Longleftrightarrow k \geq 0 \Longleftrightarrow k \in \mathbb{N} \tag{341}$$

$$\begin{gathered} R \leq Q_1 + Q_2 - 1 \Longleftrightarrow R \leq \max(Q_1, Q_2) + \min(Q_1, Q_2) - 1 \Longleftrightarrow \\ \Longleftrightarrow R - \max(Q_1, Q_2) \leq \min(Q_1, Q_2) - 1 \Longleftrightarrow k \leq \min(Q_1, Q_2) - 1 \Longleftrightarrow \\ \Longleftrightarrow \left\{ \begin{array}{c} k \leq Q_1 - 1 \\ k \leq Q_2 - 1 \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} k+1 \leq Q_1 \\ k+1 \leq Q_2 \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \end{gathered} \tag{342}$$

By taking Eq 341 and Eq 342 together, we get

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 - 1 \Longleftrightarrow \left\{ \begin{array}{l} k \in \mathbb{N} \\ Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \tag{343}$$

But, we know from previous Lemma, that

$$\left\{ \begin{array}{l} k \in \mathbb{N} \\ Q_1 \geq k+1 \\ Q_2 \geq k+1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1,Q_2,k+\max(Q_1,Q_2)} = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,R} = 0 \tag{344}$$

Thus, the last two statements taken together imply

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 - 1 \Longrightarrow Y_{0,0,Q_1,Q_2,R} = 0 \tag{345}$$

as claimed. **QED**

Lemma 4

$$Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \tag{346}$$

Proof We notice that

$$\begin{gathered} \min(Q_1, Q_2) \geq 1 \Longrightarrow Q_1 + Q_2 = \min(Q_1, Q_2) + \max(Q_1, Q_2) \geq \\ \geq 1 + \max(Q_1, Q_2) = \max(Q_1 + 1, Q_2 + 1) \geq \max(Q_1 + 1, Q_2) \Longrightarrow \\ \Longrightarrow \max(Q_1 + 1, Q_2) \leq Q_1 + Q_2 = (Q_1 + 1) + Q_2 - 1 \Longrightarrow \\ \Longrightarrow \max(Q_1 + 1, Q_2) \leq Q_1 + Q_2 \leq (Q_1 + 1) + Q_2 - 1 \Longrightarrow Y_{0,0,Q_1+1,Q_2,Q_1+Q_2} = 0 \end{gathered} \tag{347}$$

But, we know from Theorem 1 that

$$(Q_1 + Q_2)Y_{0,0,Q_1+1,Q_2,Q_1+Q_2} = Y_{0,0,Q_1,Q_2,Q_1+Q_2} - Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} \tag{348}$$

Thus, combining it with statement 347, we have

$$\begin{aligned} \min(Q_1, Q_2) \geq 1 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} - Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} = 0 \Longrightarrow \\ \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} \end{aligned} \tag{349}$$

Now let us define the set U as

$$U = \{u | Q_1 + Q_2 = U \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1\} \tag{350}$$

We observe that

$$Q_1 + Q_2 = 0 \Longrightarrow Q_1 = Q_2 = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = Y_{0,0,0,0,0} = 1 \tag{351}$$

and, therefore

$$0 \in U \tag{352}$$

Now, notice that

$$\begin{aligned} \left\{ \begin{array}{l} u \in U \\ \min(Q_1, Q_2) \geq 1 \\ Q_1 + Q_2 = u + 1 \end{array} \right\} \Longrightarrow^{\text{apply 349 to 2−nd line}} \left\{ \begin{array}{l} u \in U \\ Y_{0,0,Q_1,Q_2,Q_1+Q_2} = Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} \\ Q_1 + (Q_2 - 1) = u \end{array} \right\} \Longrightarrow \\ \Longrightarrow^{\text{Apply 1−st line to 3−rd line}} \left\{ \begin{array}{l} u \in U \\ Y_{0,0,Q_1,Q_2,Q_1+Q_2} = Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} \\ Y_{0,0,Q_1,Q_2-1,Q_1+Q_2-1} = 1 \end{array} \right\} \Longrightarrow \\ \Longrightarrow^{\text{apply 3−rd line to 2−nd line}} Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \end{aligned} \tag{353}$$

