Even More
Ordinary Differential Equations
Easy Way

Just as I have shown earlier that $y = e^{kx}$ is not just the solution for the constant coefficients, linear homogeneous ordinary differential equation (LHODE), but for any LHODE of the form:
$y'' + Py' + [-k(P+k)]y = 0$; the solution to the general elementary HLODE is not limited to that LHODE.

Again, as shown earlier:
$s = aP + b$:

$$y = e^{a\int Pdx + bx} \Rightarrow y'' + Py' + Qy = 0 \;\; ; \;\; Q = -aP' - a(a+1)P^2 - b(2a+1)P - b^2$$

But, also:

$$y' = (aP+b)y$$
$$y'' = \left[(aP+b)^2 + aP'\right]y$$

So, for any function R (even non-continuous, or even non-integrable function R):

$$y'' + Ry' = \left[(aP+b)^2 + aP' + R(aP+b)\right]y$$
$$\Rightarrow y'' + Ry' + \left[-(aP+b)^2 - aP' - R(aP+b)\right]y = 0$$

Next, consider:

$$y_1'' + P_1y_1' + Q_1y_1 = 0$$

and:

$$y_2'' + P_2y_2' + Q_2y_2 = 0$$

Let:

$$y_2 = uy_1$$

Then:

$$\begin{aligned} 0 &= y_2'' + P_2y_2' + Q_2y_2 = u_1y_1'' + 2u'y_1' + u''y_1 + P_2uy_1' + P_2u'y_1 + Q_2uy_1 \\ &= u(y_1'' + P_1y_1' - P_1y_1' + P_2y_1' + Q_1y_1 - Q_1y_1 + Q_2y_1) + u'(2y_1' + P_2y_1) + u''y_1 \\ &= u(P_2y_1' - P_1y_1' + Q_2y_1 - Q_1y_1) + u'(2y_1' + P_2y_1) + u''y_1 \\ &= u'' + \left(2\frac{y_1'}{y_1} + P_2\right)u' + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]u \end{aligned}$$

This leads to Theorem #1.

Theorem #1: If $y_1'' + P_1y_1' + Q_1y_1 = 0$ and $y_2'' + P_2y_2' + Q_2y_2 = 0$ and:

$$u = \frac{y_2}{y_1}$$

then

$$0 = u'' + \left(2\frac{y_1'}{y_1} + P_2\right)u' + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]u$$

Proof:

The following are given:

$$y_1'' + P_1y_1' + Q_1y_1 = 0$$
$$y_2'' + P_2y_2' + Q_2y_2 = 0$$
$$u = \frac{y_2}{y_1}$$

$$\Rightarrow u' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1}$$

$$\begin{aligned} \Rightarrow u'' &= \frac{y_1y_2'' - y_1'y_2'}{y_1^2} - \frac{y_1y_1'' - y_1'y_1'}{y_1^2}\frac{y_2}{y_1} - \frac{y_1'}{y_1}\left(\frac{y_2}{y_1}\right)' \\ &= \frac{y_2''}{y_1} - \frac{y_1'}{y_1^2}y_2' - \frac{y_1''}{y_1}\frac{y_2}{y_1} + \frac{y_1'}{y_1}\frac{y_1'}{y_1}\frac{y_2}{y_1} - \frac{y_1'}{y_1}\left(\frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1}\right) \\ &= \frac{1}{y_1}\left(y_2'' - \frac{y_1'}{y_1}y_2' - \frac{y_1''}{y_1}y_2 + \frac{y_1'}{y_1}\frac{y_1'}{y_1}y_2 - \frac{y_1'}{y_1}y_2' + \frac{y_1'}{y_1}\frac{y_1'}{y_1}y_2\right) \\ &= \frac{1}{y_1}\left[y_2'' - 2\frac{y_1'}{y_1}y_2' + \frac{1}{y_1}\left(-y_1'' + 2\frac{y_1'}{y_1}y_1'\right)y_2\right] \end{aligned}$$

