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The Pioneer Anomaly in Covariant Theory of Gravitation

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Abstract: The difference of equations of motion in the covariant theory of gravitation and in the general theory of relativity is used to explain the Pioneer anomaly. Calculation shows that the velocities of a spacecraft in both theories at equal distances can differ by several centimetres per second. This leads also to a possible explanation of the flyby anomaly and comet disturbances which are not taken into account by the general theory of relativity.

Keywords: Pioneer anomaly, covariant theory of gravitation, general theory of relativity, equation of motion, flyby anomaly.

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Résumé: Les differences entre les équations de mouvement dans la théorie covariante de la gravitation et dans la théorie générale de la gravitation sont utilisées pour expliquer “l'anomalie Pioneer”. Le calcul montre que les vélocités des sondes spatiales dans les deux theories aux distances différentes peuvent se différencier à plusieurs cm/s. Cela amène à l’explication possible de l'anomalie “flyby” et aux perturbations de la comète qui ne sont pas pris en compte dans la théorie générale de la relativité.

1. Introduction

The stories of the American spacecrafts Pioneer 10 and Pioneer 11 began on 2 March 1972, and, respectively, on 6 April 1973, respectively, at the times of their launches. Both spacecrafts passed in the ecliptic plane of the entire Solar system in opposite directions, passing close to different planets. Pioneer 10 on 4 December 1973, reached Jupiter, located at a distance of 5.2 a.u. from the Sun (1 a.u. = $1.496 \cdot 10^{11}$ m), in June 1983 it passed Pluto (39.4 a.u.), in May 2001 it was at the distance of 78 a.u., moving at a speed of nearly 13 km/s.

Starting from a distance of about 20 a.u., when it was evident from the Doppler signal from Pioneer 10 that the shift of the speed significantly decreased, caused by the pressure of the solar plasma on the spacecraft, after taking into account all other possible causes of acceleration, the residual signal from the spacecraft started to show the presence of an anomalous acceleration towards the Sun, of the order of $8 \cdot 10^{-10}$ m/s^2 [1]. For Pioneer 11 a similar acceleration was of about $8.6 \cdot 10^{-10}$ m/s^2; for the spacecraft Ulysses at distances of 1.3 – 5.2 a.u. the acceleration reached $(12 \pm 3) \cdot 10^{-10}$ m/s^2, while for the spacecraft Galileo – $8 \cdot 10^{-10}$ m/s^2.

There are some possible explanations for anomalous acceleration of the spacecrafts. One of them for the Pioneer 10 and 11 spacecraft is due to the recoil force associated with an anisotropic emission of thermal radiation off the vehicles [2-3]. The other explanations of the Pioneer anomaly include new gravitational physical mechanisms [4-10].

The covariant theory of gravitation (CTG) is an alternative theory to the general theory of relativity (GTR) and we present further CTG approach to the problem of the Pioneer anomaly by comparing of calculations of CTG and GTR.

2. Metric tensor in CTG

The metric tensor in spherical coordinates $x^0 = ct$, $x^1 = r$, $x^2 = Q$, $x^3 = \varphi$ has the following form:

$$
g_{ik} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E\sin^2 Q \end{pmatrix}, \qquad (1)
$$

and for the functions B, K, E we assume that they are the functions only of the radial coordinate r as the distance from the center of the massive body (where we placed the origin) to the observation point, located outside the body.

The components of the metric tensor g_{ik} are as follows [11-12]:

$$
B = g_{00} = 1 + \frac{GM\alpha}{rc^2} - \frac{\beta G^2 M^2}{r^2 c^4}, \qquad E = -g_{22} = r^2, \qquad (2)
$$

$$
K = -g_{11} = \frac{1}{B} = \frac{1}{1 + \frac{GM\alpha}{rc^2} - \frac{\beta G^2 M^2}{r^2 c^4}}, \qquad g_{33} = -r^2 \sin^2 Q,
$$

$$
g^{00} = \frac{1}{B}, \qquad g^{11} = -B, \qquad g^{22} = -\frac{1}{r^2}, \qquad g^{33} = -\frac{1}{r^2 \sin^2 Q},
$$

where α and β are values that cannot be determined from the equations for the metric, which leads to their possible dependence on the properties of test particles in the gravitational field,

M is the body mass, near which the metric is determined,

G is the gravitational constant,

$c = c_g$ is the speed of gravitation propagation.

We shall express the metric tensor g^{in} in terms of Cartesian coordinates. For the relation of the Cartesian and the spherical coordinates we have:

$$x = r\sin Q\cos\varphi, \qquad y = r\sin Q\sin\varphi, \qquad z = r\cos Q. \tag{3}$$

$$x^2 + y^2 + z^2 = r^2 . \tag{4}$$

$$dx = dr\sin Q\cos\varphi + dQ\, r\cos Q\cos\varphi - d\varphi\, r\sin Q\sin\varphi ,$$

$$dy = dr\sin Q\sin\varphi + dQ\, r\cos Q\sin\varphi + d\varphi\, r\sin Q\cos\varphi ,$$

$$dz = dr\cos Q - dQ\, r\sin Q . \tag{5}$$

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + (r\,dQ)^2 + (r\sin Q\, d\varphi)^2 ,$$

$$d\vec{\ell} = (dx,\ dy,\ dz), \qquad d\vec{\ell} = (dr,\ r\,dQ,\ r\sin Q\, d\varphi) . \tag{6}$$

Relations (3) are the rules with the help of which by the known the spherical coordinates r, Q, φ the Cartesian coordinates of the point are found. Because of this definition, (4) for the Cartesian coordinates will hold in the Riemannian space.

In (6) the three-vector of displacement of the test particle $d\vec{\ell}$ has projections on three mutually perpendicular axes of the Cartesian coordinate system, equal to dx, dy and dz. The similar projections of the three-vector $d\vec{\ell}$ on three mutually perpendicular axes of the spherical coordinate system are equal to dr, $r\,dQ$ and $r\sin Q\, d\varphi$. One unit vector in the spherical coordinate system is directed along the radial coordinate r and the other two are perpendicular to it and are directed along the meridians and parallels, where the changes of the angles dQ and $d\varphi$ are measured.

In view of (6) the four-vector of displacement which is symmetrical with respect to dimensions (the four-vector of the distance differential) in spherical coordinates has the form:

$$dx^i = (cdt, dr, r\,dQ, r\sin Q\, d\varphi) . \qquad (7)$$

To find g^{in} in the Cartesian coordinates through its form in the spherical coordinates (2) we must take into account the existing relationship between the coordinates and the components of the four-vector of displacement. In the Cartesian coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and $dx^i = (cdt, dx, dy, dz)$, so to obtain the four-vector of displacement it is sufficient to take the differentials of the coordinates. For the spherical coordinates $x^0 = ct$, $x^1 = r$, $x^2 = Q$, and $x^3 = \varphi$, but to obtain the corresponding four-vector of displacement it is not enough just to use the differentials of the coordinates, we must also multiply them by some functions of the coordinates, as seen in (7). Only in this case it becomes possible to compare the four-vectors of displacement, expressed in different reference frames.

