Testing 4-critical plane and projective plane multiwheels using *Mathematica*

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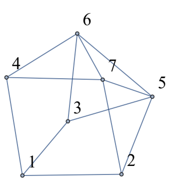
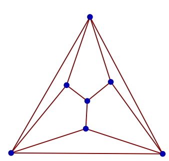
Abstract

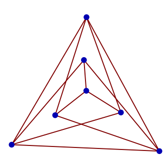
In this article we explore 4-critical graphs using *Mathematica*. We generate graph patterns according [1]. Using the base graph, minimal planar multiwheel and in the same time minimal according projective pattern built multiwheel, we build minimal multiwheels according [1], We forward two conjectures according graphs augmented according considered patterns and their 4-criticallity, and argue them to be proved here if the paradigmatic examples of this article are accepted to be parts of proofs.

Introduction

We build 4-critical plane and projective plane multiwheels from wheels, which should be odd. We use building technique described in the article [1]. Multiwheel $w_{k_1 k_2 \ldots k_s}$ is built from s wheels of order $2k_i + 1$ each. Thus, the smallest built by us multiwheel is w_{111} built from three wheels W_3. Note, we design ordinary wheels with capital letters W, and multiwheels with small letters w. For Grötsch graphs we should use letter g instead of w.





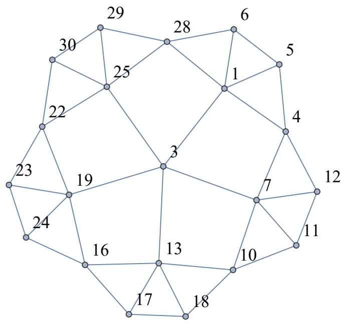


Mathematica draws the base graph non-planar. By hand is done what we see right and in two versions: planar and as Grötsch series graph.

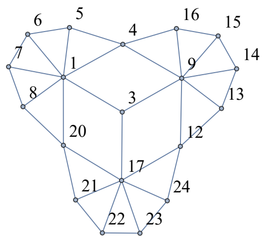
Smallest of our multiwheels w_{111}. Central hub is vertex 1, section hubs are 2,4,7, and rim vertices are 5,6,7. This graph is g_{111} two and on projective plane it quadrangulates it. Central hub spikes unite central hub with section hubs, Section hub spikes unite section hubs with rim vertices, and rim spikes unite rim vertices between themselves. This we call the base graph, taking it as paradigmical, because all other multiwheels are built increaseing numbers s and k_i-s. The only variation from this is that sections may be implemented in four ways, see page 6. [1].
Further in the article we test multiwheels that we build on being 4-critical. It must be said that we as if proof this criticality in the article [1], but we relay on article [2] that is not yet published and even ready. This for we test builted multiwheels on being 4-critical using *Mathematica*, and do this test on paradigmatical graphs, leaving to conclude that there isn't left space for arguments of [2] to be false.

Building plane multiwheels

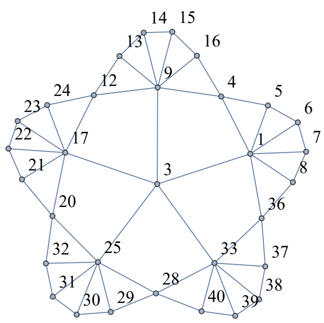
Here we build some samples of plane multiwheels.



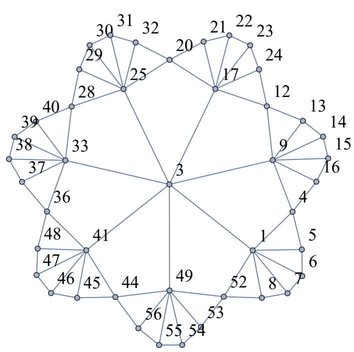
The graph $w_{22\,222}$ or w_{2^3}. It is tested by *Mathematica* in time 2.42 seconds. Here anihilation ocurred on rim-rim pair of edges.



The graph w_{1^3} of 19 vertices. The time of testing 1.07.

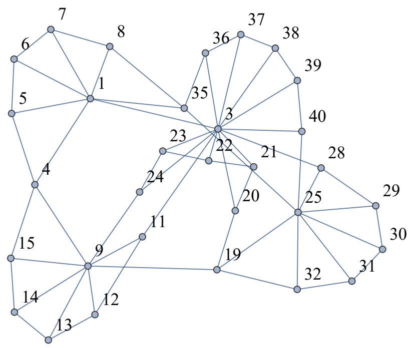


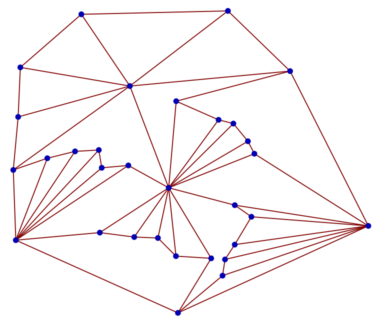
The graph w_{2^3} of 31 vertices and 60 edges. Time about 31 secundes.



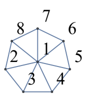
The graph w_{3^3} of 43 vertices and 84 edges.Time about 31 secundes.

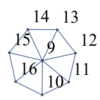
Now comes a test graph built from a sequence of marked graphs. We are marking graphs with pairs of edges that should be anihiliated by building the corresponding multiwheel, see [1]. To get a plane multiwheel marked edges should be with common vertex and plane in the wheel they thaken from, see [1].

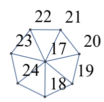




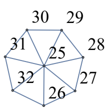
The graph w_{2^3} of 31 vertices but with non-regular orientation of sections. The section orientation is taken from the sequence of marked graphs below. Time for test of 4-criticality about 28 seconds.

