# Investigation of Nonlinear Dynamics and Chaos in Driven Systems derived from Einstein Functions

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**Abstract**

The present work pertains to the formulation and generation of signal based frequency controlled chaos based on Einstein Functions. Specifically, the variable in the four Einstein functions is taken as an additively coupled sum of sinusoids with competing frequencies. By adapting the four Einstein functions into signals, the time derivatives are computed and used to form the iterative maps and phase portraits. It is seen that the four phase portraits display to varying degrees, ornamental patterns, characteristic of quasiperiodicity and chaos. Using these, bifurcation diagrams are plotted, and the chaotic behavior is quantitatively characterized using largest Lyapunov Exponents. It is seen that the nature of chaos in the generated signals depend on the frequency ratio of the driving signals, thus pertaining to a case of signal based chaos, which has the key advantage of easy tunability, forming the novelty of the present work.

*Keywords:* Nonlinear Dynamics, Einstein Functions, Frequency Controlled Chaos, Bifurcation Analysis

## 1. Introduction

Nonlinear Dynamics, and subsequently Chaos Theory, with the characteristic signature 'sensitive dependence on initial conditions', has undoubtedly set one of the defining paradigms of twentieth century science, with applications covering diverse fields including biology, astrophysics, engineering and mechanics [1]-[16]. The development of tools and techniques to characterize and study nonlinear dynamics such as Bifurcation Plots, Iterative Maps and Lyapunov Exponents, stemming from the development of computation and visualization technologies have enabled visualizations of complex and intricate patterns of long term evolution [1, 2].

In electronics, tremendous development has been made following the ability to generate chaotic signals using op-amp based physical realizations of nonlinear differential equations, such as the Chua Circuits, and this has promptly been translated into real-time applications such as secure communications and cryptography [17]-[21]. However, such implementations use system-based parameters such as resistors and capacitors as the initial conditions, with the obvious disadvantage of difficulty in tuning when implemented at high frequencies as Integrated Circuits (IC) [22, 23].

In the present work, this issue is addressed in a radical and innovative way using the concept of signal based chaos, where the initial conditions are not physical parameters, but rather the signal based properties (amplitude, frequency and phase) of the inputs in a driven chaotic system. With this motivation, the present work focuses on the four forms of the Einstein Function, often classified as a special mathematical function, finding applications alongside the Gruneisen function in solid state physics to compute the electrical resistivity [24]-[28]. For ach of the four forms denoted by a generalized $E(x)$, the variable $x$ is set to an additively coupled sinusoidal signal with competing frequencies, becoming the 'driving signal' of the system, where the frequency ratio between the input signals acting as the control parameter $r$. The phase portrait is plotted for an $r$ value of $\pi$, wherein the presence of ornamental patterns indicate quasi-periodicity and chaos in the system. Following this, the iterative map is formed by computing the derivative of $E(x)$ and expressing it as a difference equation. Using the iterative map, the Bifurcation plots are then plotted, which indicate the regions of order and chaos in these signal forms. Finally, the computation of Largest Lyapunov Exponents assertively establishes the trends of order and chaos in these signal forms as a function of $r$.

The results discussed in the present work indicate that for specific values of $r$ in signals derived from the Einstein Functions, chaotic behavior is observed, thus pertaining to the case of signal based chaos, which is physically realized using Field Programmable Gate Arrays (FPGA), with much simpler circuitry and easier tunability than conventional chaos generator circuits, and this forms the novelty of the present work.

## 2. Phase Portrait Analysis of The Einstein Functions

There exist four Einstein Functions, $E_1(x)$, $E_2(x)$, $E_3(x)$ and $E_4(x)$, denoted for generality as $E(x)$ [24]-[28]. Based on this notation, the following procedure is used for the investigation of nonlinear dynamics in the Einstein Functions:

1. The variable $x$ is denoted as an additively coupled signal of two sinusoids of frequencies $f_1$ and $f_2 = rf_1$, as
$$x = sin(2\pi f_1 t) + sin(2\pi r f_1 t) \tag{1}$$
with $r$ denoting the ratio between the frequencies, and acting as the key control parameter.
2. Using this substitution, $E(x)$ is rewritten as a time-varying signal $E(t)$, and its time derivative $E'(t)$ is computed.
3. The dynamics of $E(x)$ are studied using the Phase Portrait, which is a plot of $E'(t)$ in terms of $E(t)$ for a given $r$, illustrating the phase space dynamics and qualitatively serving as a tool to assess sensitivity and ergodicity. Since $r$ denotes the frequency ratio of the driving signals, an irrational number such as $\pi$ is set as the value of $r$, in order to maximize the frequency and phase mismatches between the driving signals. The detection of ornamental and rich patterns in a phase portrait is a clear indicator of the presence of chaos.
4. In order to form the iterative map, $E(t)$ and $E'(t)$ are discretized into $E(i)$ and $E'(i)$ respectively and a difference equation of the form $E'(i) = E(i+1) - E(i)$ is formed. This difference equation is rearranged to give the expression of 'next' sample $E(i+1)$ in terms of 'current' samples $E(i)$, as $E(i+1) = E(i) + E'(i)$, this equation termed the 'Iterative Map' due to its recurrence nature. For systems depicting phase portraits indicative of chaos, the bifurcation diagram, plotting the output $E$ as a function of $r$ is obtained. This diagram clearly illustrates for what values of $r$, the system exhibits chaotic and non-chaotic behavior.

The four Einstein Functions are given as follows:

$$E_1(x) = \frac{x^2 e^x}{(e^x - 1)^2} \tag{2}$$

$$E_2(x) = \frac{x}{e^x - 1} \tag{3}$$

$$E_3(x) = ln(1 - e^{-x}) \tag{4}$$

$$E_4(x) = E_2(x) - E_3(x) \tag{5}$$

By substituting $x$ with the additive sum of sinusoids as mentioned above, we obtain the 'Einstein Function Signals' $E(t)$. Using this, the time derivative $E'(t)$ is computed, and is plotted for an $r$ value of $\pi$ Fig. (1).

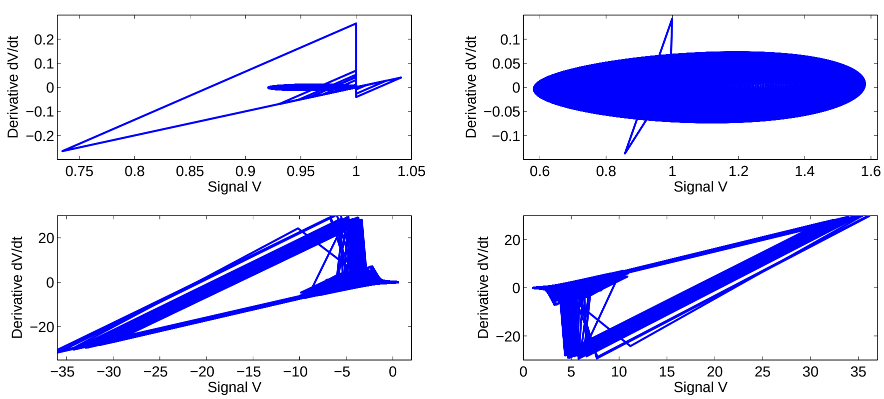


Figure 1: Phase Portraits of $E_1(t)$ (top left), $E_2(t)$ (top right), $E_3(t)$ (bottom left) and $E_4(t)$ (bottom right)

As seen from the phase portraits, it can be observed that all the four orders show, to varying degrees, ornamental patterns characteristic of either quasi-periodic or chaotic behavior, where to distinguish between these two, further bifurcation analysis is required. It is noteworthy that the phase portrait of $E_2(t)$ shows the most characteristic signatures of quasiperiodicity among the four, whereas phase portraits of $E_3(t)$ and $E_4(t)$ are more indicative of chaos.

## 3. Bifurcation Analysis

In this section, the bifurcation analysis for $E(t)$ is explored. It must be noted that, on account of the coupled frequency formation of $x$, these forms do not give rise to bifurcations in the traditional sense; rather they exhibit a quasi-periodic route to chaos, as typically seen in other coupled phase based chaotic systems such as the standard circle map [29, 30, 31, 32, 33, 34].