However, as follows from (2), the various components of the metric tensor in the spherical coordinates, as well as the corresponding Christoffel coefficients have different dimensions. This means that the four-vector of displacement in the spherical coordinates should be asymmetrical with respect to dimension and have the form:

$$dx^i = (cdt, dr, dQ, d\varphi) . \qquad (8)$$

In general, the transformation of four-vectors and tensors from one frame to another is performed by using the transformation matrices of the form $A^i_k = \frac{\partial x'^i}{\partial x^k}$ and $B_k{}^i = \frac{\partial x^i}{\partial x'^k}$, so for an arbitrary tensor the transformation of four -coordinates x^i into the four -coordinates x'^i is valid:

$$T^{mn...}{}_{rs...}{}'(x'^i) = A^m{}_k A^n{}_l ... B_r{}^p B_s{}^q ... T^{kl...}{}_{pq...}(x^i) \,. \qquad (9)$$

We shall find the transformation matrix A^i_k , with the help of which the four -vector (8) can be transformed into the four -vector of displacement in Cartesian coordinates. If we take into account the relations for the differentials (5), which give the expressions for the corresponding partial derivatives, standing before the differentials, then we shall obtain:

$$A^i_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin Q \cos\varphi & r\cos Q \cos\varphi & -r\sin Q \sin\varphi \\ 0 & \sin Q \sin\varphi & r\cos Q \sin\varphi & r\sin Q \cos\varphi \\ 0 & \cos Q & -r\sin Q & 0 \end{pmatrix}, \qquad (10)$$

$$dx^i(ct, x, y, z) = A^i_k \, dx^k(ct, r, Q, \varphi) \,, \qquad (11)$$

where dx^k is from (8).

To complete the transition from the spherical to Cartesian variables, the angles Q and φ in (10) should be expressed through x, y and z with the help of (3).

Applying to the tensor g^{ik} from (2) the transformation (9) with the help of A^i_k from (10) we find the corresponding metric tensor in the Cartesian variables:

$$g^{mn}(ct,x,y,z) = A_i^m A_k^n g^{ik}(ct,r,Q,\varphi), \quad (12)$$

$$g^{mn} = \begin{pmatrix} \frac{1}{B} & 0 & 0 & 0 \\ 0 & C\sin^2 Q\cos^2\varphi - 1 & C\sin^2 Q\sin\varphi\cos\varphi & C\sin Q\cos Q\cos\varphi \\ 0 & C\sin^2 Q\sin\varphi\cos\varphi & C\sin^2 Q\sin^2\varphi - 1 & C\sin Q\cos Q\sin\varphi \\ 0 & C\sin Q\cos Q\cos\varphi & C\sin Q\cos Q\sin\varphi & C\cos^2 Q - 1 \end{pmatrix},$$

where $C = -\frac{GM\alpha}{rc^2} + \frac{\beta G^2 M^2}{r^2 c^4} = 1 - B$.

After replacing the trigonometric functions of the angles Q and φ through x, y and z with the help of (3) the metric tensor (12) in the Cartesian coordinates become as follows:

$$g^{mn} = \begin{pmatrix} \frac{1}{B} & 0 & 0 & 0 \\ 0 & \frac{Cx^2}{r^2} - 1 & \frac{Cxy}{r^2} & \frac{Cxz}{r^2} \\ 0 & \frac{Cyx}{r^2} & \frac{Cy^2}{r^2} - 1 & \frac{Cyz}{r^2} \\ 0 & \frac{Czx}{r^2} & \frac{Czy}{r^2} & \frac{Cz^2}{r^2} - 1 \end{pmatrix}. \quad (13)$$

Because for the metric tensor the equality holds: $g_{sm} g^{mn} = \delta_s{}^n$, where $\delta_s{}^n = \begin{cases} 1, & s = n \\ 0, & s \neq n \end{cases}$, it allows us to find g_{sm} by the known form g^{mn}. In particular, for each component of the metric tensor with covariant indices we can write:

$$g_{sm} = \frac{D^{sm}}{g},$$

where D^{sm} is the algebraic supplement to the components of the metric tensor g^{mn} with contravariant indices, which is the minor of the matrix of the tensor with the corresponding sign,

g is the determinant of the metric tensor g^{mn}, in our case $g = -1$.

Using this rule, we find g_{sm}:

$$g_{sm} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & -1-\dfrac{Cx^2}{Br^2} & -\dfrac{Cxy}{Br^2} & -\dfrac{Cxz}{Br^2} \\ 0 & -\dfrac{Cyx}{Br^2} & -1-\dfrac{Cy^2}{Br^2} & -\dfrac{Cyz}{Br^2} \\ 0 & -\dfrac{Czx}{Br^2} & -\dfrac{Czy}{Br^2} & -1-\dfrac{Cz^2}{Br^2} \end{pmatrix}. \tag{14}$$

With the components of metric tensor (13) and (14) we find the non-zero Christoffel symbols for the Cartesian coordinates:

$$\Gamma^s_{ik} = \frac{1}{2} g^{sm} \left(\partial_i g_{mk} + \partial_k g_{mi} - \partial_m g_{ik} \right),$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{B'x}{2Br}, \qquad \Gamma^0_{02} = \Gamma^0_{20} = \frac{B'y}{2Br}, \qquad \Gamma^0_{03} = \Gamma^0_{30} = \frac{B'z}{2Br}, \qquad \Gamma^1_{00} = \frac{B'Bx}{2r},$$

$$\Gamma^1_{11} = \frac{x(1-B)}{r^2} + \frac{x^3 B}{r^4} - \frac{x^3(B'r + 2B)}{2Br^4}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{x^2 yB}{r^4} - \frac{x^2 y(B'r + 2B)}{2Br^4},$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{x^2 zB}{r^4} - \frac{x^2 z(B'r + 2B)}{2Br^4}, \quad \Gamma^1_{22} = \frac{x(1-B)}{r^2} + \frac{xy^2 B}{r^4} - \frac{xy^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{xyzB}{r^4} - \frac{xyz(B'r + 2B)}{2Br^4}, \quad \Gamma^1_{33} = \frac{x(1-B)}{r^2} + \frac{xz^2 B}{r^4} - \frac{xz^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^2_{11} = \frac{y(1-B)}{r^2} + \frac{x^2 yB}{r^4} - \frac{x^2 y(B'r + 2B)}{2Br^4}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{xy^2 B}{r^4} - \frac{xy^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \frac{xyzB}{r^4} - \frac{xyz(B'r + 2B)}{2Br^4}, \quad \Gamma^2_{22} = \frac{y(1-B)}{r^2} + \frac{y^3 B}{r^4} - \frac{y^3 (B'r + 2B)}{2Br^4},$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{y^2 zB}{r^4} - \frac{y^2 z(B'r + 2B)}{2Br^4}, \quad \Gamma^2_{33} = \frac{y(1-B)}{r^2} + \frac{yz^2 B}{r^4} - \frac{yz^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^2_{00} = \frac{B'By}{2r}, \quad \Gamma^3_{00} = \frac{B'Bz}{2r}, \quad \Gamma^3_{11} = \frac{z(1-B)}{r^2} + \frac{x^2 zB}{r^4} - \frac{x^2 z(B'r + 2B)}{2Br^4},$$

