

# Parallel Bundles in Planar Map Geometries

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**Abstract:** Parallel lines are very important objects in Euclid plane geometry and its behaviors can be gotten by one's intuition. But in a planar map geometry, a kind of the Smarandache geometries, the situation is complex since it may contains elliptic or hyperbolic points. This paper concentrates on the behaviors of parallel bundles in planar map geometries, a generalization of parallel lines in plane geometry and obtains characteristics for parallel bundles.

**Key Words:** parallel bundle, planar map, Smarandache geometry, map geometry, classification.

**AMS(2000):** 05C15, 20H15, 51D99, 51M05

## 1 Introduction

A *map* is a connected topological graph cellularly embedded in a surface. On the past century, many works are concentrated on to find the combinatorial properties of maps, such as to determine whether exists a particularly embedding on a surface ([7][11]) or to enumerate a family of maps ([6]). All these works are on the side of algebra, not the object itself, i.e., geometry. For the later, more attentions are given to its element's behaviors, such as, the line, angle, area, curvature,  $\dots$ , see also [12] and [14]. For returning to its original face, the conception of map geometries is introduced in [10]. It is proved in [10] that the map geometries are nice model of the Smarandache geometries. They are also a new kind of intrinsic geometry of surfaces ([1]). The main purpose of this paper is to determine the behaviors of parallel bundles in planar geometries, a generalization of parallel lines in the Euclid plane geometry.

An axiom is said *Smarandachely denied* if the axiom behaves in at least two different ways within the same space, i.e., validated and invalidated, or only invalidated but in multiple distinct ways.

A *Smarandache geometry* is a geometry which has at least one Smarandachely denied axiom(1969)([5][13]).

In [3][4], Iseri presented a nice model of the Smarandache geometries, called *s-manifolds* by using equilateral triangles, which is defined as follows([3][5][9]):

*An s-manifold is any collection  $\mathcal{C}(T, n)$  of these equilateral triangular disks  $T_i, 1 \leq i \leq n$  satisfying the following conditions:*

(i) *Each edge  $e$  is the identification of at most two edges  $e_i, e_j$  in two distinct triangular disks  $T_i, T_j, 1 \leq i, j \leq n$  and  $i \neq j$ ;*

(ii) Each vertex  $v$  is the identification of one vertex in each of five, six or seven distinct triangular disks.

The conception of map geometries without boundary is defined as follows ([10]).

**Definition 1.1** For a given combinatorial map  $M$ , associates a real number  $\mu(u)$ ,  $0 < \mu(u) < \pi$ , to each vertex  $u, u \in V(M)$ . Call  $(M, \mu)$  a map geometry without boundary,  $\mu(u)$  the angle factor of the vertex  $u$  and to be orientable or non-orientable if  $M$  is orientable or not.

In [10], it has proved that map geometries are the Smarandache geometries. The realization of each vertex  $u, u \in V(M)$  in  $R^3$  space is shown in the Fig.1 for each case of  $\rho(u)\mu(u) > 2\pi$ ,  $= 2\pi$  or  $< 2\pi$ , call *elliptic point*, *euclidean point* and *hyperbolic point*, respectively.

$$\rho(u)\mu(u) < 2\pi \qquad \rho(u)\mu(u) = 2\pi \qquad \rho(u)\mu(u) > 2\pi$$

**Fig.1**

Therefore, a line passes through an elliptic vertex, an euclidean vertex or a hyperbolic vertex  $u$  has angle  $\frac{\rho(u)\mu(u)}{2}$  at the vertex  $u$ . It is not  $180^\circ$  if the vertex  $u$  is elliptic or hyperbolic. Then what is the angle of a line passes through a point on an edge of a map? It is  $180^\circ$ ? Since we wish the change of angles on an edge is smooth, the answer is not. For the Smarandache geometries, the parallel lines in them are need to be given more attention. We have the following definition.

**Definition 1.2** A family  $\mathcal{L}$  of infinite lines not intersecting each other in a planar geometry is called a parallel bundle.

In the Fig.2, we present all cases of parallel bundles passing through an edge in planar geometries, where, (a) is the case of points  $u, v$  are same type with  $\rho(u)\mu(u) = \rho(v)\mu(v)$ , (b) and (c) the cases of same types with  $\rho(u)\mu(u) > \rho(v)\mu(v)$  and (d) the case of  $u$  is elliptic and  $v$  hyperbolic.