The time derivatives of the four Einstein Functions are given as follows:

$$z = sin(2\pi f_1 t) + sin(2\pi r f_1 t) \tag{6}$$

$$z' = 2\pi f_1 cos(2\pi f_1 t) + 2\pi r f_1 cos(2\pi r f_1 t) \tag{7}$$

$$E_1'(t) = \frac{x^2 x' e^x}{(e^x - 1)^2} - \frac{2x^2 x' e^{2x}}{(e^x - 1)^3} + \frac{2x x' e^x}{(e^x - 1)^2} \tag{8}$$

$$E_2'(t) = \frac{x'}{e^x - 1} - \frac{x x' e^x}{(e^x - 1)^2} \tag{9}$$

$$E_3'(t) = \frac{x' e^{-x}}{1 - e^{-x}} \tag{10}$$

$$E_4'(t) = E_2'(t) - E_3'(t) \tag{11}$$

Thus, we obtain the following difference equation by discretizing the above equations and setting $E'(i) = E(i+1) - E(i)$, and with $n = 1, 2, 3, 4$.

$$E_n(i+1) = E_n(i) + E_n'(i) \tag{12}$$

The above equation is termed the 'iterative map', and the corresponding bifurcation diagrams are plotted for $E_n$ as a function of $r$ in Fig. (2).

From the bifurcation plots, it is seen that while first and second forms display largely non-chaotic quasiperiodic behavior, the third and fourth forms display interesting bifurcation trends with dense 'grassy' patches characteristic of chaotic behavior.

## 4. Chaotic Characterization using Lyapunov Exponents

In order to understand the dependence of the chaotic nature in the generated signals on $r$, we use the Largest Lyapunov Exponent (LLE), quantifying a systems sensitive dependence on initial conditions [35, 36]. The Rosensteins algorithm is used to compute the Lyapunov Exponents $\lambda_i$ from the signal, where the sensitive dependence is characterized by the divergence samples $d_j(i)$ between nearest trajectories represented by $j$ given as $d_j(i) = C_j e^{\lambda_i (i\delta t)}$, $C_j$ being a normalization constant [35, 36]. With a positive value of LLE assertively establishing the presence of chaos, the LLE's of the $E_n(t)$ forms with $n = 1, 2, 3, 4$ for various $r$ values from 0.1 to 1.5 are computed and plotted as a graph in Fig. (3).

As seen from the plot, the trends of LLE for the first two forms are mostly similar with very few positive values, whereas, the trends for the last two forms are similar with chaotic LLE values seen for $r$ values close to 1.

## 5. Conclusion

Motivated by issues of tunability in system based chaos generation circuits, the present work proposes a radical and innovative solution using signal based chaos, and to achieve this, the four forms of Einstein Functions are considered. These forms are adapted into signals by substituting the variable $x$ as an additively coupled sinusoidal signal,viewing the output as a representative of a driven nonlinear coupled system. Following this, the derivative of the output is computed and used to form a difference equation, which yields the iterative map of the proposed system. This iterative map is studied using phase portraits exhibiting varying degrees of ornamental patterns characteristic of quasiperiodic or chaotic behavior. Hence, the bifurcation analysis of these three forms are presented following which the dependence of chaotic nature on the frequency ratio $r$ is studied using largest Lyapunov Exponents.

Finally, it is noteworthy that since the behavior of the output signal depends on frequency ratio $r$, this ratio serves as a secure 'key', enabling the use of the Einstein Function based 'Frequency Controlled Chaos' in secure communication and encryption systems. The signal oriented approach to generating chaos from mathematical functions, coupled with the easy tunability hence obtained forms the novelty of the present work.