$$\Gamma^3_{12} = \Gamma^3_{21} = \frac{xyzB}{r^4} - \frac{xyz(B'r + 2B)}{2Br^4}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{xz^2 B}{r^4} - \frac{xz^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^3_{22} = \frac{z(1-B)}{r^2} + \frac{y^2 zB}{r^4} - \frac{y^2 z(B'r + 2B)}{2Br^4}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{yz^2 B}{r^4} - \frac{yz^2 (B'r + 2B)}{2Br^4},$$

$$\Gamma^3_{33} = \frac{z(1-B)}{r^2} + \frac{z^3 B}{r^4} - \frac{z^3 (B'r + 2B)}{2Br^4}, \tag{15}$$

where we used the equalities of the type $\frac{\partial B}{\partial x} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial x} = \frac{B'x}{r}$, as well as $r^2 = x^2 + y^2 + z^2$.

With the help of (14) and the expression for the four-vector of displacement $dx^s = (cdt, dx, dy, dz)$ we find the square of the interval:

$$(ds)^2 = c^2(d\tau)^2 = g_{sm}\,dx^s\,dx^m = Bc^2(dt)^2 - \left[(dx)^2 + (dy)^2 + (dz)^2\right] - \frac{C}{Br^2}\left(xdx + ydy + zdz\right)^2. \tag{16}$$

The expression (16) for the square of the interval according to [11] coincides with one of the two so-called normal forms for the Cartesian coordinates [13]. We obtained it without solving the equations for the metric in the Cartesian coordinates, by recalculation the metric in spherical coordinates.

From (16) for the differential of the proper time of a test particle near a massive body it follows:

$$d\tau = dt\sqrt{B - \frac{1}{c^2}\left(\frac{d\ell}{dt}\right)^2 - \frac{1-B}{Bc^2}\left(\frac{dr}{dt}\right)^2} = dt\sqrt{B - \frac{V^2}{c^2} - \frac{1-B}{Bc^2}\left(\frac{dr}{dt}\right)^2}\,, \tag{17}$$

where $V = \frac{d\ell}{dt}$ is the total velocity of the test particle,

and in the derivation of (17) we used the relations: $\left[d(x^2) + d(y^2) + d(z^2)\right]^2 = [d(r^2)]^2 = 4r^2(dr)^2$, $\quad (dx)^2 + (dy)^2 + (dz)^2 = (d\ell)^2$.

3. Equation of motion in CTG

In CTG, in contrast to GTR, there is its own equation of motion of test bodies, which changes the results of the calculations. We shall use the equation of motion of test particles in the gravitational field in the form deduced from the principle of least action for CTG [11], [14-16]:

$$\rho_0 \frac{du^i}{d\tau} + \Gamma^i_{ks} u^k J^s = g^{in} \Phi_{nk} J^k \,, \qquad (18)$$

where ρ_0 is the mass density in the reference frame associated with the test particle,

u^i is the four-velocity of the test particle,

$J^i = \rho_0 u^i$ is the mass four-current density,

$d\tau$ is the differential of the proper dynamic time of the test particle,

Φ_{nk} is the tensor of gravitational field,

Γ^i_{ks} are the Christoffel symbols.

The four-vector J^i in the Cartesian coordinates can be represented as follows:

$$J^i = \rho_0 \frac{dx^i}{d\tau} = \left(\frac{\rho_0 c dt}{d\tau}, \frac{\rho_0 d\vec{\ell}}{d\tau}\right) , \qquad (19)$$

where $d\vec{\ell} = (dx, dy, dz)$.

In the static case the four-vector of the gravitational potential has the form $D_i = \left(\frac{\psi}{c}, 0, 0, 0\right)$, where the scalar potential $\psi = -\frac{GM}{r}$. This gives the tensor of gravitational field strengths with the components:

$$\Phi_{ik} = \partial_i D_k - \partial_k D_i = \begin{pmatrix} 0 & -\frac{GMx}{cr^3} & -\frac{GMy}{cr^3} & -\frac{GMz}{cr^3} \\ \frac{GMx}{cr^3} & 0 & 0 & 0 \\ \frac{GMy}{cr^3} & 0 & 0 & 0 \\ \frac{GMz}{cr^3} & 0 & 0 & 0 \end{pmatrix}. \qquad (20)$$

Substituting (19) and (20) into the equations of motion (18), taking into account metric tensor (13) and non-zero Christoffel symbols (15), with the values of the index $i = 0,1,2,3$, we obtain four equations of motion in Cartesian coordinates:

$$\rho_0 \frac{du^0}{d\tau} + \Gamma^0_{01} u^0 J^1 + \Gamma^0_{10} u^1 J^0 + \Gamma^0_{02} u^0 J^2 + \Gamma^0_{20} u^2 J^0 + \Gamma^0_{03} u^0 J^3 + \Gamma^0_{30} u^3 J^0 = g^{00} \Phi_{0k} J^k .$$

$$\rho_0 \frac{du^1}{d\tau} + \Gamma^1_{00} u^0 J^0 + \Gamma^1_{11} u^1 J^1 + \Gamma^1_{12} u^1 J^2 + \Gamma^1_{21} u^2 J^1 + \Gamma^1_{13} u^1 J^3 + \Gamma^1_{31} u^3 J^1 + \\ + \Gamma^1_{22} u^2 J^2 + \Gamma^1_{23} u^2 J^3 + \Gamma^1_{32} u^3 J^2 + \Gamma^1_{33} u^3 J^3 = g^{1k} \Phi_{k0} J^0 .$$

$$\rho_0 \frac{du^2}{d\tau} + \Gamma^2_{00} u^0 J^0 + \Gamma^2_{11} u^1 J^1 + \Gamma^2_{12} u^1 J^2 + \Gamma^2_{21} u^2 J^1 + \Gamma^2_{13} u^1 J^3 + \Gamma^2_{31} u^3 J^1 + \\ + \Gamma^2_{22} u^2 J^2 + \Gamma^2_{23} u^2 J^3 + \Gamma^2_{32} u^3 J^2 + \Gamma^2_{33} u^3 J^3 = g^{2k} \Phi_{k0} J^0 .$$

$$\rho_0 \frac{du^3}{d\tau} + \Gamma^3_{00} u^0 J^0 + \Gamma^3_{11} u^1 J^1 + \Gamma^3_{12} u^1 J^2 + \Gamma^3_{21} u^2 J^1 + \Gamma^3_{13} u^1 J^3 + \Gamma^3_{31} u^3 J^1 + \\ + \Gamma^3_{22} u^2 J^2 + \Gamma^3_{23} u^2 J^3 + \Gamma^3_{32} u^3 J^2 + \Gamma^3_{33} u^3 J^3 = g^{3k} \Phi_{k0} J^0 ,$$

here the nonzero terms are indicated, and by the repeated index k, with the values $k = 1,2,3$ summation is made as usual.

