

# Palindromic Permutations and Generalized Smarandache Palindromic Permutations <sup>\*†</sup>

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## Abstract

The idea of left(right) palindromic permutations(LPPs,RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs,RGSPPs) are introduced in symmetric groups  $S_n$  of degree  $n$ . It is shown that in  $S_n$ , there exist a LPP and a RPP and they are unique(this fact is demonstrated using  $S_2$  and  $S_3$ ). The dihedral group  $D_n$  is shown to be generated by a RGSP and a LGSPP(this is observed to be true in  $S_3$ ) but the geometric interpretations of a RGSP and a LGSP are found not to be rotation and reflection respectively. In  $S_3$ , each permutation is at least a RGSP or a LGSP. There are 4 RGSPs and 4 LGSPs in  $S_3$ , while 2 permutations are both RGSPs and LGSPs. A permutation in  $S_n$  is shown to be a LPP or RPP(LGSP or RGSP) if and only if its inverse is a LPP or RPP(LGSP or RGSP) respectively. Problems for future studies are raised.

## 1 Introduction

According to Ashbacher and Neiryneck [1], an integer is said to be a palindrome if it reads the same forwards and backwards. For example, 12321 is a palindromic number. They also stated that it is easy to prove that the density of the palindromes is zero in the set of positive integers and they went ahead to answer the question on the density of generalized Smarandache palindromes (GSPs) by showing that the density of GSPs in the positive integers is approximately 0.11. Gregory [2], Smarandache [8] and Ramsharan [7] defined a generalized Smarandache palindrome (GSP) as any integer or number of the form

$$a_1 a_2 a_3 \cdots a_n a_n \cdots a_3 a_2 a_1 \quad \text{OR} \quad a_1 a_2 a_3 \cdots a_{n-1} a_n a_{n-1} \cdots a_3 a_2 a_1$$

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where all  $a_1, a_2, a_3, \dots, a_n \in \mathbb{N}$  having one or more digits. On the other hand, Hu [3] calls any integer or number of this form a Smarandache generalized palindrome(SGP). His naming will not be used here the first naming will be adopted.

Numbers of this form have also been considered by Khoshnevisan [4], [5] and [6]. For the sake of clarification, it must be mentioned that the possibility of the trivial case of enclosing the entire number is excluded. For example, 12345 can be written as (12345). In this case, the number is simply said to be a palindrome or a palindromic number as it was mentioned earlier on. So, every number is a GSP. But this possibility is eliminated by requiring that each number be split into at least two segments if it is not a regular palindrome. Trivially, since each regular palindrome is also a GSP and there are GSPs that are not regular palindromes, there are more GSPs than there are regular palindromes. As mentioned by Gregory [2], very interesting GSPs are formed from smarandacheian sequences. For an illustration he cited the smarandacheian sequence

$$11, 1221, 123321, \dots, 123456789987654321, 1234567891010987654321, \\ 12345678910111110987654321, \dots$$

and observed that all terms are all GSPs. He also mentioned that it has been proved that the GSP 1234567891010987654321 is a prime and concluded his work by posing the question of 'How many primes are in the GSP sequence above?'

Special mappings such as morphisms(homomorphisms, endomorphisms, automorphisms, isomorphisms e.t.c) have been useful in the study of the properties of most algebraic structures(e.g groupoids, quasigroups, loops, semigroups, groups e.tc.). In this work, the notion of palindromic permutations and generalized Smarandache palindromic permutations are introduced and studied using the symmetric group on the set  $\mathbb{N}$  and this can now be viewed as the study of some palindromes and generalized Smarandache palindromes of numbers.

The idea of left(right) palindromic permutations(LPPs,RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs, RGSPPs) are introduced in symmetric groups  $S_n$  of degree  $n$ . It is shown that in  $S_n$ , there exist a LPP and a RPP and they are unique. The dihedral group  $D_n$  is shown to be generated by a RGSPP and a LGSPP but the geometric interpretations of a RGSPP and a LGSPP are found not to be rotation and reflection respectively. In  $S_3$ , each permutation is at least a RGSPP or a LGSPP. There are 4 RGSPPs and 4 LGSPPs in  $S_3$ , while 2 permutations are both RGSPPs and LGSPPs. A permutation in  $S_n$  is shown to be a LPP or RPP(LGSPP or RGSPP) if and only if its inverse is a LPP or RPP(LGSPP or RGSPP) respectively. Some of these results are demonstrated with  $S_2$  and  $S_3$ . Problems for future studies are raised.