$$\begin{aligned} \Rightarrow u'' + \left(2\frac{y_1'}{y_1} + P_2\right)u' + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]u &= \\ = \frac{1}{y_1}\left[y_2'' - 2\frac{y_1'}{y_1}y_2' + \frac{1}{y_1}\left(-y_1'' + 2\frac{y_1'}{y_1}y_1'\right)y_2\right] &+ \\ + \left(2\frac{y_1'}{y_1} + P_2\right)\left[\frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1}\right] + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]\frac{y_2}{y_1} \end{aligned}$$

$$= \frac{1}{y_1}\Bigg(y_2'' - 2\frac{y_1'}{y_1}y_2' + \frac{1}{y_1}\left(-y_1'' + 2\frac{y_1'}{y_1}y_1'\right)y_2 + \left(2\frac{y_1'}{y_1} + P_2\right)\left(y_2' - \frac{y_1'}{y_1}y_2\right) +$$
$$+ \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]y_2\Bigg)$$

$$= \frac{1}{y_1}\Bigg(y_2'' - 2\frac{y_1'}{y_1}y_2' - \frac{1}{y_1}y_1''y_2 + 2\frac{1}{y_1}\frac{y_1'}{y_1}y_1'y_2 + \left(2\frac{y_1'}{y_1} + P_2\right)y_2' - \left(2\frac{y_1'}{y_1} + P_2\right)\frac{y_1'}{y_1}y_2 +$$
$$+ \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]y_2\Bigg)$$

$$= \frac{1}{y_1}\Bigg(y_2'' - 2\frac{y_1'}{y_1}y_2' - \frac{1}{y_1}y_1''y_2 + 2\frac{1}{y_1}\frac{y_1'}{y_1}y_1'y_2 + 2\frac{y_1'}{y_1}y_2' + P_2y_2' - 2\frac{y_1'}{y_1}\frac{y_1'}{y_1}y_2 - P_2\frac{y_1'}{y_1}y_2 +$$

$$+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]y_2\Big)$$
$$=\frac{1}{y_1}\Big(y_2''-2\frac{y_1'}{y_1}y_2'+2\frac{y_1'}{y_1}y_2'-\frac{1}{y_1}y_1''y_2+2\frac{y_1'}{y_1}\frac{y_1'}{y_1}y_2-2\frac{y_1'}{y_1}\frac{y_1'}{y_1}y_2+P_2y_2'-P_2\frac{y_1'}{y_1}y_2+$$
$$+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]y_2\Big)$$
$$=\frac{1}{y_1}\left(y_2''-\frac{1}{y_1}y_1''y_2+P_2y_2'-P_2\frac{y_1'}{y_1}y_2+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]y_2\right)$$
$$=\frac{1}{y_1}\left[y_2''+P_2y_2'+Q_2y_2-\frac{1}{y_1}y_1''y_2-P_2\frac{y_1'}{y_1}y_2+P_2\frac{y_1'}{y_1}y_2-P_1\frac{y_1'}{y_1}y_2-Q_1y_2\right]$$
$$=\frac{1}{y_1}\left[y_2''+P_2y_2'+Q_2y_2-\frac{1}{y_1}y_1''y_2-P_1\frac{y_1'}{y_1}y_2-Q_1y_2\right]$$
$$=\frac{1}{y_1}\left[(y_2''+P_2y_2'+Q_2y_2)-\frac{y_2}{y_1}(y_1''+P_1y_1'+Q_1y_1)\right]=\frac{1}{y_1}\left[0-\frac{y_2}{y_1}0\right]=0$$

□

Each set of LHODEs:

$$y_1''+P_1y_1'+Q_1y_1=0$$

and:

$$y_2''+P_2y_2'+Q_2y_2=0$$

is, thus, a **generating pair** of LHODEs for an LHODE:

$$=u''+\left(2\frac{y_1'}{y_1}+P_2\right)u'+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]u\,.$$

The solution set for all generating pair for a given LHODE is a family of curves for that LHODE.