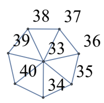
{{ 

, {2, 3, 4}}, { 

, {16, 10, 9}}, { 

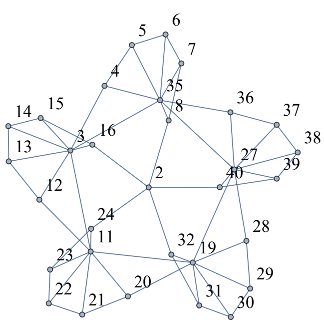
, {18, 17, 19}},

{ 

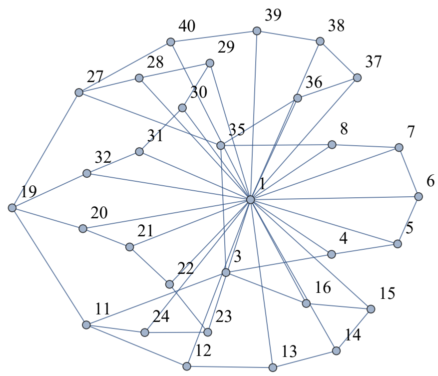
, {26, 27, 25}}, { 

, {34, 33, 35}}}

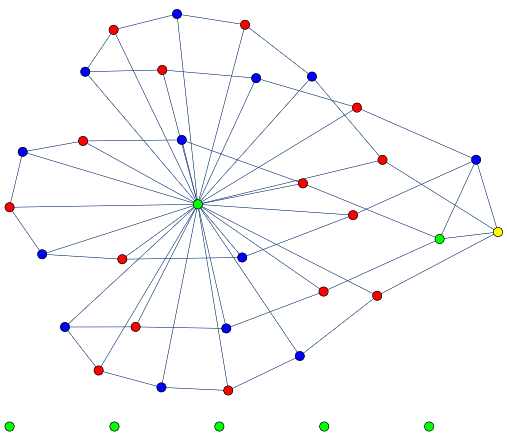
We may build these type graphs but with common type of marked edges,
those that are to be anihilated. Further comes two examples.



Here the graph w_{2^3} of 31 vertices and 60 edges but edges of anihilation are of type spike-rim. . Time to test 4-criticality about 49.5 secundes.



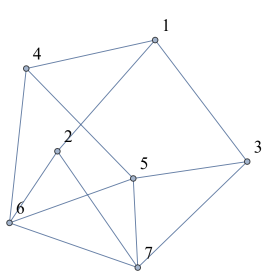
Here the graph w_{2^3} of 31 vertices and 60 edges but edges of anihilation are of type spike-spike. . Time about 119 secundes

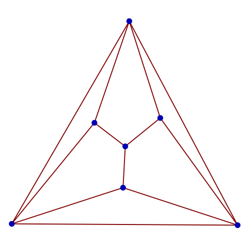


The same graph but colored.

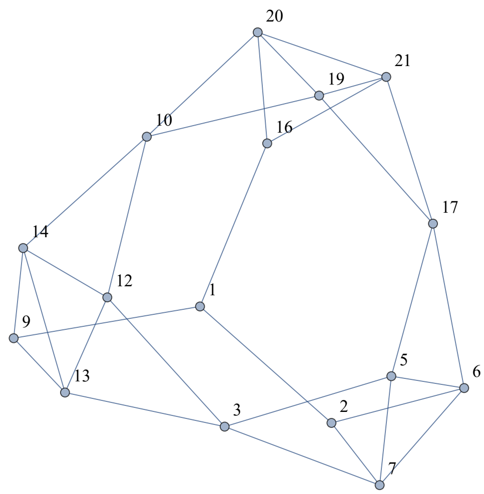
Building multiwheels with multiwheel sections

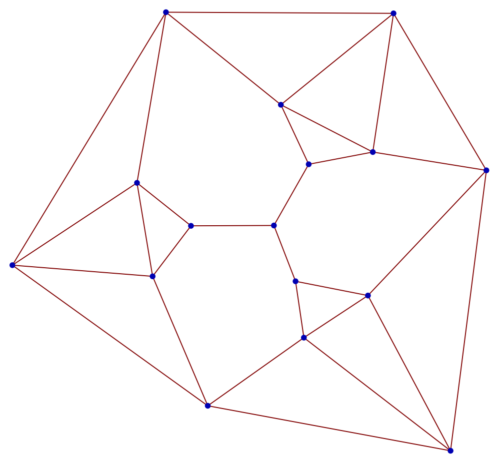
Let us build .multiwheels where sections themselves are multiwheels. Let us take the base graph and replace each simple section with the base graph. We do in all possible ways for section orientation in different variations.
Let us fix notation for base graph edges: the base has three types of edges: central spike or c-spike, section spike or s-spike and rim edge or r-edge, thus, c-spike, s-spike and r-edge. Central hub is vertex 1, section hubs are 2,3,4, and rim vertices are 5,6,7. Remprezentatives of the edges are, e.g., c-spike 1-2, s-spike 2-7, and r-edge 5-6.





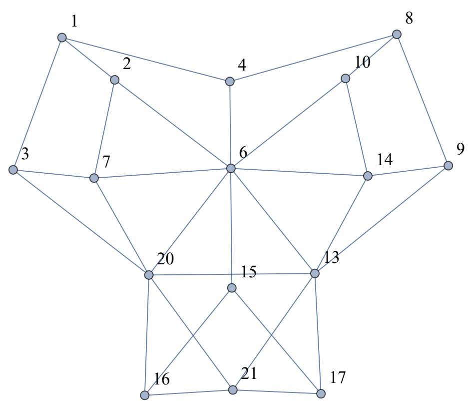
The first generated sample we get taking as section base graph with marked edges c-spike s-spike, 1-4 4-6.