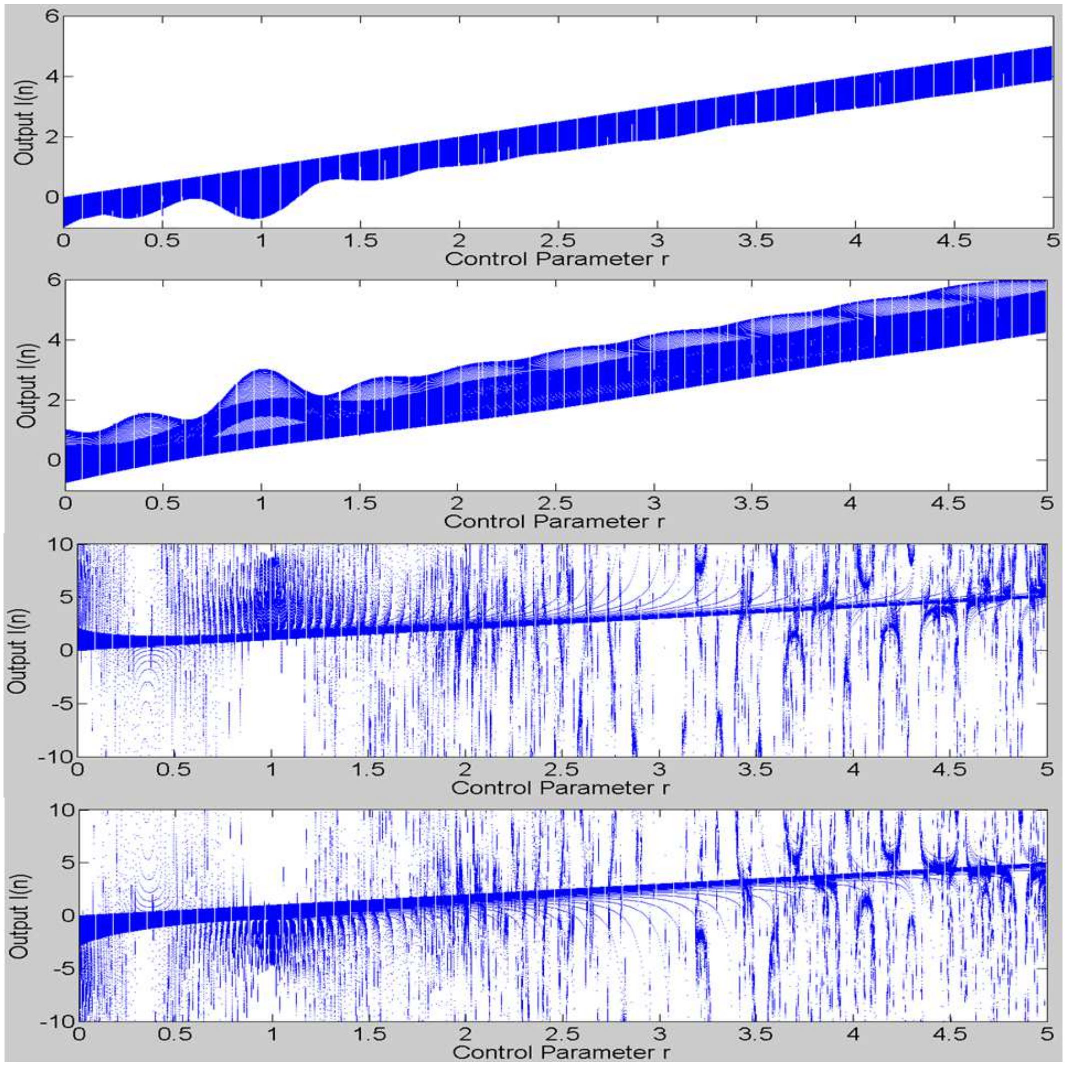


Figure 2: Bifurcation Plots of $E_n$ forms for (from top to bottom) $n = 1, 2, 3, 4$

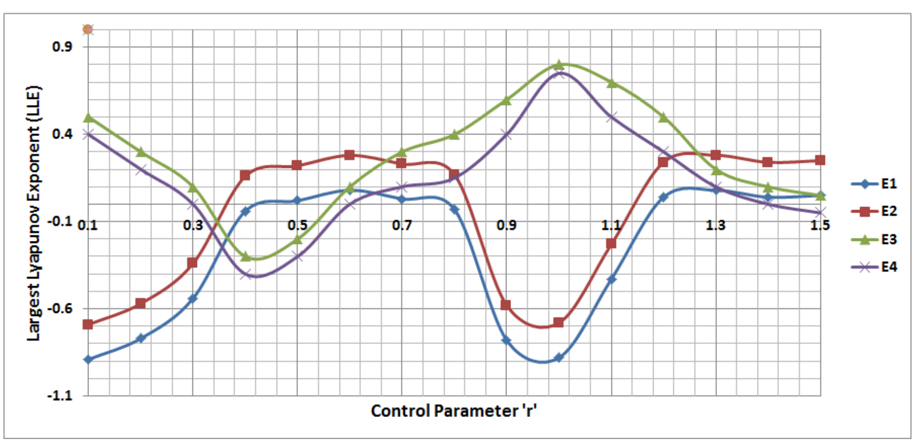


Figure 3: Largest Lyapunov Exponent Values of $E_n(t)$ for $n = 1, 2, 3, 4$ as function of $r$

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