Corollary 1.1: If : $y_1''+Py_1'+Q_1y_1=0$ and $y_2''+Py_2'+Q_2y_2=0$ and:

$$u=\frac{y_2}{y_1}$$

then

$$0=\left(\frac{1}{u}\right)''+\left(2\frac{y_2'}{y_2}+P_1\right)\left(\frac{1}{u}\right)'+\left[(P_1-P_2)\frac{y_2'}{y_2}+Q_1-Q_2\right]\left(\frac{1}{u}\right)$$

Proof:

Follows from the main theorem (Theorem #1), by interchanging y_1 and y_2 .

□

Corollary 1.2: If : $y_2''+Py_2'+Q_2y_2=0$ and:

$$u=y_2e^{\frac{1}{2}\left(\int Pdx-Bx\right)}\ ;\ \left(B\text{ constant}\right)$$

then

$$u''+Bu'+(Q_2-\tfrac{1}{2}P'-\tfrac{1}{4}P^2+\tfrac{1}{4}B^2)u=0$$

Proof:

Let: $y_1=e^{-\frac{1}{2}\left(\int Pdx-Bx\right)}$

$$y_1'=-\tfrac{1}{2}(P-B)y_1$$
$$y_1''=\left[-\tfrac{1}{2}P'+\tfrac{1}{4}(P-B)^2\right]y_1$$
$$\Rightarrow\frac{y_1'}{y_1}=-\tfrac{1}{2}(P-B)\Rightarrow 2\frac{y_1'}{y_1}+P=B$$
$$\Rightarrow Q_1=-\frac{y_1''}{y_1}-P\frac{y_1'}{y_1}=-\left[-\tfrac{1}{2}P'+\tfrac{1}{4}(P-B)^2\right]-P[-\tfrac{1}{2}(P-B)]$$
$$=\tfrac{1}{2}P'+\tfrac{1}{4}P^2-\tfrac{1}{4}B^2$$
$$\Rightarrow 0=u''+Bu'+[Q_2-\tfrac{1}{2}P'-\tfrac{1}{4}P^2+\tfrac{1}{4}B^2]u$$

□

Thus: $B=0$ and $y_2''+Py_2'+Q_2y_2=0$

$$\Rightarrow u=y_2e^{\frac{1}{2}\int Pdx}\Rightarrow u''+(Q_2-\tfrac{1}{2}P'-\tfrac{1}{4}P^2)u=0$$

is another way of transforming a LHODE with $P\neq0$ into a LHODE with $P=0$.

Corollary 1.3: If : $y_2''+P_2y_2'+Q_2y_2=0$ and:

$$u=y_2e^{\frac{1}{2}\int(P_2-R)dx}\ ;$$

then

$$u''+Ru'+[Q_2-\tfrac{1}{2}P_2'-\tfrac{1}{4}P_2^2+\tfrac{1}{2}R'+\tfrac{1}{4}R^2]u=0$$
$$u''+Ru'+\left[Q_2-\tfrac{1}{2}(P_2-R)'-\tfrac{1}{4}(P_2-R)(P_2+R)\right]u=0$$

Proof:

Let: $y_1=e^{-\frac{1}{2}\int(P_2-R)dx}$

$$y_1'=-\tfrac{1}{2}(P_2-R)y_1$$
$$y_1''=\left[-\tfrac{1}{2}(P_2-R)'+\tfrac{1}{4}(P_2-R)^2\right]y_1$$
$$\Rightarrow\frac{y_1'}{y_1}=-\tfrac{1}{2}(P_2-R)\Rightarrow 2\frac{y_1'}{y_1}+P_2=R$$
$$\Rightarrow Q_1=-\frac{y_1''}{y_1}-P_1\frac{y_1'}{y_1}=-\left[-\tfrac{1}{2}(P_2-R)'+\tfrac{1}{4}(P_2-R)^2\right]-P_1[-\tfrac{1}{2}(P_2-R)]$$
$$=\tfrac{1}{2}P_2'+\tfrac{1}{2}R'-\tfrac{1}{4}P_2^2+\tfrac{1}{2}P_2R-\tfrac{1}{4}R^2+\tfrac{1}{2}P_1P_2-\tfrac{1}{2}P_1R$$
$$=\tfrac{1}{2}P_2'-\tfrac{1}{4}P_2^2+\tfrac{1}{2}P_2R+\tfrac{1}{2}P_1P_2-\tfrac{1}{2}R'-\tfrac{1}{4}R^2-\tfrac{1}{2}P_1R$$
$$\Rightarrow 0=u''+Ru'+$$
$$+[(P_2-P_1)[-\tfrac{1}{2}(P_2-R)]+Q_2-(\tfrac{1}{2}P_2'-\tfrac{1}{4}P_2^2+\tfrac{1}{2}P_2R+\tfrac{1}{2}P_1P_2-\tfrac{1}{2}R'-\tfrac{1}{4}R^2-\tfrac{1}{2}P_1R)]u$$

$$= u'' + Ru' + \Big(-\tfrac{1}{2}P_2^2 + \tfrac{1}{2}P_2R + \tfrac{1}{2}P_1P_2 - \tfrac{1}{2}P_1R + Q_2 - \tfrac{1}{2}P_2' + + \tfrac{1}{4}P_2^2 - \tfrac{1}{2}P_2R - \tfrac{1}{2}P_1P_2 + \tfrac{1}{2}R' + \tfrac{1}{4}R^2 + \tfrac{1}{2}P_1R \Big) u$$
$$= u'' + Ru' + [Q_2 - \tfrac{1}{2}P_2' - \tfrac{1}{4}P_2^2 + \tfrac{1}{2}R' + \tfrac{1}{4}R^2]u$$
$$= u'' + Ru' + \big[Q_2 - \tfrac{1}{2}(P_2 - R)' - \tfrac{1}{4}(P_2 - R)(P_2 + R)\big]u$$

□

Corollary 1.4: If : $y_2'' + P_2y_2' + Q_2y_2 = 0$ and:

$$u = y_2e^{\int P_2dx} \;;$$

then

$$u'' - P_2u' + [Q_2 - P_2']u = 0$$

Proof:

$$R = -P_2 \Rightarrow u = y_2e^{\int P_2dx}$$
$$\Rightarrow u'' - P_2u' + [Q_2 - P_2']u = 0$$

□

This may be verified by:

$$Q - P' = (-s' - s^2 - sP) - P' = -(s+P)' - s^2 - sP$$
$$= -(s+P)' - s^2 - 2sP - P^2 + 2sP + P^2 - sP$$
$$= -(s+P)' - (s+P)^2 + 2sP + P^2 - sP$$
$$= -(s+P)' - (s+P)^2 - (s+P)(-P)$$

In fact, this points out more generally, that:

$$-(s+aP)' - (s+aP)^2 - (s+aP)[(1-2a)P] =$$
$$= -s' - aP' - s^2 - 2saP - a^2P^2 - sP + 2saP - aP^2 + 2a^2P^2$$
$$= -s' - s^2 - sP - aP' - aP^2 + a^2P^2$$
$$= (-s' - s^2 - sP) + (-a[(1-a)P^2 + P'])$$