But before then, some definitions and basic results on symmetric groups in classical group theory which shall be employed and used are highlighted first.

## 2 Preliminaries

**Definition 2.1** *Let  $X$  be a non-empty set. The group of all permutations of  $X$  under composition of mappings is called the **symmetric group** on  $X$  and is denoted by  $S_X$ . A*

subgroup of  $S_X$  is called a permutation group on  $X$ .

It is easily seen that a bijection  $X \simeq Y$  induces in a natural way an isomorphism  $S_X \cong S_Y$ . If  $|X| = n$ ,  $S_X$  is denoted by  $S_n$  and called the *symmetric group of degree  $n$* .

A permutation  $\sigma \in S_n$  can be exhibited in the form

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

consisting of two rows of integers; the top row has integers  $1, 2, \dots, n$  usually (but not necessarily) in their natural order, and the bottom row has  $\sigma(i)$  below  $i$  for each  $i = 1, 2, \dots, n$ . This is called a two-row notation for a permutation. There is a simpler, one-row notation for a special kind of permutation called *cycle*.

**Definition 2.2** Let  $\sigma \in S_n$ . If there exists a list of distinct integers  $x_1, \dots, x_r \in \mathbb{N}$  such that

$$\begin{aligned} \sigma(x_i) &= x_{i+1}, & i &= 1, \dots, r-1, \\ \sigma(x_r) &= x_1, \\ \sigma(x) &= x \text{ if } x \notin \{x_1, \dots, x_r\}, \end{aligned}$$

then  $\sigma$  is called a cycle of length  $r$  and denoted by  $(x_1 \cdots x_r)$ .

**Remark 2.1** A cycle of length 2 is called a *transposition*. In other words, a cycle  $(x_1 \cdots x_r)$  moves the integers  $x_1, \dots, x_r$  one step around a circle and leaves every other integer in  $\mathbb{N}$ . If  $\sigma(x) = x$ , we say  $\sigma$  does not move  $x$ . Trivially, any cycle of length 1 is the identity mapping  $I$  or  $e$ . Note that the one-row notation for a cycle does not indicate the degree  $n$ , which has to be understood from the context.

**Definition 2.3** Let  $X$  be a set of points in space, so that the distance  $d(x, y)$  between points  $x$  and  $y$  is given for all  $x, y \in X$ . A permutation  $\sigma$  of  $X$  is called a **symmetry** of  $X$  if

$$d(\sigma(x), \sigma(y)) = d(x, y) \quad \forall x, y \in X.$$

Let  $X$  be the set of points on the vertices of a regular polygon which are labelled  $\{1, 2, \dots, n\}$  i.e

The group of symmetries of a regular polygon  $P_n$  of  $n$  sides is called the **dihedral group of degree  $n$**  and denoted  $D_n$ .

**Remark 2.2** It must be noted that  $D_n$  is a subgroup of  $S_n$  i.e  $D_n \leq S_n$ .

**Definition 2.4** Let  $S_n$  be a symmetric group of degree  $n$ . If  $\sigma \in S_n$  such that

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

then

1. the number  $N_\lambda(\sigma) = 12 \cdots n\sigma(n) \cdots \sigma(1)$  is called the left palindromic value(LPV) of  $\sigma$ .
2. the number  $N_\rho(\sigma) = 12 \cdots n\sigma(1) \cdots \sigma(n)$  is called the right palindromic value(RPV) of  $\sigma$ .