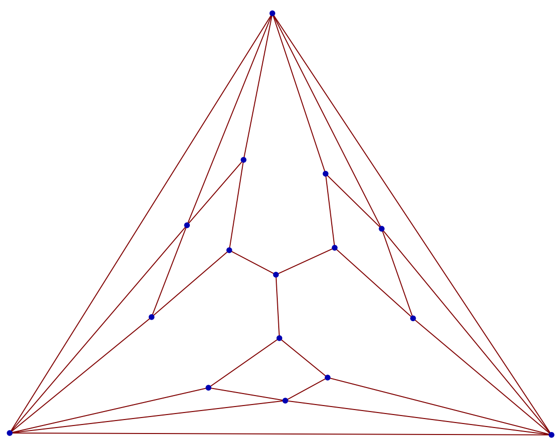




The graph is tested as 4-critical. *Mathematica* draws the graph non-planar and not quite regular. Right, with hand, graph is recognized as planar and wiith some regularity.

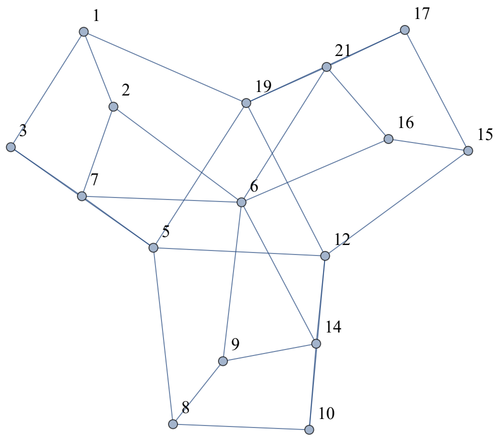
Next generated sample we get taking as section base graph with marked edges s-spike s-spike (from one section), 5-4 4-6.

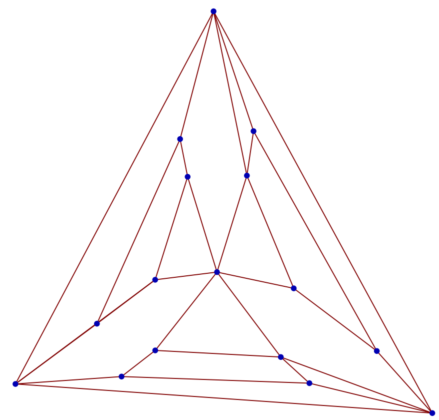




The graph is tested as 4-critical. *Mathematica* draws the graph non-planar and partly regular but otherwise as in our setting. Right, with hand, graph is recognized as planar and wiith some regularity. The vertex 4 becomes new hub, but *Mathematica* didn't recognize it.

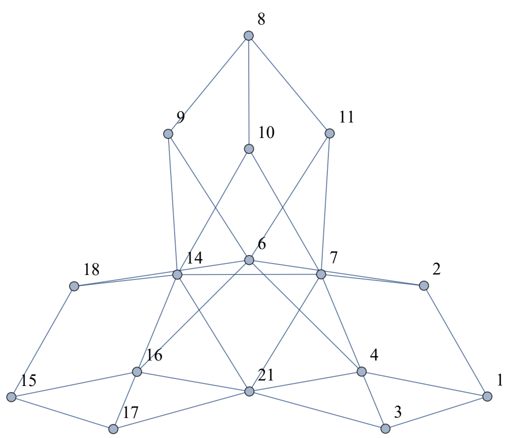
Next generated sample we get taking as section base graph with marked edges s-spike r-edge, 4-6 6-5.

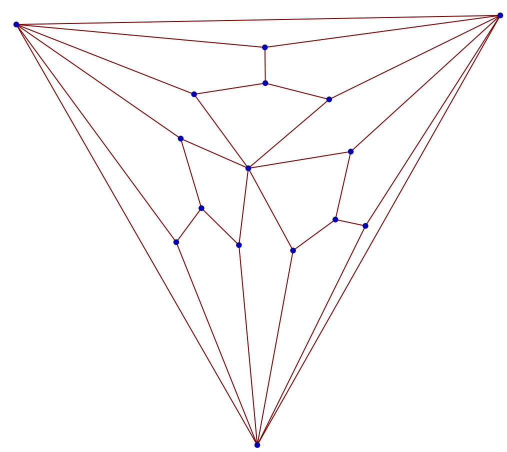




The graph is tested as 4-critical. *Mathematica* draws the graph non-planar and otherwise regular than we would expect. Right, with hand, graph is recognized as planar and wiith some regularity. The vertex 6 becomes new hub, and this time *Mathematica* recognized it.

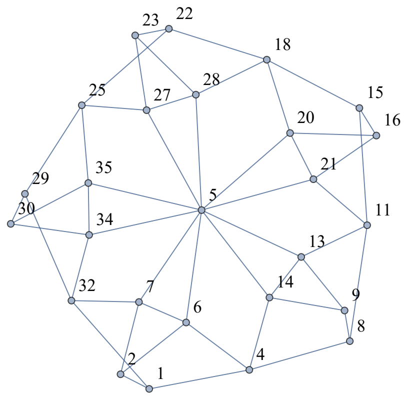
Next generated sample we get taking as section base graph with marked edges r-edge r-edge, 5-6 6-7.

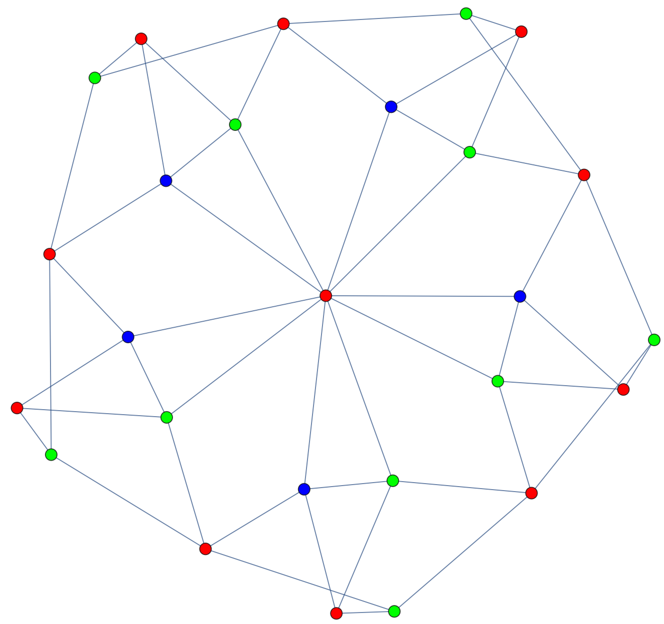


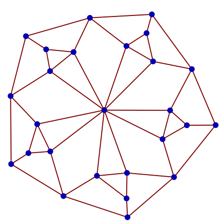


The graph is tested as 4-critical. *Mathematica* draws the graph non-planar and more or less correctly regular. Right, with hand, graph is recognized as planar and wiith some regularity. The vertex 6 becomes new hub, and this time *Mathematica* recognized it as regular drawing it non-planar. This sample contains octahedron as minor, thus, octahedral bracket here doesn't work. This case violates the expectance that 4-critical multiwheels has octahedral bracket as invariant.