On the other hand, from Lemma 1 we know that

$$Q_1 = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = Y_{0,0,0,Q_2,Q_2} = Y_{0,0,0,Q_2,\max(0,Q_2)} =^{\text{Lemma 1}} \delta_0^0 = 1 \tag{354}$$

and, by symmetry between Q_1 and Q_2, we automatically know that

$$Q_2 = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \tag{355}$$

and, therefore,

$$\min(Q_1, Q_2) = 0 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \tag{356}$$

On the other hand, we know from 354 that

$$\left\{ \begin{array}{l} u \in U \\ \min(Q_1, Q_2) \geq 1 \\ Q_1 + Q_2 = u + 1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \tag{357}$$

Thus, we can use 356 to enable ourselves to "drop" the requirement $\min(Q_1, Q_2) \geq 1$ from left hand side of 357, thus obtaining

$$\left\{ \begin{array}{l} u \in U \\ Q_1 + Q_2 = u + 1 \end{array} \right\} \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1 \tag{358}$$

This can be rewritten as

$$\forall u \in U \forall Q_1 \in \mathbb{N} \forall Q_2 \in \mathbb{N}(Q_1 + Q_2 = u + 1 \Longrightarrow Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1) \tag{359}$$

Now, everything to the right of "$\forall u \in U$" matches the definition of a statement $u + 1 \in U$. Thus, we have

$$\forall u \in U(u + 1 \in U) \tag{360}$$

And, together with $0 \in U$ that we read off from 352, we obtain

$$U = \mathbb{N} \tag{361}$$

Therefore,

$$\forall Q_1 \forall Q_2 (Q_1 + Q_2 \in U) \tag{362}$$

and, therefore

$$\forall Q_1 \forall Q_2 (Y_{0,0,Q_1,Q_2,Q_1+Q_2} = 1) \tag{363}$$

which completes the proof. **QED.**

Lemma 5

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow Y_{0,0,Q_1,Q_2,R} = \delta^R_{Q_1+Q_2} \tag{364}$$

Proof: This is direct consequence of combining Lemma 3 with Lemma 4. **QED.**

Lemma 6

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow Y_{0,P,Q_1,Q_2,R} = \frac{\delta^R_{Q_1+Q_2}}{P!} \tag{365}$$

Proof: From Theorem 2 we know that

$$Y_{0,P,Q_1,Q_2,R} = \frac{1}{P!} Y_{0,0,Q_1,Q_2,R} \tag{366}$$

and from Lemma 5 we know that

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow Y_{0,0,Q_1,Q_2,R} = \delta^R_{Q_1+Q_2} \tag{367}$$

The combination of the above two statements tells us

$$\max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow Y_{0,P,Q_1,Q_2,R} = \frac{\delta^R_{Q_1+Q_2}}{P!} \tag{368}$$

which completes the proof. **QED.**

Theorem

$$\min(P_1,P_2)+\max(Q_1,Q_2)\le R\le Q_1+Q_2\Longrightarrow Y_{P_1,P_2,Q_1,Q_2,RQ_1+Q_2}=\frac{1}{Q_1!Q_2!} \tag{369}$$

Proof We note that

$$\min(P_1,P_2)+\max(Q_1,Q_2)\le R\le Q_1+Q_2\Longrightarrow$$
$$\Longrightarrow \min(P_1-1,P_2-1)+\max(Q_1,Q_2)\le R-1\le Q_1+Q_2-1\Longrightarrow$$
$$\Longrightarrow Y_{P_1-1,P_2-1,Q_1,Q_2,R-1}=0\Longrightarrow$$
$$\Longrightarrow P_1Y_{P_1P_2Q_1Q_2R}=Y_{P_1-1,P_2,Q_1,Q_2,R}-Y_{P_1-1,P_2-1,Q_1,Q_2,R-1}=Y_{P_1-1,P_2,Q_1,Q_2,R} \tag{370}$$