**Definition 2.5** Let  $\sigma \in S_X$  such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

If  $X = \mathbb{N}$ , then

1.  $\sigma$  is called a left palindromic permutation(LPP) if and only if the number  $N_\lambda(\sigma)$  is a palindrome.

$$PP_\lambda(S_X) = \{\sigma \in S_X : \sigma \text{ is a LPP}\}$$

2.  $\sigma$  is called a right palindromic permutation(RPP) if and only if the number  $N_\rho(\sigma)$  is a palindrome.

$$PP_\rho(S_X) = \{\sigma \in S_X : \sigma \text{ is a RPP}\}$$

3.  $\sigma$  is called a palindromic permutation(PP) if and only if it is both a LPP and a RPP.

$$PP(S_X) = \{\sigma \in S_X : \sigma \text{ is a LPP and a RPP}\} = PP_\lambda(S_X) \cap PP_\rho(S_X)$$

**Definition 2.6** Let  $\sigma \in S_X$  such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

If  $X = \mathbb{N}$ , then

1.  $\sigma$  is called a left generalized Smarandache palindromic permutation(LGSPP) if and only if the number  $N_\lambda(\sigma)$  is a GSP.

$$GSPP_\lambda(S_X) = \{\sigma \in S_X : \sigma \text{ is a LGSPP}\}$$

2.  $\sigma$  is called a right generalized Smarandache palindromic permutation(RGSPP) if and only if the number  $N_\rho(\sigma)$  is a GSP.

$$GSPP_\rho(S_X) = \{\sigma \in S_X : \sigma \text{ is a RGSPP}\}$$

3.  $\sigma$  is called a generalized Smarandache palindromic permutation(GSPP) if and only if it is both a LGSPP and a RGSPP.

$$GSPP(S_X) = \{\sigma \in S_X : \sigma \text{ is a LGSPP and a RGSPP}\} = GSPP_\lambda(S_X) \cap GSPP_\rho(S_X)$$

**Theorem 2.1** (*Cayley Theorem*)

*Every group is isomorphic to a permutation group.*

**Theorem 2.2** *The dihedral group  $D_n$  is a group of order  $2n$  generated by two elements  $\sigma, \tau$  satisfying  $\sigma^n = e = \tau^2$  and  $\tau\sigma = \sigma^{n-1}\tau$ , where*

$$\sigma = (1 \ 2 \ \cdots \ n) \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

### 3 Main Results

**Theorem 3.1** *In any symmetric group  $S_n$  of degree  $n$ , there exists*

1. *a LPP and it is unique.*
2. *a RPP and it is unique.*

*But there does not exist a PP.*

**Proof**

Let  $\sigma \in S_n$ , then

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

1. When

$$\sigma(n) = n, \sigma(n-1) = n-1, \dots, \sigma(2) = 2, \sigma(1) = 1$$

then the number

$$N_\lambda(\sigma) = 12 \cdots n \sigma(n) \cdots \sigma(2) \sigma(1) = 12 \cdots nn \cdots 21$$

is a palindrome which implies  $\sigma \in PP_\lambda(S_n)$ . So, there exists a LPP. The uniqueness is as follows. Observe that

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = I.$$

Since  $S_n$  is a group for all  $n \in \mathbb{N}$  and  $I$  is the identity element(mapping), then it must be unique.

2. When

$$\sigma(1) = n, \sigma(2) = n-1, \dots, \sigma(n-1) = 2, \sigma(n) = 1$$

then the number

$$N_\rho(\sigma) = 12 \cdots n \sigma(1) \cdots \sigma(n-1) \sigma(n) = 12 \cdots nn \cdots 21$$

is a palindrome which implies  $\sigma \in PP_\rho(S_n)$ . So, there exists a RPP. The uniqueness is as follows. If there exist two of such, say  $\sigma_1$  and  $\sigma_2$  in  $S_n$ , then

$$\sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1(1) & \sigma_1(2) & \cdots & \sigma_1(n) \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_2(1) & \sigma_2(2) & \cdots & \sigma_2(n) \end{pmatrix}$$

such that

$$N_\rho(\sigma_1) = 12 \cdots n \sigma_1(1) \cdots \sigma_1(n-1) \sigma_1(n)$$

and

$$N_\rho(\sigma_2) = 12 \cdots n \sigma_2(1) \cdots \sigma_2(n-1) \sigma_2(n)$$

are palindromes which implies

$$\sigma_1(1) = n, \sigma_1(2) = n-1, \dots, \sigma_1(n-1) = 2, \sigma_1(n) = 1$$

and

$$\sigma_2(1) = n, \sigma_2(2) = n-1, \dots, \sigma_2(n-1) = 2, \sigma_2(n) = 1.$$

So,  $\sigma_1 = \sigma_2$ , thus  $\sigma$  is unique.