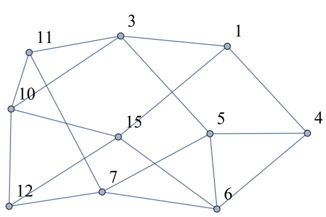
Next generated sample we get taking as section base graph with marked edges s-pike s-spike (but different sections), 3-5 5-4.

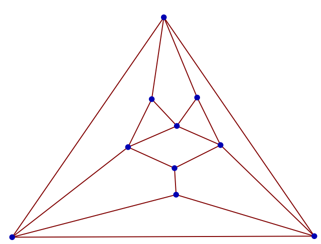






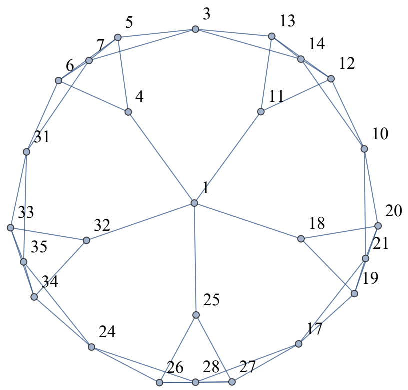
This time we get 3-chromatic graph and of course not 4-critical. The graph doesn’t have octahedron as minor. To see this one must examine more simple case below, where base graph as sectinon is put only in one section of the base graph. Thus, these cases are the only when 4-criticality doesn’t take place and octahedral bracket doesn’t work.

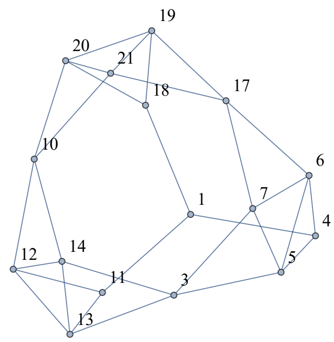


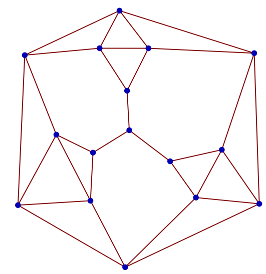


It is easy to see that the graph has octahedron as a minor. The graph is 3-chromatic, thus not 4-critical.

Next generated samples (central hub degree 3 and 5) we get taking as section base graph with marked edges c-spike c-spike, 3-5 5-4.



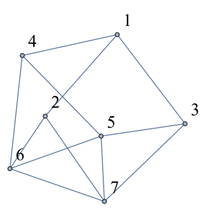


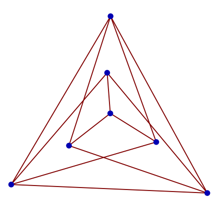


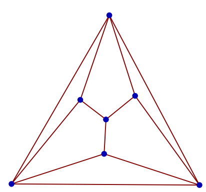
The graphs are tested as 4-critical.

Grötsch graph series
Basic samples and examples of the series:

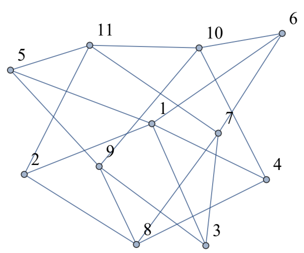
Here we may give some Grötsch graph exemplars. All these graphs are of type g_{1^k}, where the sections are wheel W_3. Therefore we denote them here g_k in place of g_{1^k}. We remind that g_{1^k} means g with k indices 1.

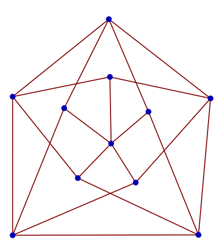




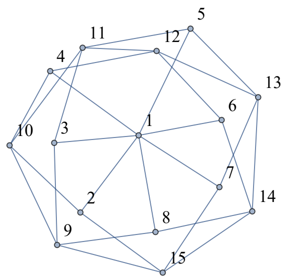


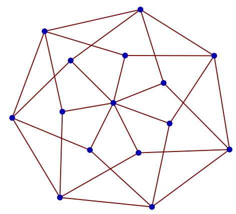
The base graph, minimal multiwheel graph but also minimal Grötsch graph g_1, thus g_1 is isomorphic with w_1. We see that *Mathematica* draws this minimal graph as non-planar and not very regular.



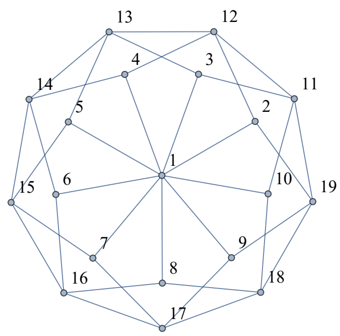


Next Grötsch graph is the graph g_2 with 11 vertices and 20 edges. This the classical Grotsch graph discovered by Grötsch. Alas, *Mathematica* depicts this fameous graph not very symmetrical.