Now, for any given values of P_2, Q_1 and Q_2, let us define the set $V_{P_2Q_1Q_2}$ to be

$$V_{P_2Q_1Q_2}=\left\{P_1\middle|\forall R\in\mathbb{N}(P_1+\max(Q_1,Q_2)\le R\le Q_1+Q_2\Longrightarrow\right.$$
$$\left.\Longrightarrow\left(Y_{P_1P_2Q_1Q_2R}=\frac{\delta^R_{Q_1+Q_2}}{P_1!P_2!}\right)\right\} \tag{371}$$

We then argue that

$$\left\{\begin{array}{l}P_1-1\in V_{P_2Q_1Q_2}\\ P_1+\max(Q_1,Q_2)\le R\le Q_1+Q_2\end{array}\right\}\Longrightarrow^{\text{apply 370 to 2−nd line}} \tag{372}$$

$$\Longrightarrow\left\{\begin{array}{l}P_1-1\in V_{P_2Q_1Q_2}\\ P_1Y_{P_1P_2Q_1Q_2R}=Y_{P_1-1,P_2,Q_1,Q_2,R}\end{array}\right\}\Longrightarrow^{\text{apply 371 to 1−st line}} \tag{373}$$

$$\Longrightarrow\left\{\begin{array}{l}Y_{P_1-1,P_2,Q_1,Q_2,R}=\frac{1}{(P_1-1)!P_2!}\\ P_1Y_{P_1P_2Q_1Q_2R}=Y_{P_1-1,P_2,Q_1,Q_2,R}\end{array}\right\}\Longrightarrow^{\text{substitute 1−st line into 2−nd line}} \tag{374}$$

$$\Longrightarrow P_1Y_{P_1P_2Q_1Q_2R}=\frac{1}{(P_1-1)!P_2!}\Longrightarrow^{\text{move P}_1\text{ to the right}} \tag{375}$$

$$\Longrightarrow Y_{P_1P_2Q_1Q_2R}=\frac{1}{P_1}\frac{1}{(P_1-1)!P_2!}=\frac{1}{P_1!P_2!} \tag{376}$$

We can rewrite the above as

$$\forall P_1\left[P_1-1\in V_{P_2Q_1Q_2}\Longrightarrow\right.$$
$$\left.\Longrightarrow\forall R\left(P_1+\max(Q_1,Q_2)\le R\le Q_1+Q_2\Longrightarrow Y_{P_1P_2Q_1Q_2R}=\frac{1}{P_1!P_2!}\right)\right] \tag{377}$$

Now, the second line of the above matches the definition fo a statement $P_1\in V_{P_2Q_1Q_2}$. Thus, we can rewrite the above as

$$\forall P_1(P_1-1\in V_{P_2Q_1Q_2}\longrightarrow P_1\in V_{P_2Q_1Q_2}) \tag{378}$$

Furthermore, the statement of Lemma 6 can be rewritten as

$$0 \in V(P_2 Q_1 Q_2) \tag{379}$$

And, therefore, we conclude by induction that

$$V_{P_2 Q_1 Q_2} = \mathbb{N} \tag{380}$$

At first, this might appear odd: after all, if it happens that $\min(P_1, P_2) > \min(Q_1, Q_2)$, then the condition $\min(P_1, P_2) + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2$ can't possibly hold. However, from the formal logical point of view, this doesn't pose any obstacles with regards to "truthfulness" of the statement. After all, any statement pertaining to "all elements of empty set" is always true by default. In other words, the "non-trivial" part of our statement pertains to the domain $\min(P_1, P_2) \leq \min(Q_1, Q_2)$; but we are always free to "add" a "trivial" statement pertaining to domain $\min(P_1, P_2) > \min(Q_1, Q_2)$ (the latter being trivial due to its reference to "all elements of empty set"). So, if we combine non-trivial together with trivial statements, we would indeed have a statement pertaining to *all* P_1, P_2, Q_1 and Q_2.