We shall write down the equations for the motion in time and for the motion along the axis OX in the explicit form:

$$\rho_0 \frac{d}{d\tau}\left(\frac{cdt}{d\tau}\right)+\frac{B'\rho_0 c}{Br}\frac{dt}{d\tau}\left(\frac{xdx}{d\tau}+\frac{ydy}{d\tau}+\frac{zdz}{d\tau}\right)=-\frac{GM\rho_0}{Bcr^3}\left(\frac{xdx}{d\tau}+\frac{ydy}{d\tau}+\frac{zdz}{d\tau}\right). \qquad (21)$$

$$\rho_0 \frac{d}{d\tau}\left(\frac{dx}{d\tau}\right)+\frac{B'B\rho_0 xc^2}{2r}\left(\frac{dt}{d\tau}\right)^2+\frac{(1-B)\rho_0 x}{r^2}\left[\left(\frac{dx}{d\tau}\right)^2+\left(\frac{dy}{d\tau}\right)^2+\left(\frac{dz}{d\tau}\right)^2\right]+$$
$$+\frac{\rho_0 x}{4}\left(\frac{B}{r^4}-\frac{B'r+2B}{2Br^4}\right)\left[\frac{d(x^2)}{d\tau}+\frac{d(y^2)}{d\tau}+\frac{d(z^2)}{d\tau}\right]^2=-\frac{BGM\rho_0 x}{r^3}\frac{dt}{d\tau}.$$

(22)

We shall further cancel ρ_0 in (21)-(22). By putting x, y, z in (21) under the signs of the differentials and further summation, taking into account the equality $x^2+y^2+z^2=r^2$, we can transform (21). Then after multiplying all the parts of (21) by B we shall obtain:

$$B\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right)+B'\frac{dt}{d\tau}\frac{dr}{d\tau}=\frac{d}{d\tau}\left(B\frac{dt}{d\tau}\right)=-\frac{GM}{c^2r^2}\frac{dr}{d\tau}, \qquad (23)$$

here we have used equality $\frac{dB}{d\tau}=\frac{dB}{dr}\frac{dr}{d\tau}=B'\frac{dr}{d\tau}$.

Equation (22), taking into account:

$$\left[d(x^2)+d(y^2)+d(z^2)\right]^2=[d(r^2)]^2=4r^2(dr)^2, \quad (dx)^2+(dy)^2+(dz)^2=(d\ell)^2,$$

can be transformed to the following form:

$$\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right)+\frac{B'Bxc^2}{2r}\left(\frac{dt}{d\tau}\right)^2+\frac{(1-B)x}{r^2}\left(\frac{d\ell}{d\tau}\right)^2+x\left(\frac{B}{r^2}-\frac{B'r+2B}{2Br^2}\right)\left(\frac{dr}{d\tau}\right)^2=-\frac{BGMx}{r^3}\frac{dt}{d\tau}. \tag{24}$$

For the motion along the axes OY and OZ, respectively, we obtain:

$$\frac{d}{d\tau}\left(\frac{dy}{d\tau}\right)+\frac{B'Byc^2}{2r}\left(\frac{dt}{d\tau}\right)^2+\frac{(1-B)y}{r^2}\left(\frac{d\ell}{d\tau}\right)^2+y\left(\frac{B}{r^2}-\frac{B'r+2B}{2Br^2}\right)\left(\frac{dr}{d\tau}\right)^2=-\frac{BGMy}{r^3}\frac{dt}{d\tau}. \tag{25}$$

$$\frac{d}{d\tau}\left(\frac{dz}{d\tau}\right)+\frac{B'Bzc^2}{2r}\left(\frac{dt}{d\tau}\right)^2+\frac{(1-B)z}{r^2}\left(\frac{d\ell}{d\tau}\right)^2+z\left(\frac{B}{r^2}-\frac{B'r+2B}{2Br^2}\right)\left(\frac{dr}{d\tau}\right)^2=-\frac{BGMz}{r^3}\frac{dt}{d\tau}. \tag{26}$$

In (24) – (26) the value $\frac{d\ell}{d\tau}$ is the total velocity and $\frac{dr}{d\tau}$ is the radial velocity of the test particle. Further we shall consider the case of motion of a test body near the Sun, when the orbit is in the equatorial plane of the spherical coordinate system, and correspondingly in the plane XOY of the Cartesian coordinate system. Then for the test body $z=0$, the velocity $\frac{dz}{d\tau}=0$, in (26) $\frac{d}{d\tau}\left(\frac{dz}{d\tau}\right)=0$, and over time the coordinate z does not change.

After cancelling $d\tau$ (23) can be integrated:

$$B\frac{dt}{d\tau}=\frac{GM}{c^2 r}+A_1. \tag{27}$$

At infinity the gravitational influence of the Sun can be neglected, and we can assume that the test body moves inertially. Then the coordinate time t differs from the proper time of the test body τ only by the Lorentz factor, so we can determine the value of the constant: $A_1 = \left(\frac{dt}{d\tau}\right)_\infty = \frac{1}{\sqrt{1-V_0^2/c^2}}$, where V_0 is the velocity of the test body at infinity. We can also specify that the velocity V_0 at infinity must be, at least to a small degree, directed to the Sun, otherwise the test body will never get close to it.