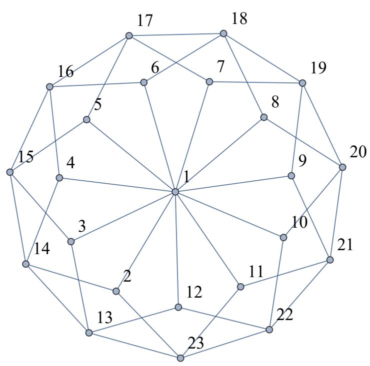




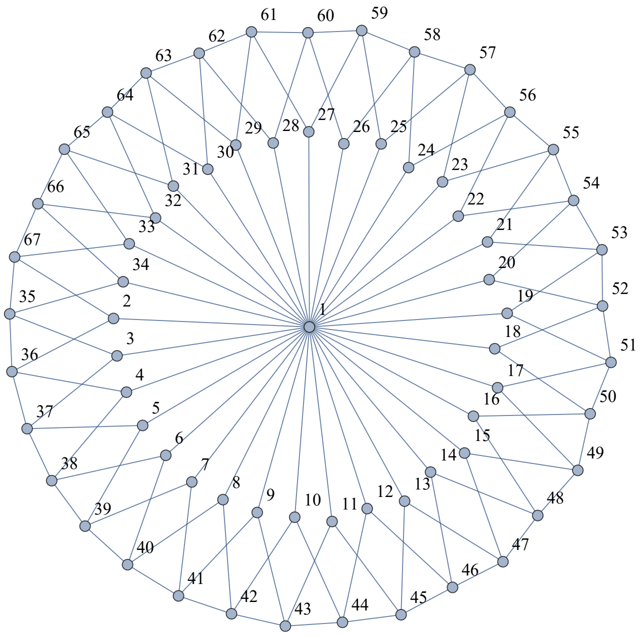
The third Grötsch graph, graph g_3 is with 15 vertices and 28 edges. 4-critical test takes 2 secunds.



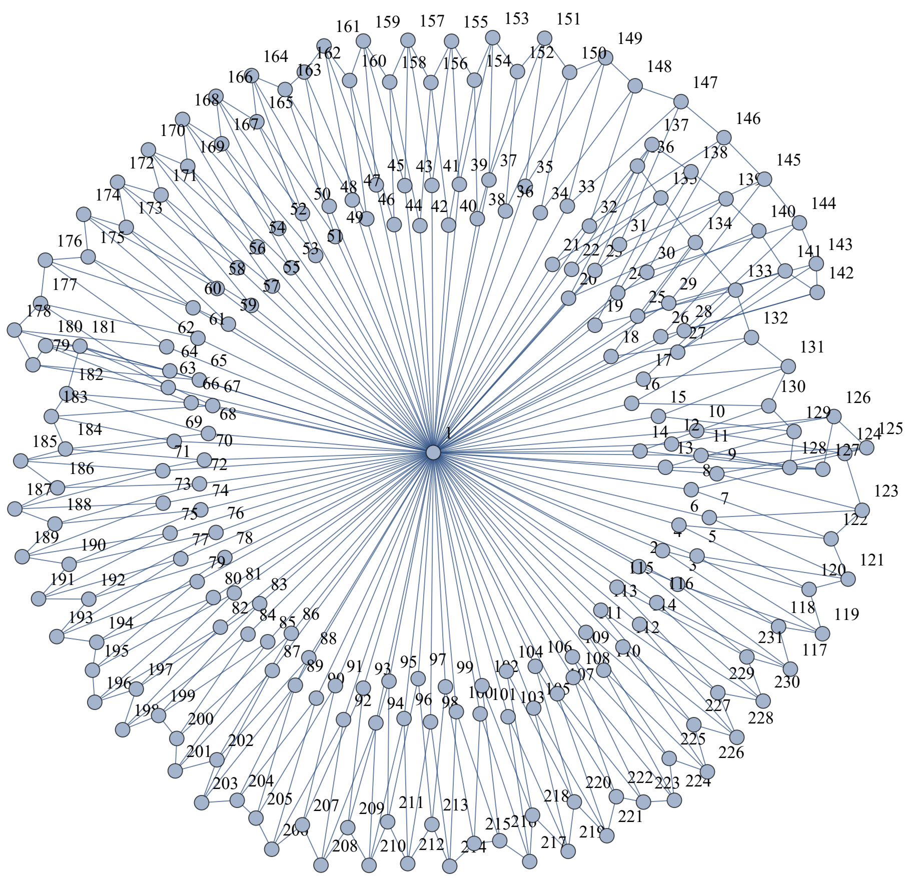
The graph g_4 with 19 vertices and 36 edges. 4-critical test takes 16 secunds.