In any case, to continue with our proof, since the Eq 380 was derived for arbitrary P_2, Q_1 and Q_2, we can rewrite it as

$$\forall P_2 \forall Q_1 \forall Q_2 (V_{P_2 Q_1 Q_2} = \mathbb{N}) \tag{381}$$

and, by substituting the definition of V given in Eq 371, we have

$$\forall P_2 \forall Q_1 \forall Q_2 \forall P_1 \forall R \bigg(P_1 + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow \bigg(Y_{P_1 P_2 Q_1 Q_2 R} = \frac{\delta^R_{Q_1+Q_2}}{P_1! P_2!} \bigg) \bigg) \tag{382}$$

This statement is true both for $P_1 \leq P_2$ and $P_2 \leq P_1$. Thus, we have separately proven two things:

$$\min(P_1, P_2) + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow \bigg(Y_{P_1 P_2 Q_1 Q_2 R} = \frac{\delta^R_{Q_1+Q_2}}{P_1! P_2!} \bigg) \tag{383}$$

and

$$\max(P_1, P_2) + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow \bigg(Y_{P_1 P_2 Q_1 Q_2 R} = \frac{\delta^R_{Q_1+Q_2}}{P_1! P_2!} \bigg) \tag{384}$$

However, the first statement is stronger than the second. Thus, we will only have the first statement as our final answer:

$$\min(P_1, P_2) + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longrightarrow \bigg(Y_{P_1 P_2 Q_1 Q_2 R} = \frac{\delta^R_{Q_1+Q_2}}{P_1! P_2!} \bigg) \tag{385}$$

which is what we wanted to show. **QED.**

Theorem

$$Y_{\frac{p-L}{2}, \frac{p+L}{2}, \frac{q-L}{2}, \frac{q+L}{2}, \frac{p+q}{2}} = \frac{\delta^p_q}{\sqrt{\left(\frac{p-L}{2}\right)! \left(\frac{p+L}{2}\right)! \left(\frac{q-L}{2}\right)! \left(\frac{q+L}{2}\right)!}} \text{ if } p+q \text{ is even} \tag{386}$$

Proof We introduce the following notation:

$$P_1 = \frac{p-L}{2} \ , \ P_2 = \frac{p+L}{2} \ , \ Q_1 = \frac{q-L}{2} \ , \ Q_2 = \frac{q+L}{2} \ , \ R = \frac{p+q}{2} \tag{387}$$

Thus, we have

$$min(P_1, P_2) = \frac{p-|L|}{2} \ , \ \max(P_1, P_2) = \frac{p+|L|}{2} \tag{388}$$

$$\min(Q_1, Q_2) = \frac{q-|L|}{2} \ , \ \max(Q_1, Q_2) = \frac{q+|L|}{2} \tag{389}$$

and, therefore,

$$\min(P_1, P_2) + \max(Q_1, Q_2) = \frac{p-|L|}{2} + \frac{q+|L|}{2} = \frac{p+q}{2} = R \tag{390}$$

On the other hand,

$$Q_1 + Q_2 = \frac{q-L}{2} + \frac{q+L}{2} = q \tag{391}$$

and, therefore,

$$R \leq Q_1 + Q_2 \Longleftrightarrow \frac{p+q}{2} \leq q \Longleftrightarrow 0 \leq q - \frac{p+q}{2} = \frac{q-p}{2} \Longleftrightarrow p \leq q \tag{392}$$

In light of Eq 390 we know that $\min(P_1, P_2) + \max(Q_1, Q_2) \leq R$ is always true and, therefore, can be either included or dropped at will. Therefore,

$$\min(P_1, P_2) + \max(Q_1, Q_2) \leq R \leq Q_1 + Q_2 \Longleftrightarrow R \leq Q_1 + Q_2 \Longleftrightarrow p \leq q \tag{393}$$

