# A NOTE ON $q$-ANALOGUE OF SÁNDOR'S FUNCTIONS

by

**Taekyun Kim[1], C. Adiga[2] and Jung Hun Han[2]**

[1]Department of Mathematics Education
Kongju National University, Kongju 314-701
South Korea
e-mail:tkim@kongju.ac.kr / tkim64@hanmail.net

[2]Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysore 570006, India
e-mail:c-adiga@hotmail.com

Dedicated to Sun-Yi Park on 90th birthday

**ABSTRACT**

The additive analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals have been recently studied by J. Sándor. In this note, we obtain $q$-analogues of Sándor's theorems [6].

**Keywords and Phrases:** $q$-gamma function, Pseudo-Smarandache function, Smarandache-simple function, Asymtotic formula.

**2000 AMS Subject Classification:** 33D05,40A05.

## 1 Introduction

The additive analogues of Smarandache functions $S$ and $S_*$ have been introduced by Sándor [5] as follows:

$$S(x) = \min\{m \in N : x \leq m!\}, \quad x \in (1, \infty),$$

and

$$S_*(x) = \max\{m \in N : m! \leq x\}, \quad x \in [1, \infty).$$

He has studied many important properties of $S_*$ relating to continuity, differentiability and Riemann integrability and also proved the following theorems:

**Theorem 1.1**

$$S_*(x) \sim \frac{\log x}{\log\log x} \quad (x \to \infty).$$

**Theorem 1.2** The series

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

In [1], Adiga and Kim have obtained generalizations of Theorems 1.1 and 1.2 by the use of Euler's gamma function. Recently Adiga-Kim-Somashekara-Fathima [2] have established a $q$-analogues of these results on employing the $q$-analogue of Stirling's formula. In [6], Sándor defined the additive analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals as follows:

$$Z(x) = \min\left\{m \in N : x \leq \frac{m(m+1)}{2}\right\}, \quad x \in (0, \infty),$$

$$Z_*(x) = \max\left\{m \in N : \frac{m(m+1)}{2} \leq x\right\}, \quad x \in [1, \infty),$$

$$P(x) = \min\{m \in N : p^x \leq m!\}, \quad p > 1, x \in (0, \infty),$$

and

$$P_*(x) = \max\{m \in N : m! \leq p^x\}, \quad p > 1, x \in [1, \infty).$$

He has also proved the following theorems:

**Theorem 1.3**

$$Z_*(x) \sim \frac{1}{2}\sqrt{8x+1} \quad (x \to \infty).$$

**Theorem 1.4** The series

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^{\alpha}}$$

is convergent for $\alpha > 2$ and divergent for $\alpha \leq 2$. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^{\alpha}}$$

is convergent for all $\alpha > 0$.

**Theorem 1.5**

$$\log P_*(x) \sim \log x \quad (x \to \infty).$$

**Theorem 1.6** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\log\log n}{\log P_*(n)} \right)^{\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

The main purpose of this note is to obtain $q$-analogues of Sándor's Theorems 1.3 and 1.5. In what follows, we make use of the following notations and definitions. F. H. Jackson defined a $q$-analogue of the gamma function which extends the $q$-factorial

$$(n!)_q = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}), \quad \text{cf [3]},$$

which becomes the ordinary factorial as $q \to 1$. He defined the $q$-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, \quad q>1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary gamma function.

## 2 Main Theorems

We now define the $q$-analogues of $Z$ and $Z_*$ as follows:

$$Z_q(x) = \min\left\{\frac{1-q^m}{1-q} : x \le \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right\}, \quad m \in N, x \in (0,\infty),$$

and

$$Z_q^*(x) = \max\left\{\frac{1-q^m}{1-q} : \frac{\Gamma_q(m+2)}{2\Gamma_q(m)} \le x\right\}, \quad m \in N, x \in \left[\frac{\Gamma_q(3)}{2\Gamma_q(1)}, \infty\right),$$

where $0<q<1$. Clearly, $Z_q(x) \to Z(x)$ and $Z_q^*(x) \to Z_*(x)$ as $q \to 1^-$. From the definitions of $Z_q$ and $Z_q^*$, it is clear that

$$Z_q(x) = \left\{\begin{array}{ll} 1, \quad \text{if} & x \in \left(0, \frac{\Gamma_q(3)}{2\Gamma_q(1)}\right] \\ \frac{1-q^m}{1-q}, \quad \text{if} & x \in \left(\frac{\Gamma_q(m+1)}{2\Gamma_q(m-1)}, \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right], m \ge 2, \end{array}\right] \tag{2.1}$$

and

$$Z_q^*(x) = \frac{1-q^m}{1-q} \quad \text{if} \quad x \in \left[\frac{\Gamma_q(m+2)}{2\Gamma_q(m)}, \frac{\Gamma_q(m+3)}{2\Gamma_q(m+1)}\right). \tag{2.2}$$

Since

$$\frac{1-q^{m-1}}{1-q} \le \frac{1-q^m}{1-q} = \frac{1-q^{m-1}}{1-q} + q^{m-1} \le \frac{1-q^{m-1}}{1-q} + 1,$$

(2.1) and (2.2) imply that for $x \ge \frac{\Gamma_q(3)}{2\Gamma_q(1)}$,

$$Z_q^*(x) \le Z_q(x) \le Z_q^*(x) + 1.$$

Hence it suffices to study the function $Z_q^*$. We now prove our main theorems.