### Fig.2

Here, we assume the angle at the intersection point is in clockwise, that is, a line passing through an elliptic point will bend up and a hyperbolic point will bend down, such as the cases (b),(c) in the Fig.2. For a vector  $\vec{O}$  on the Euclid plane, call it an *orientation*. We classify parallel bundles in planar map geometries along an orientation  $\vec{O}$ .

### 2. A condition for parallel bundles

We investigate the behaviors of parallel bundles in the planar map geometries. For this object, we define a function  $f(x)$  of angles on an edge of a planar map as follows.

**Definition 2.1** Denote by  $f(x)$  the angle function of a line  $L$  passing through an edge  $uv$  at the point of distance  $x$  to  $u$  on the edge  $uv$ .

Then we get the following result.

**Proposition 2.1** A family  $\mathcal{L}$  of parallel lines passing through an edge  $uv$  is a parallel bundle iff

$$\left. \frac{df}{dx} \right|_+ \geq 0.$$

*Proof* If  $\mathcal{L}$  is a parallel bundle, then any two lines  $L_1, L_2$  will not intersect after them passing through the edge  $uv$ . Therefore, if  $\theta_1, \theta_2$  are the angles of  $L_1, L_2$  at the intersect points of  $L_1, L_2$  with  $uv$  and  $L_2$  is far from  $u$  than  $L_1$ , then we know that  $\theta_2 \geq \theta_1$ . Whence, for any point with  $x$  distance from  $u$  and  $\Delta x > 0$ , we have that

$$f(x + \Delta x) - f(x) \geq 0.$$

Therefore, we get that

$$\left. \frac{df}{dx} \right|_+ = \lim_{\Delta x \rightarrow +0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

As the cases in the Fig.1.

Now if  $\left. \frac{df}{dx} \right|_+ \geq 0$ , then  $f(y) \geq f(x)$  if  $y \geq x$ . Since  $\mathcal{L}$  is a family of parallel lines before meeting  $uv$ , whence, any two lines in  $\mathcal{L}$  will not intersect each other after them passing through  $uv$ . Therefore,  $\mathcal{L}$  is a parallel bundle.  $\spadesuit$

A general condition for a family of parallel lines passing through a cut of a planar map being a parallel bundle is the following.

**Proposition 2.2** *Let  $(M, \mu)$  be a planar map geometry,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$  and the angle functions on them are  $f_1, f_2, \dots, f_l$ , respectively, also see the Fig.3.*

**Fig.3**

*Then a family  $\mathcal{L}$  of parallel lines passing through  $C$  is a parallel bundle iff for any  $x, x \geq 0$ ,*

$$\begin{aligned}
 & f'_1(x) \geq 0 \\
 & f'_{1+}(x) + f'_{2+}(x) \geq 0 \\
 & f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) \geq 0 \\
 & \dots\dots\dots \\
 & f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) \geq 0.
 \end{aligned}$$

*Proof* According to the Proposition 2.1, see the following Fig.4,

**Fig.4**

we know that any lines will not intersect after them passing through  $u_1v_1$  and  $u_2v_2$  iff for  $\forall \Delta x > 0$  and  $x \geq 0$ ,

$$f_2(x + \Delta x) + f'_{1+}(x)\Delta x \geq f_2(x).$$

That is,

$$f'_{1+}(x) + f'_{2+}(x) \geq 0.$$

Similarly, any lines will not intersect after them passing through  $u_1v_1, u_2v_2$  and  $u_3v_3$  iff for  $\forall \Delta x > 0$  and  $x \geq 0$ ,

$$f_3(x + \Delta x) + f'_{2+}(x)\Delta x + f'_{1+}(x)\Delta x \geq f_3(x).$$

That is,

$$f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) \geq 0.$$

Generally, any lines will not intersect after them passing through  $u_1v_1, u_2v_2, \dots, u_{l-1}v_{l-1}$  and  $u_lv_l$  iff for  $\forall \Delta x > 0$  and  $x \geq 0$ ,

$$f_l(x + \Delta x) + f'_{l-1+}(x)\Delta x + \dots + f'_{1+}(x)\Delta x \geq f_l(x).$$

Whence, we get that

$$f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) \geq 0.$$

Therefore, a family  $\mathcal{L}$  of parallel lines passing through  $C$  is a parallel bundle iff for any  $x, x \geq 0$ , we have that

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ &\dots\dots\dots \\ f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) &\geq 0. \end{aligned}$$

This completes the proof.  $\square$ .