The proof of the last part is as follows. Let us assume by contradiction that there exists a PP  $\sigma \in S_n$ . Then if

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

$$N_\lambda(\sigma) = 12 \cdots n \sigma(n) \cdots \sigma(2) \sigma(1)$$

and

$$N_\rho(\sigma) = 12 \cdots n \sigma(1) \cdots \sigma(n-1) \sigma(n)$$

are palindromes. So that  $\sigma \in S_n$  is a PP. Consequently,

$$n = \sigma(n) = 1, n-1 = \sigma(n-1) = 2, \dots, 1 = \sigma(1) = n,$$

so that  $\sigma$  is not a bijection which means  $\sigma \notin S_n$ . This is a contradiction. Hence, no PP exist.

**Example 3.1** *Let us consider the symmetric group  $S_2$  of degree 2. There are two permutations of the set  $\{1, 2\}$  given by*

$$I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

$$N_\rho(I) = 1212 = (12)(12), N_\lambda(I) = 1221 \text{ or } N_\lambda(I) = 1(22)1,$$

$$N_\rho(\delta) = 1221 \text{ or } N_\rho(\delta) = (12)(21) \text{ and } N_\lambda(\delta) = 1212 = (12)(12).$$

So,  $I$  and  $\delta$  are both RGSPPs and LGSPPs which implies  $I$  and  $\delta$  are GSPPs i.e  $I, \delta \in GSPP_\rho(S_2)$  and  $I, \delta \in GSPP_\lambda(S_2) \Rightarrow I, \delta \in GSPP(S_2)$ . Therefore,  $GSPP(S_2) = S_2$ . Furthermore, it can be seen that the result in Theorem 3.1 is true for  $S_2$  because only  $I$  is a LPP and only  $\delta$  is a RPP. There is definitely no PP as the theorem says.

**Example 3.2** Let us consider the symmetric group  $S_3$  of degree 3. There are six permutations of the set  $\{1, 2, 3\}$  given by

$$e = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

As claimed in Theorem 3.1, the unique LPP in  $S_3$  is  $I$  while the unique RPP in  $S_3$  is  $\tau_2$ . There is no PP as the theorem says.

**Lemma 3.1** In  $S_3$ , the following are true.

1. At least  $\sigma \in GSPP_\rho(S_3)$  or  $\sigma \in GSPP_\lambda(S_3) \forall \sigma \in S_3$ .
2.  $|GSPP_\rho(S_3)| = 4$ ,  $|GSPP_\lambda(S_3)| = 4$  and  $|GSPP(S_3)| = 2$ .

**Proof**

Observe the following :

$$N_\lambda(I) = 123321, N_\rho(I) = 123123 = (123)(123).$$

$$N_\lambda(\sigma_1) = 123132, N_\rho(\sigma_1) = 123231 = 1(23)(23)1.$$

$$N_\lambda(\sigma_2) = 123213, N_\rho(\sigma_2) = 123312 = (12)(33)(12).$$

$$N_\lambda(\tau_1) = 123231 = 1(23)(23)1, N_\rho(\tau_1) = 123132.$$

$$N_\lambda(\tau_2) = 123123 = (123)(123), N_\rho(\tau_2) = 123321 = 123321.$$

$$N_\lambda(\tau_3) = 123312 = (12)(33)(12), N_\rho(\tau_3) = 123213.$$

So,  $GSPP_\lambda(S_3) = \{I, \tau_1, \tau_2, \tau_3\}$  and  $GSPP_\rho(S_3) = \{I, \sigma_1, \sigma_2, \tau_2\}$ . Thus, 1. is true. Therefore,  $|GSPP_\rho(S_3)| = 4$ ,  $|GSPP_\lambda(S_3)| = 4$  and  $|GSPP(S_3)| = |GSPP_\rho(S_3) \cap GSPP_\lambda(S_3)| = 2$ . So, 2. is true.