To simplify the further solution we shall convert the equations (27), (24) and (25) to the polar coordinates in the plane of motion of the test body XOY, with the Sun at the origin. Substituting $x = r\cos\varphi$ and $y = r\sin\varphi$ into (24) and (25), expressing the total velocity in terms of the radial and tangential velocity components in the form: $\left(\frac{d\ell}{d\tau}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2$, we find:

$$\begin{aligned} -2\sin\varphi\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + \cos\varphi\frac{d^2r}{d\tau^2} - r\sin\varphi\frac{d^2\varphi}{d\tau^2} + \frac{B'Bc^2\cos\varphi}{2}\left(\frac{dt}{d\tau}\right)^2 - \\ - \frac{B'\cos\varphi}{2B}\left(\frac{dr}{d\tau}\right)^2 - Br\cos\varphi\left(\frac{d\varphi}{d\tau}\right)^2 = -\frac{BGM\cos\varphi}{r^2}\frac{dt}{d\tau}. \end{aligned} \tag{28}$$

$$\begin{aligned} 2\cos\varphi\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + \sin\varphi\frac{d^2r}{d\tau^2} + r\cos\varphi\frac{d^2\varphi}{d\tau^2} + \frac{B'Bc^2\sin\varphi}{2}\left(\frac{dt}{d\tau}\right)^2 - \\ - \frac{B'\sin\varphi}{2B}\left(\frac{dr}{d\tau}\right)^2 - Br\sin\varphi\left(\frac{d\varphi}{d\tau}\right)^2 = -\frac{BGM\sin\varphi}{r^2}\frac{dt}{d\tau}. \end{aligned} \tag{29}$$

We can get rid of sines and cosines, if we multiply (28) by $\cos\varphi$ and (29) by $\sin\varphi$, and then, respectively, add the two equations. We can also multiply (28) by $\sin\varphi$ and (29) by $\cos\varphi$ and subtract the equations from each other. The results will be as follows:

$$\frac{d^2r}{d\tau^2}+\frac{B'Bc^2}{2}\left(\frac{dt}{d\tau}\right)^2-\frac{B'}{2B}\left(\frac{dr}{d\tau}\right)^2-Br\left(\frac{d\varphi}{d\tau}\right)^2=-\frac{BGM}{r^2}\frac{dt}{d\tau}. \qquad (30)$$

$$2\frac{d\varphi}{d\tau}\frac{dr}{d\tau}+r\frac{d^2\varphi}{d\tau^2}=0. \qquad (31)$$

Equation (31) is immediately integrated:

$$\frac{r^2\,d\varphi}{d\tau}=L=const. \qquad (32)$$

From (32) we see that during the motion of the test body the quantity L is preserved, which is proportional to the density of the orbital angular momentum. Dividing (32) by (27), we find:

$$\frac{d\varphi}{dt}=\frac{BL}{r^2\left(\frac{GM}{c^2r}+A_1\right)}. \qquad (33)$$

Since the square of the total velocity V of the test body in the polar coordinates is composed of the square of the radial component $\dot{r}=\frac{dr}{dt}$ and the square of the tangential component $V_\tau = r\dfrac{d\varphi}{dt}$ in the form: $V^2 = \dot{r}^2 + V_\tau^2$, then the differential of the proper dynamic time (17), taking into account (33) will equal:

$$d\tau = dt\sqrt{B - \frac{\dot{r}^2}{Bc^2} - \frac{B^2L^2}{r^2c^2\left(\frac{GM}{c^2r} + A_1\right)^2}}\,. \qquad (34)$$

From (34) and (27) we find $\dot{r} = \dfrac{dr}{dt}$ and then $\dfrac{dr}{d\tau}$:

$$\frac{dr}{dt} = \pm B\sqrt{c^2 - \frac{BL^2}{r^2\left(\frac{GM}{c^2r} + A_1\right)^2} - \frac{Bc^2}{\left(\frac{GM}{c^2r} + A_1\right)^2}}\,. \qquad (35)$$

$$\frac{dr}{d\tau} = \pm\sqrt{c^2\left(\frac{GM}{c^2r} + A_1\right)^2 - \frac{BL^2}{r^2} - Bc^2}\,. \qquad (36)$$

After using (32) in (30) we obtain:

$$\frac{d^2r}{d\tau^2} + \frac{B'Bc^2}{2}\left(\frac{dt}{d\tau}\right)^2 - \frac{B'}{2B}\left(\frac{dr}{d\tau}\right)^2 - \frac{BL^2}{r^3} = -\frac{BGM}{r^2}\frac{dt}{d\tau}\,. \qquad (37)$$

We shall substitute $\dfrac{dt}{d\tau}$ from (27) into (37):

$$\frac{d^2r}{d\tau^2} + \frac{B'c^2}{2B}\left(\frac{GM}{c^2r} + A_1\right)^2 - \frac{B'}{2B}\left(\frac{dr}{d\tau}\right)^2 - \frac{BL^2}{r^3} = -\frac{GM}{r^2}\left(\frac{GM}{c^2r} + A_1\right). \qquad (38)$$

In fact, we have already found $\frac{dr}{d\tau}$ in (36) through the interval, and it is easy to check that its value is the solution of the equation of motion (38).

Because according to (32) $\frac{d\varphi}{d\tau}=\frac{L}{r^2}$, then dividing $\frac{dr}{d\tau}$ from (36) by $\frac{d\varphi}{d\tau}$, we find the equation of motion of the test body near the Sun in polar coordinates:

$$\frac{dr}{d\varphi}=\pm\frac{r^2}{L}\sqrt{c^2\left(\frac{GM}{c^2 r}+A_1\right)^2-\frac{BL^2}{r^2}-Bc^2}\,. \quad (39)$$

$$\varphi=\pm L\int\frac{dr}{r^2\sqrt{c^2\left(\frac{GM}{c^2 r}+A_1\right)^2-\frac{BL^2}{r^2}-Bc^2}}+A_2\,. \quad (40)$$

Relation (40) is the solution of the problem in the general case. There is a special case in which the initial velocity V_0 of the test body is zero, or is directed straight to the Sun. In this case the angular momentum of the test body is zero, $L=0$, and the angle of incidence of the test body does not change with time. In other cases, during the motion of the test body, it may, depending on the direction and the magnitude of the initial velocity, get close to the Sun for the minimal distance R and then again move away from the Sun, deflecting at some angle. With the distance R the radial velocity becomes equal to zero: $\dot{r}=\frac{dr}{dt}=0$. At this point, the total velocity of the test body V is perpendicular to the radius-vector directed from the Sun, and is equal to the tangential component of velocity. From (35) with $\dot{r}=0$ taking into account relations (2) for B we see that the constant L can be found through R and the initial velocity, which is included through A_1 in (27):

$$L = Rc\sqrt{\frac{\left(\frac{GM}{c^2 R}+A_1\right)^2}{1+\frac{GM\alpha}{Rc^2}-\frac{\beta G^2 M^2}{R^2 c^4}}-1}\,. \tag{41}$$

We can compare the relativistic solution (40) with the formula for the motion of the particle in gravitational field of the central type in the classical case [17]:

$$\varphi = \pm L\int \frac{dr}{r^2\sqrt{2(\varepsilon-\psi)-\frac{L^2}{r^2}}}+A_2\,, \tag{42}$$

where ε is proportional to the total energy of the particle and at infinity is equal to $\frac{V_0^2}{2}$, $\psi = -\frac{GM}{r}$ is the potential of the gravitational field.