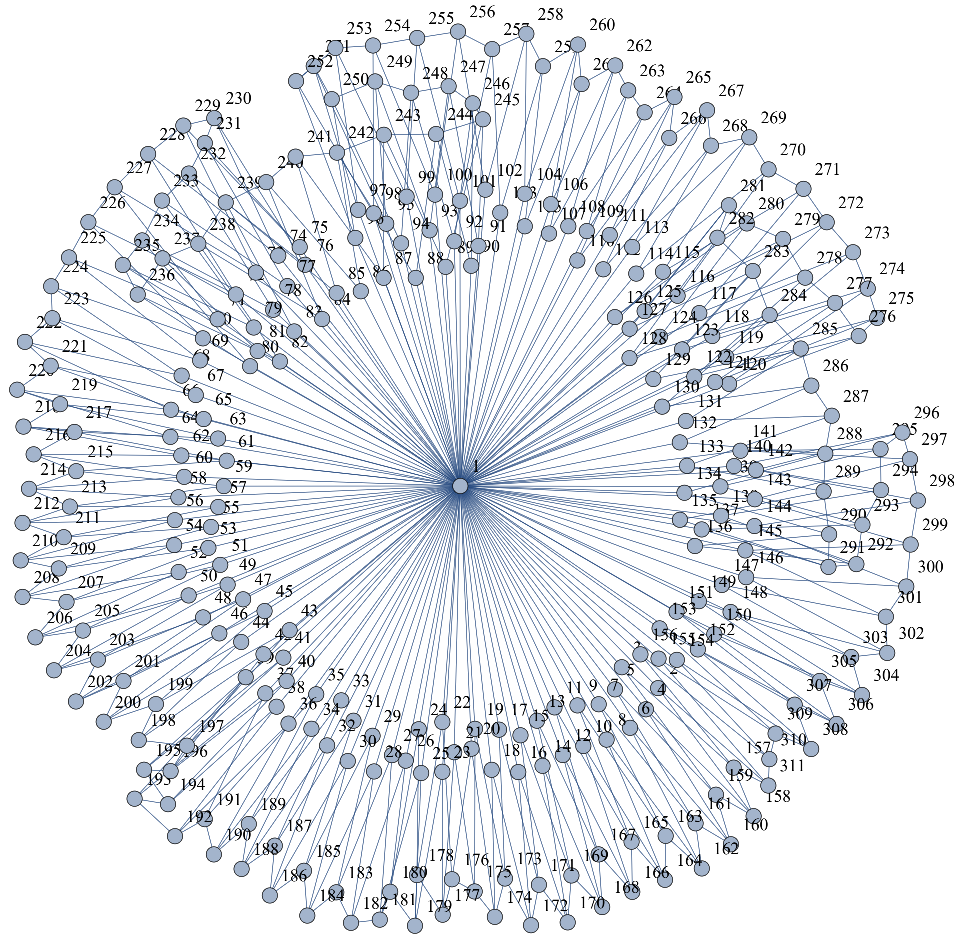
The graph g_5 with 23 vertices and 44 edges. 4-critical test takes 95 secunds.



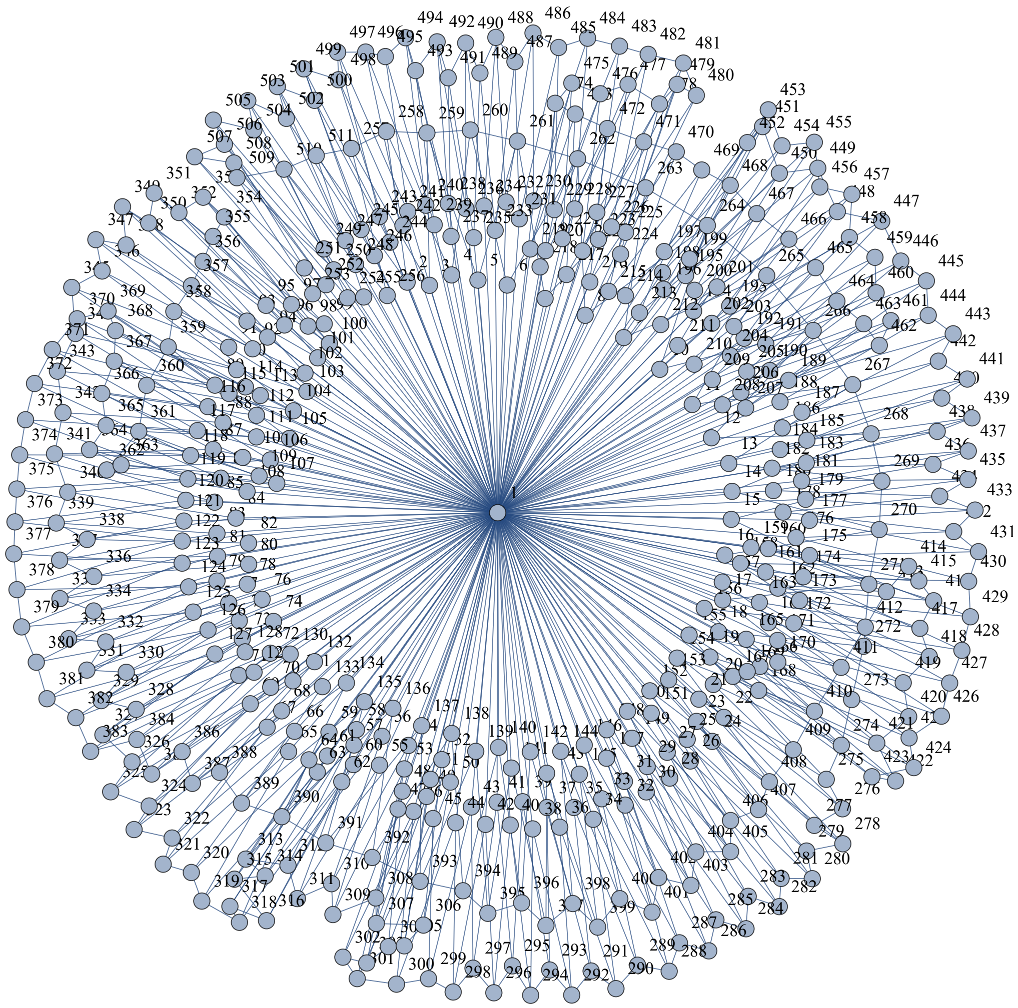
The graph g_{11} with 67 vertices and 132 edges.Drawing of the graph takes 0.2 secunds.