**Lemma 3.2**  $S_3$  is generated by a RGSPP and a LGSPP.

**Proof**

Recall from Example 3.2 that

$$S_3 = \{I = e, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}.$$

If  $\sigma = \sigma_1$  and  $\tau = \tau_1$ , then it is easy to verify that

$$\sigma^2 = \sigma_2, \sigma^3 = e, \tau^2 = e, \sigma\tau = \tau_3, \sigma^2\tau = \tau_2 = \tau\sigma \text{ hence,}$$

$$S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau_3\} \Rightarrow S_3 = \langle \sigma, \tau \rangle.$$

From the proof Lemma 3.1,  $\sigma$  is a RGSPP and  $\tau$  is a LGSPP. This justifies the claim.

**Remark 3.1** In Lemma 3.2,  $S_3$  is generated by a RGSP and a LGSP. Could this statement be true for all  $S_n$  of degree  $n$ ? Or could it be true for some subgroups of  $S_n$ ? Also, it is interesting to know the geometric meaning of a RGSP and a LGSP. So two questions are posed and the two are answered.

**Question 3.1** 1. Is the symmetric group  $S_n$  of degree  $n$  generated by a RGSP and a LGSP? If not, what permutation group(s) is generated by a RGSP and a LGSP?  
2. Are the geometric interpretations of a RGSP and a LGSP rotation and reflection respectively?

**Theorem 3.2** The dihedral group  $D_n$  is generated by a RGSP and a LGSP i.e  $D_n = \langle \sigma, \tau \rangle$  where  $\sigma \in GSPP_\rho(S_n)$  and  $\tau \in GSPP_\lambda(S_n)$ .

**Proof**

Recall from Theorem 2.2 that the dihedral group  $D_n = \langle \sigma, \tau \rangle$  where

$$\sigma = ( 1 \ 2 \ \cdots \ n ) = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

Observe that

$$N_\rho(\sigma) = 123 \cdots n 23 \cdots n 1 = 1(23 \cdots n)(23 \cdots n)1, \quad N_\lambda(\sigma) = 123 \cdots n 1 n \cdots 32.$$

$$N_\rho(\tau) = 12 \cdots n 1 n \cdots 2, \quad N_\lambda(\tau) = 12 \cdots n 2 \cdots n 1 = 1(2 \cdots n)(2 \cdots n)1.$$

So,  $\sigma \in GSPP_\rho(S_n)$  and  $\tau \in GSPP_\lambda(S_n)$ . Therefore, the dihedral group  $D_n$  is generated by a RGSP and a LGSP.

**Remark 3.2** In Lemma 3.2, it was shown that  $S_3$  is generated by a RGSP and a LGSP. Considering Theorem 3.2 when  $n = 3$ , it can be deduced that  $D_3$  will be generated by a RGSP and a LGSP. Recall that  $|D_3| = 2 \times 3 = 6$ , so  $S_3 = D_3$ . Thus Theorem 3.2 generalizes Lemma 3.2.

**Rotations and Reflections** Geometrically, in Theorem 3.2,  $\sigma$  is a rotation of the regular polygon  $P_n$  through an angle  $\frac{2\pi}{n}$  in its own plane, and  $\tau$  is a reflection (or a turning over) in the diameter through the vertex 1. It looks like a RGSP and a LGSP are formed by rotation and reflection respectively. But there is a contradiction in  $S_4$  which can be traced from a subgroup of  $S_4$  particularly the Klein four-group. The Klein four-group is the group of symmetries of a four sided non-regular polygon(rectangle). The elements are:

$$e = I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

and  $\delta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$



Observe the following:

$$N_\rho(\delta_1) = 12343412 = (12)(34)(34)(12), \quad N_\lambda(\delta_1) = 12342143.$$

$$N_\rho(\delta_2) = 12342143 = 12342143, \quad N_\lambda(\delta_2) = 12343412 = (12)(34)(34)(12).$$

$$N_\rho(\delta_3) = 12344321 = 123(44)321, \quad N_\lambda(\delta_3) = 12341234 = (1234)(1234).$$