So that: If : $y'' + Py' + Qy = 0$ and:

$$u = ye^{\int (aP)dx} = e^{\int (s+aP)dx} \;;$$

then

$$u'' + [(1-2a)P]u' + [Q + (-a[(1-a)P^2 + P'])]u = 0$$

And, even more generally, that:

$$-(s+R)' - (s+R)^2 - (s+R)[P - 2R] =$$
$$= -s' - R' - s^2 - 2sR - R^2 - s[P - 2R] - R[P - 2R]$$
$$= -s' - s^2 - sP + sP - R' - 2sR - R^2 - s[P - 2R] - R[P - 2R]$$
$$= -s' - s^2 - sP + sP - R' - 2sR - R^2 - sP + 2sR - RP + 2R^2$$
$$= -s' - s^2 - sP - R' - RP + R^2$$
$$= (-s' - s^2 - sP) + [-R(P - R) - R']$$

So that: If : $y'' + Py' + Qy = 0$ and:

$$u = ye^{\int (R)dx} = e^{\int (s+R)dx} \;;$$

then

$$u'' + [P - 2R]u' + [Q + (-R[P - R] - R')]u = 0$$

And, even more generally, that:

$$-(s+R)' - (s+R)^2 - (s+R)V =$$
$$= -s' - R' - s^2 - 2sR - R^2 - sV - RV$$
$$= -s' - s^2 - sP + sP - R' - 2sR - R^2 - sV - RV$$
$$= (-s' - s^2 - sP) + \big[(\log y)'(P - 2R - V) - R(R + V) - R'\big]$$

So that: If : $y'' + Py' + Qy = 0$ and:

$$u = ye^{\int (R)dx} = e^{\int (s+R)dx} \;;$$

then

$$u'' + Vu' + \Big(Q + \big[(\log y)'(P - 2R - V) - R(R + V) - R'\big]\Big)u = 0$$

Corollary 1.4a: If : $y_2'' + Ay_2' + Q_2y_2 = 0$ $\big(A \text{ constant}\big)$ and:

$$u = y_2e^{Ax} \;;$$

then

$$u'' - Au' + Q_2u = 0$$

Proof:

$$A = P_2 \Rightarrow P_2' = 0 \text{ and } u = y_2e^{Ax}$$
$$\Rightarrow u'' - Au' + Q_2u = 0$$

□

Corollary 1.5: If: $y = e^{\int Pdx}$, for any function s,

$$y'' + (-P + 2s)y' + (-P' - 2sP)y = 0$$

Proof 1:

From corollary 1.4 & theorem 1:

$$y_1'' + Py_1' + Qy_1 = 0 \text{ and } y_2'' - Py_2' + (Q - P')y_2 = 0 \text{ and:}$$
$$y_2 = y_1e^{\int Pdx} \text{ and } u = \frac{y_2}{y_1} = e^{\int Pdx}$$

then

$$0 = u'' + \left(2\frac{y_1'}{y_1} + (-P)\right)u' + \left[((-P) - P)\frac{y_1'}{y_1} + Q - (Q - P')\right]u$$

$$\Rightarrow 0 = u'' + \left(2\frac{y_1'}{y_1} - P\right)u' + \left[-2P\frac{y_1'}{y_1} - P'\right]u$$

Since Q is arbitrary, so is y_1, so, let $s = \frac{y_1'}{y_1}$ be an arbitray function.

Then: $u = \frac{y_2}{y_1} = e^{\int Pdx} \Rightarrow u'' + (2s - P)u' + [-2sP - P']u = 0$

□

Proof 2:

$$y = e^{\int Pdx}$$

$$\Rightarrow y'' + (-P + 2s)y' + (-P' - 2sP)y = \left(e^{\int Pdx}\right)'' + (-P + 2s)\left(e^{\int Pdx}\right)' +$$

$$+ (-P' - 2sP)\left(e^{\int Pdx}\right)$$

$$= \left([P' + P^2]e^{\int Pdx}\right) + (-P + 2s)\left(Pe^{\int Pdx}\right) + (-P' - 2sP)\left(e^{\int Pdx}\right)$$

$$= [P' + P^2 - P^2 + 2sP - P' - 2sP]e^{\int Pdx} = 0$$

□

From corollary 1.4 & theorem 1:

$y_1'' + Py_1' + Qy_1 = 0$ and $y_2'' - Py_2' + (Q - P')y_2 = 0$ and:

This may also be verified by direct substitution.