If in (40) we neglect the curvature of space-time, assuming $B=1$, eliminating the small terms of the form $\frac{\beta G^2 M^2}{c^4 R^2}$ and $\frac{V_0^4}{c^4}$, and if we subtract under the root the rest energy of the unit mass, equal to c^2, then (40) turns into (42).

4. Equation of radial motion in GTR

The standard equations of motion of the test particle near the massive body in the GTR was described, for example, in [18]:

$$\frac{d^2x^i}{d\tau^2}+\Gamma^i_{km}\frac{dx^k}{d\tau}\frac{dx^m}{d\tau}=0\,, \tag{43}$$

where τ is the proper time of the moving particle as it determined in GTR.

Since the interval can be expressed through the differential of the proper time of the test particle in the form $ds=cd\tau$, then (16) can be written as follows:

$$c^2=g_{ik}\frac{dx^i}{d\tau}\frac{dx^k}{d\tau}\,. \tag{44}$$

The non-zero Christoffel symbols for the spherical coordinates are:

$$\Gamma^0_{01}=\Gamma^0_{10}=\frac{B'}{2B},\quad \Gamma^1_{00}=\frac{B'}{2K},\quad \Gamma^1_{11}=\frac{K'}{2K},\quad \Gamma^1_{22}=-\frac{E'}{2K},\quad \Gamma^1_{33}=-\frac{E'\sin^2 Q}{2K}\,,$$

$$\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^3_{13}=\Gamma^3_{31}=\frac{E'}{2E},\qquad \Gamma^2_{33}=-\sin Q\cos Q,\qquad \Gamma^3_{23}=\Gamma^3_{32}=\text{ctg}\,Q\,. \tag{45}$$

With $i=0$ in (43) and metric tensor (1) in (45) only two Christoffel symbols are nonzero: $\Gamma^0_{01}=\Gamma^0_{10}=\frac{B'}{2B}$. Using $x^0=ct$ and $x^1=r$, taking into account the definition $B'=\frac{dB}{dr}$ and multiplying (43) by $\frac{B}{c}$, we find:

$$B\frac{d^2t}{d\tau^2}+\frac{dB}{d\tau}\frac{dt}{d\tau}=0\,,\qquad \frac{d}{d\tau}\left(B\frac{dt}{d\tau}\right)=0\,,\qquad B\frac{dt}{d\tau}=A_3\,, \tag{46}$$

where A_3 is some constant, which can be conveniently associated with the initial velocity at infinity. Indeed, at infinity $B=1$, and the coordinate time t is the time of the inertial reference frame in which the particle is moving at the constant velocity V_0. Then, according to the special theory of relativity, $d\tau = dt\sqrt{1-V_0^2/c^2}$ and $A_3 = \frac{1}{\sqrt{1-V_0^2/c^2}}$.

For the motion of the particle along the radius the angular coordinates Q and φ do not change, and $dQ = d\varphi = 0$. In this case, from (44) for the coordinates $x^0 = ct$ and $x^1 = r$ taking into account (1) and $d\tau$ from (46) we obtain:

$$c^2 = \frac{A_3^2 c^2}{B} - \frac{A_3^2 K}{B^2}\left(\frac{dr}{dt}\right)^2.$$

According to the Schwarzschild metric $B_o = B = g_{00} = \frac{1}{K} = 1 - \frac{2GM}{rc^2}$, so with $A_3 = \frac{1}{A_4}$ for radial motion we have:

$$\frac{dr}{dt} = \pm B_o c\sqrt{1-A_4^2 B_o} = \pm c\left(1-\frac{2GM}{rc^2}\right)\sqrt{1-A_4^2\left(1-\frac{2GM}{rc^2}\right)}. \qquad (47)$$

In a more general case, converting from spherical coordinates to Cartesian coordinates and then to polar coordinates, as in the previous section, (43) can be reduced to the following form:

$$\frac{dr}{d\tau} = \pm\sqrt{c^2 A_3^2 - \frac{B_o L^2}{r^2} - B_o c^2}\,. \qquad (48)$$

$$\frac{dr}{dt}=\pm\frac{B_o}{A_3}\sqrt{c^2A_3^2-\frac{B_oL^2}{r^2}-B_oc^2}\ , \qquad \frac{dr}{d\varphi}=\pm\frac{r^2}{L}\sqrt{c^2A_3^2-\frac{B_oL^2}{r^2}-B_oc^2}\ .$$

These equations are used in GTR to describe the planar motion of the bodies relative to the fixed center in the polar coordinates.

5. Pioneer anomaly

5.1. Qualitative approach

We shall assume that the spacecraft moves away from Earth and the Sun almost radially, transmitting to the tracking station the radio signal of known frequency ν_0. Because of the Doppler effect, the frequency received on Earth will change to:

$$\nu=\frac{\nu_0\sqrt{1-V^2/c^2}}{1-V/c\cos\theta}\approx\nu_0\left(1-\frac{V}{c}\right), \tag{49}$$

where V is the velocity of the spacecraft relative to the Earth,

$\theta\approx\pi$ is the angle between the velocity and the direction to the radiation detector.

As the spacecraft gets farther from the Sun with turned-off engines, under the influence of solar attraction the velocity V gradually decreases, so that the frequency ν should increase. From (49) we can obtain the change of the velocity of the spacecraft and the relative change of the frequency during the time Δt in which the signal goes from the spacecraft to the Earth:

$$\frac{\Delta\nu}{\nu_0}=-\frac{\Delta V}{c}=-\frac{a_f\,\Delta t}{c}\,, \tag{50}$$

where a_f is the total acceleration of the spacecraft.

The acceleration a_f is negative, mostly caused by the Sun and directed towards the Sun, and the velocity V is directed at the angle $\approx \pi$ away from the direction from the spacecraft to the Sun. We shall further assume that the relative change in the frequency of the signal (50) is of such kind that it takes into account all the possible sources of acceleration and the factors influencing the result. Then the residual signal, which is not simulated by anything, can also be represented by (50), in which in the place of acceleration the anomalous acceleration a_p stands:

$$\left(\frac{\Delta \nu}{\nu_0}\right)_p = -\frac{a_p \, \Delta t}{c} \,. \tag{51}$$

We can estimate the velocity of the spacecraft depending on the radial distance r from the equation of its free radial motion in classical mechanics:

$$\frac{d^2 r}{dt^2} = \frac{1}{2}\frac{d}{dr}\left(\frac{dr}{dt}\right)^2 = -\frac{GM}{r^2} \,, \tag{52}$$

where M is the Sun's mass.

Assuming in the first approximation that the motion of the spacecraft is purely radial, we shall integrate: $\frac{dr}{dt} = V = \sqrt{\frac{2GM}{r} + A_5}$. We shall assume the velocity of the spacecraft at the distance of 87 a.u. was 12.2 km/s, from this we find $A_5 = 1.28 \cdot 10^8$ m^2/c^2. Consequently, at the distance $R = 20$ a.u. for the velocity of the spacecraft in the approximation of the free radial motion we should assume about 14.7 km/s.