So,  $\delta_1$  is a RGSPP while  $\delta_2$  is a LGSPP and  $\delta_3$  is a GSPP. Geometrically,  $\delta_1$  is a rotation through an angle of  $\pi$  while  $\delta_2$  and  $\delta_3$  are reflections in the axes of symmetry parallel to the sides. Thus  $\delta_3$  which is a GSPP is both a reflection and a rotation, which is impossible. Therefore, the geometric meaning of a RGSPP and a LGSPP are not rotation and reflection respectively. It is difficult to really ascertain the geometric meaning of a RGSPP and a LGSPP if at all it exist.

How beautiful will it be if  $GSPP_\rho(S_n)$ ,  $PP_\rho(S_n)$ ,  $GSPP_\lambda(S_n)$ ,  $PP_\lambda(S_n)$ ,  $GSPP(S_n)$  and  $PP(S_n)$  form algebraic structures under the operation of map composition.

**Theorem 3.3** *Let  $S_n$  be a symmetric group of degree  $n$ . If  $\sigma \in S_n$ , then*

1.  $\sigma \in PP_\lambda(S_n) \Leftrightarrow \sigma^{-1} \in PP_\lambda(S_n)$ .
2.  $\sigma \in PP_\rho(S_n) \Leftrightarrow \sigma^{-1} \in PP_\rho(S_n)$ .
3.  $I \in PP_\lambda(S_n)$ .

**Proof**

1.  $\sigma \in PP_\lambda(S_n)$  implies

$$N_\lambda(\sigma) = 12 \cdots n\sigma(n) \cdots \sigma(2)\sigma(1)$$

is a palindrome. Consequently,

$$\sigma(n) = n, \sigma(n-1) = n-1, \cdots, \sigma(2) = 2, \sigma(1) = 1.$$

So,

$$N_\lambda(\sigma^{-1}) = \sigma(1)\sigma(2) \cdots \sigma(n)n \cdots 21 = 12 \cdots nn \cdots 21 \Rightarrow \sigma^{-1} \in PP_\lambda(S_n).$$

The converse is similarly proved by carrying out the reverse of the procedure above.

2.  $\sigma \in PP_\rho(S_n)$  implies

$$N_\rho(\sigma) = 12 \cdots n\sigma(1) \cdots \sigma(n-1)\sigma(n)$$

is a palindrome. Consequently,

$$\sigma(1) = n, \sigma(2) = n-1, \cdots, \sigma(n-1) = 2, \sigma(n) = 1.$$

So,

$$N_\rho(\sigma^{-1}) = \sigma(1) \cdots \sigma(n-1)\sigma(n)12 \cdots n = n \cdots 2112 \cdots n \Rightarrow \sigma^{-1} \in PP_\rho(S_n).$$

The converse is similarly proved by carrying out the reverse of the procedure above.

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$$N_\lambda(I) = 12 \cdots nn \cdots 21 \Rightarrow I \in PP_\lambda(S_n).$$

**Theorem 3.4** *Let  $S_n$  be a symmetric group of degree  $n$ . If  $\sigma \in S_n$ , then*

1.  $\sigma \in GSPP_\lambda(S_n) \Leftrightarrow \sigma^{-1} \in GSPP_\lambda(S_n)$ .
2.  $\sigma \in GSPP_\rho(S_n) \Leftrightarrow \sigma^{-1} \in GSPP_\rho(S_n)$ .
3.  $I \in GSPP(S_n)$ .