Thus, the statement of the previous theorem becomes

$$p \leq q \Longrightarrow Y_{\frac{p-L}{2}, \frac{p+L}{2}, \frac{q-L}{2}, \frac{q+L}{2}, \frac{p+q}{2}} = \frac{\delta_q^{\frac{p+q}{2}}}{\frac{p-L}{2}! \frac{p+L}{2}!} \tag{394}$$

Now, since Y is symmetric with respect to $p \leftrightarrow q$, we can drop the assumption $p \leq q$, thus obtaining a general statement

$$Y_{\frac{p-L}{2}, \frac{p+L}{2}, \frac{q-L}{2}, \frac{q+L}{2}, \frac{p+q}{2}} = \frac{\delta_q^{\frac{p+q}{2}}}{\frac{p-L}{2}! \frac{p+L}{2}!} \tag{395}$$

Furthermore, we observe that

$$\frac{p+q}{2} = q \Longleftrightarrow 0 = q - \frac{p+q}{2} = \frac{q-p}{2} \Longleftrightarrow p = q \tag{396}$$

and, therefore,

$$\delta_q^{\frac{p+q}{2}} = \delta_q^p \tag{397}$$

Thus, we have

$$Y_{\frac{p-L}{2}, \frac{p+L}{2}, \frac{q-L}{2}, \frac{q+L}{2}, \frac{p+q}{2}} = \frac{\delta_q^p}{\frac{p-L}{2}! \frac{p+L}{2}!} \tag{398}$$

Now, we observe that

$$p = q \Longrightarrow \frac{1}{\frac{p-L}{2}!\frac{p+L}{2}!} = \frac{1}{\sqrt{\frac{p-L}{2}!\frac{p+L}{2}!\frac{q-L}{2}!\frac{q+L}{2}!}} \tag{399}$$

and, therefore,

$$\frac{\delta_q^p}{\frac{p-L}{2}!\frac{p+L}{2}!} = \frac{\delta_q^p}{\sqrt{\frac{p-L}{2}!\frac{p+L}{2}!\frac{q-L}{2}!\frac{q+L}{2}!}} \tag{400}$$

Thus, we rewrite Y as

$$Y_{\frac{p-L}{2},\frac{p+L}{2},\frac{q-L}{2},\frac{q+L}{2},\frac{p+q}{2}} = \frac{\delta_q^p}{\sqrt{\frac{p-L}{2}!\frac{p+L}{2}!\frac{q-L}{2}!\frac{q+L}{2}!}} \tag{401}$$

which coincides with our desired answer. **QED.**

11. Exercises

As one is about to see, the orthonormality is by far not the only thing that needs to be verified. However, this one example is hopefully enough to convince the reader that just because something looks mysterious it doesn't mean the answer is wrong; rather, it simply means that there are some difficult theorems to be proven. Therefore, we will leave the verifications of other things as exercises for the reader. The following list of things to be verified is by no means exclusive but it will inform the reader regarding the direction in which to think:

1. Note that in Section 7 we have started from the solution in Cartesian coordinates from which we then obtained various solutions in polar coordinates. Thus, we have never directly used a_{++} and a_{+-} in inductively obtaining such solutions from vacuum state. It would now be interesting to do so, and verify that we would get the same answer. Since doing that is too complicated, there is a short cut: we can take the list of explicit solutions we already have, and check that acting on them with $\{a_{++}, a_{+-}, a_{-+}, a_{--}\}$ (which are defined in terms of $\{r, \partial/\partial r, \theta, \partial/\partial\theta\}$) would indeed "take" us from one solution to the other in expected manner.

2. Write down Schrodinger's equation in polar coordinates and verify that ψ taken from Eq 201 and appropriately rescaled via $\partial(x, y)/\partial(r, \theta)$ obeys it.