**Corollary 2.1** *Let  $(M, \mu)$  be a planar map geometry,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$  and the angle functions on them are  $f_1, f_2, \dots, f_l$ . Then a family  $\mathcal{L}$  of parallel lines passing through  $C$  is still parallel lines after them leaving  $C$  iff for any  $x, x \geq 0$ ,*

$$\begin{aligned}
& f'_1(x) \geq 0 \\
& f'_{1+}(x) + f'_{2+}(x) \geq 0 \\
& f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) \geq 0 \\
& \dots\dots\dots \\
& f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l-1+}(x) \geq 0 \\
& f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) = 0.
\end{aligned}$$

*Proof* According to the Proposition 2.2, we know the condition is a necessary and sufficient condition for  $\mathcal{L}$  being a parallel bundle. Now since lines in  $\mathcal{L}$  are parallel lines after them leaving  $C$  iff for any  $x \geq 0$  and  $\Delta x \geq 0$ , there must be that

$$f_l(x + \Delta x) + f'_{l-1+}(x)\Delta x + \dots + f'_{1+}(x)\Delta x = f_l(x).$$

Therefore, we get that

$$f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) = 0 \quad \spadesuit$$

When do the parallel lines parallel the initial parallel lines after them passing through a cut  $C$  in a planar map geometry? The answer is in the following result.

**Proposition 2.3** *Let  $(M, \mu)$  be a planar map geometry,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$  and the angle functions on them are  $f_1, f_2, \dots, f_l$ . Then the parallel lines parallel the initial parallel lines after them passing through  $C$  iff for  $\forall x \geq 0$ ,*

$$\begin{aligned}
& f'_1(x) \geq 0 \\
& f'_{1+}(x) + f'_{2+}(x) \geq 0 \\
& f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) \geq 0 \\
& \dots\dots\dots \\
& f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l-1+}(x) \geq 0
\end{aligned}$$

and

$$f_1(x) + f_2(x) + \dots + f_l(x) = l\pi.$$

*Proof* According to the Proposition 2.2 and Corollary 2.1, we know the parallel lines passing through  $C$  is a parallel bundle.

We calculate the angle  $\alpha(i, x)$  of a line  $L$  passing through an edge  $u_iv_i$ ,  $1 \leq i \leq l$  with the line before it meeting  $C$  at the intersection of  $L$  with the edge  $u_iv_i$ , where  $x$  is the distance of the intersection point to  $u_1$  on  $u_1v_1$ , see also the Fig.4. By the

definition, we know the angle  $\alpha(1, x) = f(x)$  and  $\alpha(2, x) = f_2(x) - (\pi - f_1(x)) = f_1(x) + f_2(x) - \pi$ .

Now if  $\alpha(i, x) = f_1(x) + f_2(x) + \cdots + f_i(x) - (i - 1)\pi$ , then similar to the case  $i = 2$ , we know that  $\alpha(i + 1, x) = f_{i+1}(x) - (\pi - \alpha(i, x)) = f_{i+1}(x) + \alpha(i, x) - \pi$ . Whence, we get that

$$\alpha(i + 1, x) = f_1(x) + f_2(x) + \cdots + f_{i+1}(x) - i\pi.$$

Notice that a line  $L$  parallel the initial parallel line after it passing through  $C$  iff  $\alpha(l, x) = \pi$ , i.e.,

$$f_1(x) + f_2(x) + \cdots + f_l(x) = l\pi.$$

This completes the proof.  $\spadesuit$

### 3. Linear condition and combinatorial realization for parallel bundles

For the simplicity, we can assume the function  $f(x)$  is linear and denoted it by  $f_l(x)$ . We can calculate  $f_l(x)$  as follows.

**Proposition 3.1** *The angle function  $f_l(x)$  of a line  $L$  passing through an edge  $uv$  at the point with distance  $x$  to  $u$  is*

$$f_l(x) = \left(1 - \frac{x}{d(uv)}\right) \frac{\rho(u)\mu(v)}{2} + \frac{x}{d(uv)} \frac{\rho(v)\mu(v)}{2},$$

where,  $d(uv)$  is the length of the edge  $uv$ .