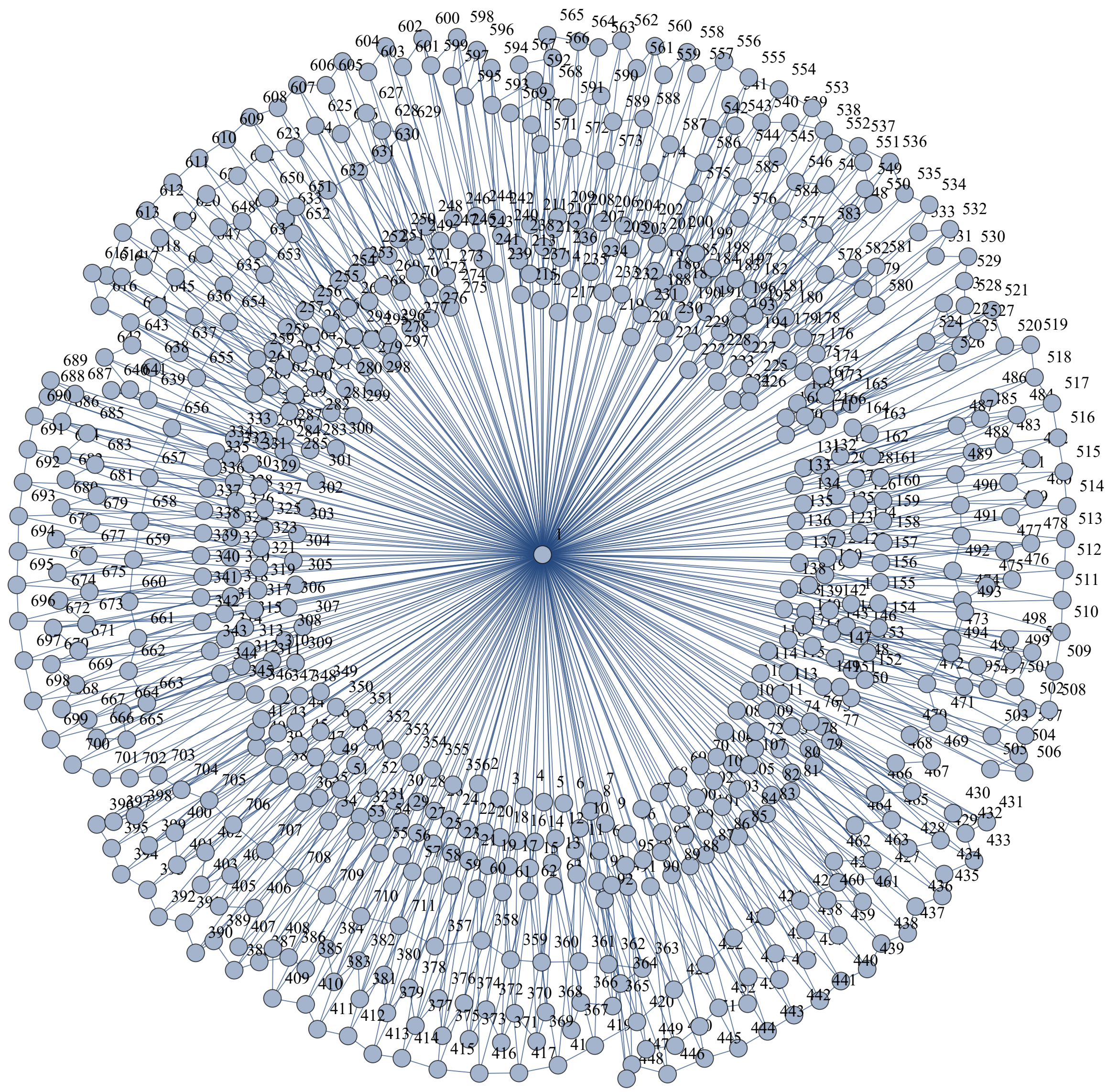
The graph g_{57} with 231 vertices and 460 edges.Drawing of the graph takes 8 secunds.



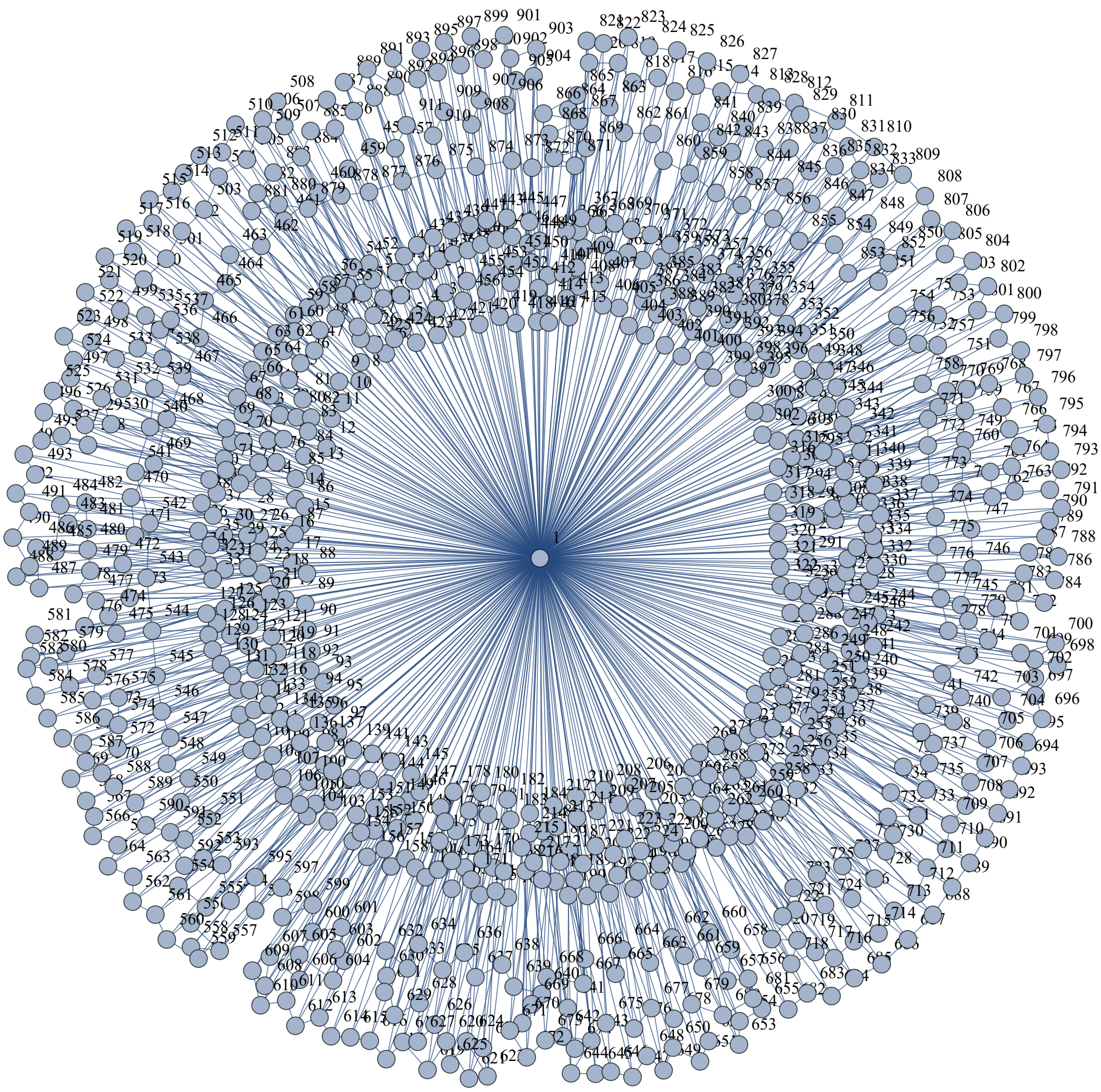
The graph g_{77} with 311 vertices and 620 edges.Drawing of the graph takes 20 secunds.