3. Write down a general superposition of all possible states derived in Cartesian coordinates by means of separation of variables. After that, derive the formula for "rotation" of those states from (x, y) to (x', y') coordinate system. The rotation would take the form $(a_{x'}^\dagger)^i(a_{y'}^\dagger)^j|0\rangle = T_{ijkl}(\theta)(a_x^\dagger)^k(a_y^\dagger)^l|0\rangle$, where θ is the angle between the two coordinates systems. Verify that

a) $T_{ijkl}(\theta_1)T_{ijkl}(\theta_2) = T_{ijkl}(\theta_1 + \theta_2)$

b) If we put (x', y') in place of (x, y) and visa versa then $T_{ijkl}(\theta)$ will become $T_{ijkl}(-\theta)$

c) $T_{ijkl}(0) = 1$

d) Eigenstates of T_{ijkl} coincide with Eq 201 converted from polar to Cartesian coordinates.

4. Since the Hamiltonian is symmetric in position and momentum, one should expect that the Fourier transform of n-th excited state should, again, be n-th excited state, with appropriately chained coefficients.

a) Determine how the coefficients should change by mere inspection of the Hamiltonian

b) Carry out Fourier transform of Eq 155 and check that it matches Eq 155 with coefficients changed according to what was "predicted" in part a.

5. Repeat the same thing for 2D case, replacing Eq 155 with Eq 201, n with (n, L), and so forth.

6. Forget the wave function altogether. Instead, define $\hat{x}$ as $(a + a^\dagger)/\sqrt{2m\omega}$ where $[a, a^\dagger] = 1$ (this has nothing to do with derivatives; indeed, derivatives do not exist and the above commutation relation is merely an axiom). Postulate state $|0\rangle$ (which has nothing to do with wave function; it is merely "abstract entity" satisfying $a|0\rangle = 0$) then define $|n\rangle = C_n (a^\dagger)^n |0\rangle$ where C_n is selected in such a way that $\langle n|n\rangle = 0$. Use induction to derive the "eigenstate" of $\hat{x}$ given by $\hat{x}|x_0\rangle = x_0|x_0\rangle$; such eigenstate would take the form $|x_0\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + \cdots$. Prove that $c_n = \psi_n(x_0)$ where ψ_n is given in Eq 155.

7. Postulate algebraic structure given in Eq 41-45; once again, there is no coordinate system at all. Inspired by Eq 58, define an "object" $|r, \theta\rangle$ as an eigenstate of $(a_{++} + a_{-+})/\sqrt{m\omega}$ with an eigenvalue $re^{i\theta}$. Clearly, said "object" should be a linear combination of the form

$$\sum C_{kl} a_{++}^k a_{+-}^l |0\rangle \tag{402}$$

with appropriately chosen coefficients C_{kl}. Define $|\psi_{nL}\rangle$ as

$$|\psi_{nL}\rangle = \frac{1}{\sqrt{\langle 0| a_{--}^{(n+L)/2} a_{-+}^{(n-L)/2} a_{+-}^{(n-L)/2} a_{++}^{(n+L)/2} |0\rangle}} a_{+-}^{(n-L)/2} a_{++}^{(n+L)/2} |0\rangle \tag{403}$$

where it is assumed that n and L are either both even or both odd. Show by induction that the value of $\langle r, \theta|\psi_{nL}\rangle$ coincides with the value of $\psi_{nL}(r, \theta)$ given in Eq 201, up to appropriate rescaling via $\partial(x, y)/\partial(r, \theta)$, as long as r and θ used in defining $|r, \theta\rangle$ are both real.