$y = e^{\int Pdx} \Rightarrow y' = Py \Rightarrow y'' = (P' + P^2)y \Rightarrow y'' + (2s - P)y' = [P' + P^2 + (2s - P)P]y$

Note that for $P = k$:

$R = 2s - k \Rightarrow s = \frac{1}{2}(k + R) \Rightarrow -P' - 2sP = -k(k + R)$

as the previously noted general 2nd order LHODE solution for $y = e^{kx}$.

One last tidbit, here, generalizing corollary 1.6 which may be useful:

Corollary 1.6: If : $y_2'' + P_2y_2' + Q_2y_2 = 0$ and:

$$u = y_2e^{\int sdx} \;;$$

then

$$u'' + (P_2 + 2s)u' + [Q_2 + (s' + s^2 + sP_2)]u = 0$$

Proof:

From theorem 1: If $y_1'' + P_1y_1' + Q_1y_1 = 0$ and $y_2'' + P_2y_2' + Q_2y_2 = 0$ and:

$$u = \frac{y_2}{y_1}$$

then

$$0 = u'' + \left(2\frac{y_1'}{y_1} + P_2\right)u' + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]u$$

and from corollary 1.5: If: $y = e^{\int Pdx}$, for any function s ,

$$y'' + (-P + 2s)y' + (-P' - 2sP)y = 0$$

So, if $y_1 = e^{\int Pdx}$ and $y_1'' + (-P + 2s)y_1' + (-P' - 2sP)y_1 = 0$

and so, with: $y_2'' + P_2y_2' + Q_2y_2 = 0$ and: $u = \frac{y_2}{y_1} = y_2e^{-\int Pdx}$, then

$$0 = u'' + \left(2\frac{y_1'}{y_1} + P_2\right)u' + \left[(P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1\right]u$$

where: $P_1 = -P + 2s$, $Q_1 = -P' - 2sP$

but: $\frac{y_1'}{y_1} = P$

$\Rightarrow 0 = u'' + [2P + P_2]u' + [[P_2 - (-P + 2s)]P + Q_2 - (-P' - 2sP)]u$

$\Rightarrow 0 = u'' + (2P + P_2)u' + [Q_2 + [P_2 + P - 2s]P + P' + 2sP]u$

$\Rightarrow 0 = u'' + (2P + P_2)u' + [Q_2 + P' + P^2 + PP_2]u$

□

Corollary 1.6a: If : $y_2'' + P_2y_2' + Q_2y_2 = 0$ and:

$$u = y_2e^{\left(\frac{\alpha-1}{2}\right)\int P_2dx} \;;$$

then

$$u'' + (\alpha P_2)u' + \left[Q_2 + (\tfrac{\alpha-1}{2})P_2' + \left(\tfrac{\alpha^2-1}{4}\right)P_2^2\right]u = 0$$

Proof:

From corollary 1.5 , with: $s = (\frac{\alpha-1}{2})P_2$

$$u = y_2e^{\left(\frac{\alpha-1}{2}\right)\int P_2dx} \;;$$

then

$$u'' + [P_2 + 2(\tfrac{\alpha-1}{2})P_2]u' + \left[Q_2 + \left((\tfrac{\alpha-1}{2})P_2' + (\tfrac{\alpha-1}{2})^2P_2^2 + (\tfrac{\alpha-1}{2})P_2^2\right)\right]u = 0$$

$$u'' + (\alpha P_2)u' + \left[Q_2 + (\tfrac{\alpha-1}{2})P_2' + \left(\tfrac{\alpha^2-1}{4}\right)P_2^2\right]u = 0$$

□

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