The graph g_{127} with 511 vertices and 1020 edges.Drawing of the graph takes 92 secunds.



The graph g_{177} with 711 vertices and 1420 edges.Drawing of the graph takes 4 minutes.



The graph g_{227} with 911 vertices and 1820 edges.Drawing of the graph takes 10 minutes.

On the theory of 4-critical multiwheels

Mostly we test here multiwheels that in [1] are proved to be 4-critical. But the proofs in [1] relay on the promise that the article [3] should provide the truth of the fact that graph can’t remain k-critical after vertex split. We here testet base graph’s main configurations on being 4-critical, and [1] shurely argues that simpply augmented configurations with section number augmentaion without changing their type can’t disturb graph being 4-critical. Thus, testing base graph configurations here we conclude the proofs of [1] without assumtion of [3]. Here even more, we test multiwheel configurations with sections being multiwheels, here, base graph. We must take into considerations that we can augment graph as planar of as Grötsch graph because the base graph belongs to both clases, i.e., is the minimal graph in both series, and both series augmentations should equaly work in producing 4-critical graphs.

But we found two exeptional graphs, situations. Section with s-spike rim vertex s-spike marking gave 3-chromatic graph that of course isn’t 4-

critical. But in the same time it is easy to see that this graph, and all augments from it, doesn't contain octahedral bracket as graph invariant: simply, octahedron is their minor. Alas, one more configuraion gives graphs with O as minor. This was r-edge r-edge case, which gave 4-critical graphs. But these facts allow to forward following conjectures:

Conjecture 1:

Multiwheels received in the way discribed in [1] and here are 4-critical, or 3-chromatic and with O as minor.

Conjecture 2:

[3] holds for 4-critical multiwheels built according patterns of [1] and what we find here.

Both conjectures are actually proved here if we accept the paradigmatic computational examples here to be parts of the proofs.

References

[1] Dainis Zeps. On building 4-critical plane and projective plane multiwheels from odd wheels, arXiv:1202.4862v1, [math.CO] Feb 2012

[2] Dainis Zeps. 4-critical wheel graphs of higher order, arXiv:1106.1336v1

[3] Zeps D. Can graph remain k-critical after vertex split?, in preparation, 2012

Here programs that were run:

```
w1 = WheelGraph[4];
w2 = WheelGraph[8, VertexLabels → "Name"];
GraphNumShift[G_, k_] :=
  Graph[Table[(EdgeList[G][[i, 1]] + k) ⟷ (EdgeList[G][[i, 2]] + k), {i, EdgeCount[G]}],
   VertexLabels → "Name"];
RenameVertex[G_, a_, b_] :=
  Graph[Table[(If[EdgeList[G][[i, 1]] == a, b, EdgeList[G][[i, 1]]]) ⟷
     (If[EdgeList[G][[i, 2]] == a, b, EdgeList[G][[i, 2]]]), {i, EdgeCount[G]}],
   VertexLabels → "Name"];
SumEdgesMarked[g1_, g2_] :=
  {RenameVertex[
    RenameVertex[Graph[Union[EdgeList[EdgeDelete[g1[[1]], g1[[2, 2]] ⟷ g1[[2, 3]]]],
       EdgeList[EdgeDelete[g2[[1]], g2[[2, 2]] ⟷ g2[[2, 1]]]]]], g2[[2, 2]],
     g1[[2, 2]]], g2[[2, 1]], g1[[2, 3]]], {g1[[2, 1]], g1[[2, 2]], g2[[2, 3]]}};
PlaneWheelSeq[{g_, {le_, hub_, ra_}}, k_] :=
  Join[{{g, {le, hub, ra}}},
   Table[{GraphNumShift[g, i VertexCount[g]],
     {le + i VertexCount[g], hub + i VertexCount[g], ra + i VertexCount[g]}}, {i, k}]];
PlaneMultiWheel[gs_] :=
 EdgeDelete[
  EdgeDelete[
   SumEdgesMarked[Fold[SumEdgesMarked[#1, #2] &, gs],
     Fold[SumEdgesMarked[#1, #2] &, gs]][[1]], First[gs][[2, 2]] ⟷ Last[gs][[2, 3]]],
  First[gs][[2, 2]] ⟷ Last[gs][[2, 3]]]
CriticalQ[G_, k_] := (ChromaticPolynomial[G, k - 1] == 0) &&
   Apply[And, Table[ChromaticPolynomial[EdgeDelete[G, EdgeList[G][[i]]], k - 1] > 0,
     {i, EdgeCount[G]}]];

ggs = PlaneWheelSeq[{w2, {2, 1, 3}}, 4]
gr = PlaneMultiWheel[ggs]
VertexCount[gr]
EdgeCount[gr]
Timing[CriticalQ[gr, 4]]
```