**Theorem 2.1** If $0 < q < 1$, then

$$\frac{\sqrt{1+8xq} - (1+2q)}{2q^2} < Z_q^*(x) \le \frac{\sqrt{1+8xq} - 1}{2q}, \quad x \ge \frac{\Gamma_q(3)}{2\Gamma_q(1)}.$$

**Proof.** If

$$\frac{\Gamma_q(k+2)}{2\Gamma_q(k)} \le x < \frac{\Gamma_q(k+3)}{2\Gamma_q(k+1)}, \tag{2.3}$$

then

$$Z_q^*(x) = \frac{1-q^k}{1-q}$$

and

$$(1-q^k)(1-q^{k+1}) - 2x(1-q)^2 \le 0 < (1-q^{k+1})(1-q^{k+2}) - 2x(1-q)^2. \tag{2.4}$$

Consider the functions $f$ and $g$ defined by

$$f(y) = (1-y)(1-yq) - 2x(1-q)^2$$

and

$$g(y) = (1 - yq)(1 - yq^2) - 2x(1 - q)^2.$$

Note that $f$ is monotonically decreasing for $y \leq \frac{1+q}{2q}$ and $g$ is strictly decreasing for $y < \frac{1+q}{2q^2}$. Also $f(y_1) = 0 = g(y_2)$ where

$$y_1 = \frac{(1+q) - (1-q)\sqrt{1+8xq}}{2q},$$

$$y_2 = \frac{(q+q^2) - q(1-q)\sqrt{1+8xq}}{2q^3}.$$

Since $y_1 < \frac{1+q}{2q}, y_2 < \frac{1+q}{2q^2}$ and $q^k < \frac{1+q}{2q} < \frac{1+q}{2q^2}$, from (2.4), it follows that

$$f(q^k) \leq f(y_1) = 0 = g(y_2) < g(q^k).$$

Thus $y_1 \leq q^k < y_2$ and hence

$$\frac{1-y_2}{1-q} < \frac{1-q^k}{1-q} \leq \frac{1-y_1}{1-q}.$$

i.e.

$$\frac{\sqrt{1+8xq} - (1+2q)}{2q^2} < Z_q^*(x) \leq \frac{\sqrt{1+8xq} - 1}{2q}.$$

This completes the proof.

**Remark.** Letting $q \to 1^-$ in the above theorem, we obtain Sándor's Theorem 1.3.

We define the $q$-analogues of $P$ and $P_*$ as follows:

$$P_q(x) = \min\{m \in N : p^x \leq \Gamma_q(m+1)\}, \quad p > 1, x \in (0, \infty),$$

and

$$P_q^*(x) = \max\{m \in N : \Gamma_q(m+1) \leq p^x\}, \quad p > 1, x \in [1, \infty),$$

where $0 < q < 1$. Clearly, $P_q(x) \to P(x)$ and $P_q^*(x) \to P_*(x)$ as $q \to 1^-$. From the definitions of $P_q$and $P_q^*$, we have

$$P_q^*(x) \leq P_q(x) \leq P_q^*(x) + 1.$$

Hence it is enough to study the function $P_q^*$.

**Theorem 2.2** If $0 < q < 1$, then

$$P_*(x) \sim \frac{x \log p}{\log\left(\frac{1}{1-q}\right)} \quad (x \to \infty).$$

**Proof.** If $\Gamma_q(n+1) \leq p^x < \Gamma_q(n+2)$, then

$$P_q^*(x) = n$$

and

$$\log \Gamma_q(n+1) \leq \log p^x < \log \Gamma_q(n+2). \tag{2.5}$$

But by the $q$-analogue of Stirling's formula established by Moak [4], we have

$$\log \Gamma_q(n+1) \sim \left(n + \frac{1}{2}\right) \log\left(\frac{q^{n+1} - 1}{q - 1}\right) \sim n \log\left(\frac{1}{1-q}\right). \tag{2.6}$$

Dividing (2.5) throughout by $n\log(\frac{1}{1-q})$, we obtain

$$\frac{\log \Gamma_q(n+1)}{n \log(\frac{1}{1-q})} \leq \frac{x \log p}{P_q^*(x) \log(\frac{1}{1-q})} < \frac{\log \Gamma_q(n+2)}{n \log(\frac{1}{1-q})}. \tag{2.7}$$

Using (2.6) in (2.7), we deduce

$$\lim_{x\to\infty} \frac{x\log p}{P_q^*(x)\log(\frac{1}{1-q})} = 1.$$

This completes the proof.

## References

[1] C. Adiga and T. Kim, *On a generalization of Sándor's function,* Proc. Jangjeon Math. Soc., **5**, (2002), 121–124.

[2] C. Adiga, T. Kim, D. D. Somashekara and N. Fathima, *On a q-analogue of Sándor's function,* J. Inequal. Pure and Appl. Math., **4**,No.4 Art.84 (2003), 1–5.

[3] T. Kim, *Non-archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials,* Russian J. Math. Phys., **10** (2003), 91–98.

[4] D. S. Moak, *The q-analogue of Stirling's formula,* Rocky Mountain J. Math., **14** (1984), 403–413.

[5] J. Sándor, *On an additive analogue of the function S,* Notes Numb. Th. Discr. Math., **7**(2001), 91–95.

[6] J. Sándor, *On additive analogues of certain arithmetic functions,* Smarandache Notions J.,14(2004),128-133.