8*. From the "physics" point of view, it is easy to see that

$$\delta(x_1 - x_2) = \langle x_1|x_2\rangle = \langle x_1| \left(\sum_{k=0}^{\infty} |k\rangle\langle k| \right) \left(\sum_{l=0}^{\infty} |l\rangle\langle l| \right) |x_2\rangle =$$

$$= \sum_{kl} \langle x_1|k\rangle\langle k|l\rangle\langle l|x_2\rangle = \sum_{kl} \psi_x(x_1)\delta_l^k \psi_l^*(x_2) = \sum_k \psi_k(x_1)\psi_k^*(x_2) \tag{404}$$

but from the "math" point of view the idea that Eq 155 satisfies

$$\sum_k \psi_k(x_1)\psi_k^*(x_2) = \delta(x_1 - x_2) \tag{405}$$

is not obvious at all. The point of this exercise is to prove the above *by means of mathematics alone*, without any use of "physics" (such as the "physics" used in derivation of Eq 404). This involves two steps:

a) Define the "partial sum" to be

$$s_n(x_1, x_2) = \sum_{k=0}^{n} \psi_k(x_1)\psi_k^*(x_2) \tag{406}$$

and perform numeric simulations in order to find out exactly "in what sense" does it approach δ-function. In particular, the reader should investigate whether or not there are any anomalies. One example of "possible" anomaly (that I don't claim takes place or otherwise) is the following: suppose $s_n(x-y)$ is large for $x - y \in (-n - 1/n^2, -n + 1/n^2) \cup (-1/n, 1/n) \cup (n - 1/n^2, n + 1/n^2)$. Thus, the ever increasing distances between these three intervals indicate that strictly speaking s_n is not "narrow". At the same time, it is also true that *restriction* of s_n *onto* $[-X, X]$ is narrow for any *fixed* X (however large) as $n \to \infty$. And it is equally true that if the heights of the three peaks are comparable, then most of the contribution to the integral comes from the peak around 0. Since our "physics" knowledge is based on math theorems that were only proven for compact spaces (such as $[-X, X]$) one can not rule out the above scenario. *Now, I am not claiming that the above anomaly takes place*: I myself don't know what happens since I haven't performed any simulations; and the anomaly just described is simply one random example I have pulled out of the air. The exercize for the reader is to do the simulations to see whether anomalies happen or not, and what kind of anomalies if any. After that, the reader should make a guess as to what kind of mathematically rigorous statement the numeric results fall into.

b) Provide mathematical proof that the "guess" made in part a is in fact true for ψ described in Eq 155. In other words, part a was done based on numeric simulations for "large number of n-s", while the goal of part b is to prove that the statement holds "for all n-s" as opposed to us simply being "lucky".

9*. Repeat the above for 2D. In other words, prove that Eq 201 satisfies

$$\sum_{n,L} \psi^*_{n,L}(r_1, \theta_1)\psi_{n,L}(r_2, \theta_2) = \delta(r_1 \cos\theta_1 - r_2 \cos\theta_2)\delta(r_1 \sin\theta_1 - r_2 \sin\theta_2) \tag{407}$$

given the correct definition of "approaching δ-function" that you are supposed to provide.

12. Conclusion

We have successfully derived the expressions for general state of harmonic oscillator in 1D and 2D. It turns out that what we did is both easy and complicated at the same time. On the one hand, if one is interested to learn the derivation of general wavefunction in quickest possible way, one can limit themselves to Sections 2-4 and 7-8, which is relatively easy to follow. If, on the other hand, one is interested in verifying mathematics, one can read sections 6 and 9-10 which are a lot more complicated and finally do the exercises which would make them work even harder. This means that harmonic oscillator might be an excellent way of learning physics since, on the one hand, the reader can grasp the concepts on how derivations work (as opposed to merely memorizing results, as is the case for hydrogen atom) and, on the other hand, there are very difficult problems for the reader to think about. Combination of these two factors would allow the reader to tackle through difficult problems without relying on outside help, which is a key to truly learning the material. Some of the challenges directly involve the rotational symmetry (see Exercize 3) which seems to suggest that harmonic oscillator might also be a good tool to teach angular momentum (in which case the instructor would have to alter the sequence of topics).

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