We can explain the velocities of the Pioneers in the following way. From (35) in the approximation of the radial motion, when the density of the angular momentum $L = 0$, and with $B_k = B = g_{00} = 1 + \frac{GM\alpha}{rc^2} - \frac{\beta G^2 M^2}{r^2 c^4} \approx 1$, for the radial velocity of a freely flying spacecraft in CTG we can write:

$$\frac{dr}{dt} = c\sqrt{1 - \frac{1}{\left(\frac{GM}{c^2 r} + A_1\right)^2}}. \tag{53}$$

If we proceed from (52), at the distance of 1 a.u. we can assume that the initial velocity is equal to $V_1 = 4.361 \cdot 10^4$ m/s. This allows us to estimate in (53) the value of the constant $A_1 = 1.00000000071$ and to find the velocity of the spacecraft at different distances.

In GTR we have a similar formula according to (47):

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right)\sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)}. \tag{54}$$

Substituting in (54) $\frac{dr}{dt} = V_1 = 4.361 \cdot 10^4$ m/s with $r = 1$ a.u., we find $A_4^2 = 0.999999998579$. With the help of (53) and (54) we calculate the velocity of the spacecraft according to CTG and GTR at different distances for the case of conditionally radial motion. The results are shown in Table 1.

As we can see, the velocities of the spacecraft in GTR and CTG are slightly different. If the spacecraft starts with $r=1$ a.u., then the time of its motion up to $r=5$ a.u. is of the order of $\Delta t_{1-5}=1.82\cdot 10^7$ s (this approximate value is obtained by dividing the distance traveled by the average velocity). During this time up to the position with $r=5$ a.u. because of the different velocities the difference between the positions of the spacecrafts according to the equations of GTR and CTG will grow up to $\Delta r_{1-5}=3.29\cdot 10^5$ m. For the spacecraft to move from 5 to 10 a.u. the time required is, accordingly, about $\Delta t_{5-10}=3.79\cdot 10^7$ s, what is shown in Table 1.

Table 1. The data on the motion of the spacecraft

r, a.u.	V, 10^4 m/s	Δr, 10^5 m	Δt, 10^7 s	a_p, 10^{-10} m/s^2
1	4.361			
5	2.196624147 CTG 2.196620538 GTR	3.29	1.82	19.8
10	1.746714927 CTG 1.746712488 GTR	13.2	3.79	18.3
15	1.568321059 CTG 1.568319299 GTR	9.47	4.51	9.31
20	1.471033615 CTG 1.471032248 GTR	7.69	4.92	6.4

Because the velocity of the spacecraft in CTG is somewhat greater than in GTR, then in case of the measurements according to the Doppler effect at each time point the spacecraft is located farther than it is assumed according to GTR. Because of this difference in the distances the velocity of the spacecraft, always decreasing with time because of the attraction of the Sun, is less than the velocity of the spacecraft according to GTR. For example, at the distance $r+\Delta r_{1-5}$ = 5 a.e. + 3.29·10^5 m according to CTG the velocity of the spacecraft in our model calculations will be 2.196478495·10^4 m/s, whereas according to GTR the spacecraft is at the distance of 5 a.u. and has the velocity 2.196620538·10^4 m/s. As a result, with the help of the Doppler effect, the velocity of the spacecraft is registered, decreased relative to the data of GTR. This decrease is attributed to the anomalous acceleration acting in the direction towards the Sun.

In the last column of Table 1, we estimated the anomalous acceleration by the formula: $a_p=\frac{2\Delta r}{(\Delta t)^2}$. This acceleration indicates that the spacecraft is situated at the distance that seems to be smaller than expected by the value Δr, which arises during the time Δt because of the difference in velocities. The distances Δr in Table 1 are calculated by an average velocity at each interval of motion, so to obtain the total result we should add up all Δr. This will lead over time to the increase in distance between the positions of the spacecrafts according to GTR and CTG, and to decrease of the anomalous acceleration a_p with the distance as compared with the data in Table 1. As is shown in Table 1, the values of the anomalous acceleration are close enough to the data obtained for the effect of Pioneers, and at small distances up to 5–10 a.u. they are masked by the acceleration from the pressure force of the solar wind.

5.2. Analytical approach

Now let us try to derive the corresponding formula for the anomalous acceleration, again for the case of purely radial motion. Assuming in (35) $L=0$, for the velocity of the spacecraft in CTG and for its current position relative to the Sun we obtain:

$$\frac{dr}{dt} = Bc\sqrt{1 - \frac{B}{\left(\frac{GM}{c^2 r} + A_1\right)^2}}\,, \tag{55}$$

$$t + A_6 = \frac{A_1}{Bc(A_1^2 - B)}\sqrt{\frac{G^2M^2}{c^4} + \frac{2A_1 GMr}{c^2} + (A_1^2 - B)r^2} -$$
$$-\frac{GM}{c^3(A_1^2 - B)^{3/2}}\ln\left(\sqrt{A_1^2 - B}\sqrt{\frac{G^2M^2}{c^4} + \frac{2A_1 GMr}{c^2} + (A_1^2 - B)r^2} + (A_1^2 - B)r + \frac{A_1 GM}{c^2}\right), \tag{56}$$

where $B = g_{00} = 1 + \frac{GM\alpha}{rc^2} - \frac{\beta G^2M^2}{r^2c^4} \approx 1$ is the time component of the metric in CTG, $A_1 = \left(\frac{dt}{d\tau}\right)_\infty = \frac{1}{\sqrt{1 - V_0^2/c^2}}$, where V_0 is the velocity of the spacecraft at infinity, the radial coordinate r is the function of the time t of the motion from the Sun, and the constant A_6 is the parameter of integration.

If at a given time point $t = t_0$ we know the radial distance $r = r_0$ and the velocity $\frac{dr}{dt} = V_0$, it allows us to calculate the constants A_1 and A_6 in (55) and (56). Thus, in the previous section, we assumed for simplicity that $B \approx 1$, at $t = 0$ the spacecraft was at a distance $r = 1$ a.u., and the constant $A_1 = 1.00000000071$. These data can be used to estimate the constant A_6 in (56).

We will integrate now (54) for the radial motion in GTR:

$$t + A_7 = \frac{r\sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)}}{c(1 - A_4^2)} - \frac{2GM}{c^3}\ln\frac{1 + \sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)}}{1 - \sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)}} + \\ + \frac{(3A_4^2 - 2)GM}{c^3(1 - A_4^2)\sqrt{1 - A_4^2}}\ln\frac{\sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)} - \sqrt{1 - A_4^2}}{\sqrt{1 - A_4^2\left(1 - \frac{2GM}{rc^2}\right)} + \sqrt{1 - A_4^2}}. \qquad (57)$$

In (57) the constant A_7 appears, which must be found together with the constant A_4 from the initial conditions of motion.