**Proof**

If  $\sigma \in S_n$ , then

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

So,

$$N_\lambda(\sigma) = 12 \cdots n\sigma(n) \cdots \sigma(2)\sigma(1)$$

and

$$N_\rho(\sigma) = 12 \cdots n\sigma(1) \cdots \sigma(n-1)\sigma(n)$$

are numbers with even number of digits whether  $n$  is an even or odd number. Thus,  $N_\rho(\sigma)$  and  $N_\lambda(\sigma)$  are GSPs defined by

$$a_1 a_2 a_3 \cdots a_n a_n \cdots a_3 a_2 a_1$$

and not

$$a_1 a_2 a_3 \cdots a_{n-1} a_n a_{n-1} \cdots a_3 a_2 a_1$$

where all  $a_1, a_2, a_3, \cdots, a_n \in \mathbb{N}$  having one or more digits because the first has even number of digits(or grouped digits) while the second has odd number of digits(or grouped digits). The following grouping notations will be used:

$$(a_i)_{i=1}^n = a_1 a_2 a_3 \cdots a_n \quad \text{and} \quad [a_i]_{i=1}^n = a_n a_{n-1} a_{n-2} \cdots a_3 a_2 a_1.$$

Let  $\sigma \in S_n$  such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}$$

where  $x_i \in \mathbb{N} \forall i \in \mathbb{N}$ .

1. So,  $\sigma \in GSPP_\lambda(S_n)$  implies

$$N_\lambda(\sigma) = (x_{i_1})_{i_1=1}^{n_1} (x_{i_2})_{i_2=(n_1+1)}^{n_2} (x_{i_3})_{i_3=(n_2+1)}^{n_3} \cdots (x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} (x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

$$\leftrightarrow \leftrightarrow [\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n} [\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \cdots [\sigma(x_{i_3})]_{i_3=(n_2+1)}^{n_3} [\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2} [\sigma(x_{i_1})]_{i_1=1}^{n_1}$$

is a GSP where  $x_{i_j} \in \mathbb{N} \forall i_j \in \mathbb{N}, j \in \mathbb{N}$  and  $n_n = n$ . The interval of integers  $[1, n]$  is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping  $(\cdot)_{i_j}^{n_j}$  or  $[\cdot]_{i_j}^{n_j}$  is determined by the corresponding interval of integers  $[n_i + 1, n_{i+1}]$  and it is a matter of choice in other to make the number  $N_\lambda(\sigma)$  a GSP.

Now that  $N_\lambda(\sigma)$  is a GSP, the following are true:

$$\begin{aligned} (x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} &= [\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n} &\Leftrightarrow & [x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n} = (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n} \\ (x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} &= [\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} &\Leftrightarrow & [x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \\ &\vdots && \vdots \\ (x_{i_2})_{i_2=(n_1+1)}^{n_2} &= [\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2} &\Leftrightarrow & [x_{i_2}]_{i_2=(n_1+1)}^{n_2} = (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} \\ (x_{i_1})_{i_1=1}^{n_1} &= [\sigma(x_{i_1})]_{i_1=1}^{n_1} &\Leftrightarrow & [x_{i_1}]_{i_1=1}^{n_1} = (\sigma(x_{i_1}))_{i_1=1}^{n_1} \end{aligned}$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix},$$

so

$$N_\lambda(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1} (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} (\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3} \cdots (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n} [x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n} [x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \cdots [x_{i_3}]_{i_3=(n_2+1)}^{n_3} [x_{i_2}]_{i_2=(n_1+1)}^{n_2} [x_{i_1}]_{i_1=1}^{n_1}$$

is a GSP hence,  $\sigma^{-1} \in GSPP_\lambda(S_n)$ .

The converse can be proved in a similar way since  $(\sigma^{-1})^{-1} = \sigma$ .

2. Also,  $\sigma \in GSPP_\rho(S_n)$  implies

$$N_\rho(\sigma) = (x_{i_1})_{i_1=1}^{n_1} (x_{i_2})_{i_2=(n_1+1)}^{n_2} (x_{i_3})_{i_3=(n_2+1)}^{n_3} \cdots (x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} (x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} \\ (\sigma(x_{i_1}))_{i_1=1}^{n_1} (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} (\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3} \cdots (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP where  $x_{i_j} \in \mathbb{N} \forall i_j \in \mathbb{N}, j \in \mathbb{N}$  and  $n_n = n$ . The interval of integers  $[1, n]$  is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping  $(\cdot)_{i_j}^{n_j}$  is determined by the corresponding interval of integers  $[n_i + 1, n_{i+1}]$  and it is a matter of choice in order to make the number  $N_\rho(\sigma)$  a GSP.