We suppose now that we have derived from (56) the dependence of the radial distance in CTG as the function of time: $r(t) = r_k(t)$. Similarly, from (57) we can determine the dependence of the radial distance in GTR as the function of time: $r(t) = r_o(t)$. At a first approximation, the gravitational acceleration of the Sun depends on the radial distance according to Newton's formula, and we can write for the accelerations in CTG and GTR the following:

$$g_k(t) = -\frac{GM}{r_k^2(t)}, \qquad g_o(t) = -\frac{GM}{r_o^2(t)}.$$

The anomalous acceleration as a function of the time of the spacecraft's radial motion is found as the difference between these accelerations:

$$a_p(t) = g_o(t) - g_k(t) = -\frac{GM\,[r_k^2(t) - r_o^2(t)]}{r_k^2(t)\,r_o^2(t)}.$$

The meaning of this equality is that in the case of the Pioneers the acceleration $g_o(t)$, calculated in GTR, is overestimated in the absolute value as compared to the measured acceleration. If the acceleration $g_k(t)$ in CTG describes the motion more precisely and is equal to the measured acceleration, then to obtain it we should subtract the anomalous acceleration from the acceleration in GTR: $g_k(t) = g_o(t) - a_p(t)$.

6. Conclusion

In GTR the gravitational field is the same as the metric field with its metric tensor. As a result the gravitational field does not create the metric similar to electromagnetic field in equation for the metric, and the metric tensor is calibrated with the help of Newton's law of universal gravitation. We can suppose that such calibration is not accurate because Newton's law has no relativistic corrections. On the other hand in CTG the gravitational field is a fundamental field that has its stress-energy tensor and can influence the metric in the equation for the metric. The metric component $B_k = g_{00} = 1 + \frac{GM\alpha}{rc^2} - \frac{\beta G^2 M^2}{r^2 c^4}$ in CTG depends on the energy of the gravitational field and it seems it is more precise then $B_o = g_{00} = \frac{1}{K} = 1 - \frac{2GM}{rc^2}$ in GTR. The metric component g_{00} is in both equations of motion in CTG and GTR but the equations are different.

From the point of view of CTG the effect of the Pioneers is explained as the result of using an equation of motion that does not coincide with the equation of motion of GTR.

All computer calculations associated with the motion of the spacecrafts obligatorily use GTR and take into account not only the influence of the Sun, but of other planets. If the equation of motion of CTG is valid, there is no anomalous acceleration in the effect of Pioneer, and the effect is due to the use of GTR instead of CTG. The following fact also points out to the probable inaccuracy of GTR that in the signal from the Pioneers we could see not simulated periodic changes associated with the diurnal rotation of the Earth and its annual revolution around the Sun. From Table 1 it follows also that the velocities of the spacecraft in GTR and CTG at equal distances can differ by several centimetres per second. At the same time, in several articles the so-called flyby effect has been described, when the velocity of spacecrafts differs from the calculated values up to several centimetres per second [19-20].

There are also works such as [21] – [23], according to which the motion of the comets: Halley's comet, Encke and others, after their passing near the planets disturbances of unknown nature are discovered, which are not taken into account by GTR equations (48). We can assume that the recalculation of the motion of spacecrafts and comets in terms of CTG with the help of equations (35), (36), and (40) will improve the situation.

References

1. J.D. Anderson, P.A. Laing, E.L. Lau, A.S. Liu, M.M. Nieto and S.G. Turyshev. Phys. Rev. D **65**, 082004 (2002). doi:10.1103/PhysRevD.65.082004.
2. S.G. Turyshev, V.T. Toth, G. Kinsella, Siu-Chun Lee, S.M. Lok, and J. Ellis. Phys. Rev. Lett. **108**, 241101 (2012). doi:10.1103/PhysRevLett.108.241101.
3. D. Modenini and P. Tortora. Phys. Rev. D **90**, 022004 (2014). doi:10.1103/PhysRevD.90.022004.
4. G.U. Varieschi. Phys. Res. Int. **2012**, 469095 (2012). doi:10.1155/2012/469095.
5. A.F. Rañada and A. Tiemblo. Can. J. Phys. **90**, 931 (2012). doi:10.1139/p2012-086.
6. M.W. Kalinowski. CEAS Space J. **5**, 19 (2013). doi:10.1007/s12567-013-0042-9.
7. J.D. Anderson and J.R. Morris. Phys. Rev. D **86**, 064023 (2012). doi:10.1103/PhysRevD.86.064023.
8. G.S.M. Moore and R.E.M. Moore. Astrophys. Space. Sci. **347**, 235 (2013). doi:10.1007/s10509-013-1514-2.

9. P.C. Ferreira. Adv. Space Res. **51**, 1266 (2013). doi:10.1016/j.asr.2012.11.004.

10. M.R. Feldman. PLoS ONE. **8**, e78114 (2013). doi:10.1371/journal.pone.0078114.

11. S.G. Fedosin. Fizicheskie teorii i beskonechnaia vlozhennost' materii. Perm. 2009.

12. S.G. Fedosin. Int. Front. Sci. Lett. **1**, 48 (2014).

13. W. Pauli. Theory of Relativity. Dover Publications, New York. 1981.

14. S.G. Fedosin. Adv. Nat. Sci. **5**, 55 (2012). doi:10.3968%2Fj.ans.1715787020120504.2023.

15. S.G. Fedosin. Int. J. Thermo. **18**, 13 (2015). doi:10.5541/ijot.34003.

16. S.G. Fedosin. The concept of the general force vector field. vixra.org, 28 June 2014. http://vixra.org/abs/1406.0173.

17. L.D. Landau and E.M. Lifshitz. Mechanics, Vol. 1, 3th ed. Butterworth–Heinemann, Oxford, 1976.

18. L.D. Landau and E.M. Lifshitz. The Classical Theory of Fields, Vol. 2, 4th ed. Butterworth-Heinemann, Oxford, 1975.

19. J.D. Anderson and J.G. Williams. Class. Quantum Gravity. **18**, 2447 (2001). doi:10.1088/0264-9381/18/13/307.

20. J.D. Anderson, J.K. Campbell and M.M. Nieto. New Astron. **12**, 383 (2007). doi:10.1016/j.newast.2006.11.004.

21. F.L. Whipple. Astrophys. J. **111**, 375 (1950). doi:10.1086/145272.

22. B.G. Marsden. Planet. Space Sci. **57**, 1098 (2009). doi:10.1016/j.pss.2008.12.007.

23. T. Kiang. Mon. Not. R. Astron. Soc. **162**, 271. (1973). doi:10.1093/mnras/162.3.271.