Now that  $N_\rho(\sigma)$  is a GSP, the following are true:

$$\begin{aligned} (x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} &= (\sigma(x_{i_1}))_{i_1=1}^{n_1} \\ (x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} &= (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ (x_{i_2})_{i_2=(n_1+1)}^{n_2} &= (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \\ (x_{i_1})_{i_1=1}^{n_1} &= (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n} \end{aligned}$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix},$$

so

$$N_\rho(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1} (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} (\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3} \cdots (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \\ (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n} (x_{i_1})_{i_1=1}^{n_1} (x_{i_2})_{i_2=(n_1+1)}^{n_2} (x_{i_3})_{i_3=(n_2+1)}^{n_3} \cdots (x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} (x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP hence,  $\sigma^{-1} \in GSPP_\rho(S_n)$ .

The converse can be proved in a similar way since  $(\sigma^{-1})^{-1} = \sigma$ .

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$$N_\lambda(I) = 12 \cdots nn \cdots 21 = 12 \cdots (nn) \cdots 21 \Rightarrow I \in GSPP_\lambda(S_n) \text{ and}$$

$$N_\rho(I) = (12 \cdots n)(12 \cdots n) \Rightarrow I \in GSPP_\rho(S_n)$$

thus,  $I \in GSPP(S_n)$ .

## 4 Conclusion and Future studies

By Theorem 3.1, it is certainly true in every symmetric group  $S_n$  of degree  $n$  there exist at least a RGSP and a LGSP (although they are actually RPP and LPP). Following Example 3.1, there are 2 RGSPs, 2 LGSPs and 2 GSPs in  $S_2$  while from Lemma 3.1, there are 4 RGSPs, 4 LGSPs and 2 GSPs in  $S_3$ . Also, it can be observed that

$$|GSPP_\rho(S_2)| + |GSPP_\lambda(S_2)| - |GSPP(S_2)| = 2! = |S_2| \text{ and}$$

$$|GSPP_\rho(S_3)| + |GSPP_\lambda(S_3)| - |GSPP(S_3)| = 3! = |S_3|.$$

The following problems are open for further studies.

**Problem 4.1** 1. *How many RGSPs, LGSPs and GSPs are in  $S_n$ ?*

2. *Does there exist functions  $f_1, f_2, f_3 : \mathbb{N} \rightarrow \mathbb{N}$  such that  $|GSPP_\rho(S_n)| = f_1(n)$ ,  $|GSPP_\lambda(S_n)| = f_2(n)$  and  $|GSPP(S_n)| = f_3(n)$ ?*

3. *In general, does the formula*

$$|GSPP_\rho(S_n)| + |GSPP_\lambda(S_n)| - |GSPP(S_n)| = n! = |S_n|?$$

*hold. If not, for what other  $n > 3$  is it true?*

The GAP package or any other appropriate mathematical package could be helpful in investigating the solutions to them.

If the first question is answered, then the number of palindromes that can be formed from the set  $\{1, 2, \dots, n\}$  can be known since in the elements of  $S_n$ , the bottom row gives all possible permutation of the integers  $1, 2, \dots, n$ .

The Cayley Theorem (Theorem 2.1) can also be used to make a further study on generalized Smarandache palindromic permutations. In this work,  $\mathbb{N}$  was the focus and it does not contain the integer zero. This weakness can be strengthened by considering the set  $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} \forall n \in \mathbb{N}$ . Recall that  $(\mathbb{Z}_n, +)$  is a group and so by Theorem 2.1  $(\mathbb{Z}_n, +)$  is isomorphic to a permutation group particularly, one can consider a subgroup of the symmetric group  $S_{\mathbb{Z}_n}$ .

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