*Proof* Since  $f_l(x)$  is linear, we know that  $f_l(x)$  satisfies the following equation.

$$\frac{f_l(x) - \frac{\rho(u)\mu(u)}{2}}{\frac{\rho(v)\mu(v)}{2} - \frac{\rho(u)\mu(u)}{2}} = \frac{x}{d(uv)},$$

Calculation shows that

$$f_l(x) = \left(1 - \frac{x}{d(uv)}\right) \frac{\rho(u)\mu(v)}{2} + \frac{x}{d(uv)} \frac{\rho(v)\mu(v)}{2}. \quad \spadesuit$$

**Corollary 3.1** *Under the linear assumption, a family  $\mathcal{L}$  of parallel lines passing through an edge  $uv$  is a parallel bundle iff*

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}.$$

*Proof* According to the Proposition 2.1, a family of parallel lines passing through an edge  $uv$  is a parallel bundle iff for  $\forall x, x \geq 0, f'(x) \geq 0$ , i.e.,

$$\frac{\rho(v)\mu(v)}{2d(uv)} - \frac{\rho(u)\mu(u)}{2d(uv)} \geq 0.$$

Therefore, a family  $\mathcal{L}$  of parallel lines passing through an edge  $uv$  is a parallel bundle iff

$$\rho(v)\mu(v) \geq \rho(u)\mu(u).$$

Whence,

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}. \quad \spadesuit$$

For a family of parallel lines pass through a cut, we have the following condition for it being a parallel bundle.

**Proposition 3.2** *Let  $(M, \mu)$  be a planar map geometry,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$ . Then under the linear assumption, a family  $L$  of parallel lines passing through  $C$  is a parallel bundle iff the angle factor  $\mu$  satisfies the following linear inequality system*

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2v_2)}$$

.....

$$\begin{aligned} \frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} + \dots + \frac{\rho(v_l)\mu(v_l)}{d(u_lv_l)} \\ \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1, v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2, v_2)} + \dots + \frac{\rho(u_l)\mu(u_l)}{d(u_l, v_l)}. \end{aligned}$$

*Proof* Under the linear assumption, for any integer  $i, i \geq 1$ , we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_iv_i)}$$

by the Proposition 3.1. Whence, according to the Proposition 2.2, we get that a family  $L$  of parallel lines passing through  $C$  is a parallel bundle iff the angle factor  $\mu$  satisfies the following linear inequality system

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2v_2)}$$

.....



$$\begin{aligned} \frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} + \cdots + \frac{\rho(v_l)\mu(v_l)}{d(u_lv_l)} \\ \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1, v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2, v_2)} + \cdots + \frac{\rho(u_l)\mu(u_l)}{d(u_l, v_l)}. \end{aligned}$$

This completes the proof.  $\spadesuit$

For planar maps underlying a regular graph, we have the following interesting results for parallel bundles.

**Corollary 3.2** *Let  $(M, \mu)$  be a planar map geometry with  $M$  underlying a regular graph,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$ . Then under the linear assumption, a family  $L$  of parallel lines passing through  $C$  is a parallel bundle iff the angle factor  $\mu$  satisfies the following linear inequality system*

$$\begin{aligned} \mu(v_1) &\geq \mu(u_1) \\ \frac{\mu(v_1)}{d(u_1v_1)} + \frac{\mu(v_2)}{d(u_2v_2)} &\geq \frac{\mu(u_1)}{d(u_1v_1)} + \frac{\mu(u_2)}{d(u_2v_2)} \\ &\dots\dots\dots \\ \frac{\mu(v_1)}{d(u_1v_1)} + \frac{\mu(v_2)}{d(u_2v_2)} + \cdots + \frac{\mu(v_l)}{d(u_lv_l)} &\geq \frac{\mu(u_1)}{d(u_1v_1)} + \frac{\mu(u_2)}{d(u_2v_2)} + \cdots + \frac{\mu(u_l)}{d(u_lv_l)} \end{aligned}$$

and particularly, if assume that all the lengths of edges in  $C$  are the same, then

$$\begin{aligned} \mu(v_1) &\geq \mu(u_1) \\ \mu(v_1) + \mu(v_2) &\geq \mu(u_1) + \mu(u_2) \\ &\dots\dots\dots \\ \mu(v_1) + \mu(v_2) + \cdots + \mu(v_l) &\geq \mu(u_1) + \mu(u_2) + \cdots + \mu(u_l). \end{aligned}$$

Certainly, by choosing different angle factors, we can also get combinatorial conditions for existing parallel bundles under the linear assumption.

**Proposition 3.3** *Let  $(M, \mu)$  be a planar map geometry,  $C = \{u_1v_1, u_2v_2, \dots, u_lv_l\}$  a cut of the map  $M$  with order  $u_1v_1, u_2v_2, \dots, u_lv_l$  from the left to the right,  $l \geq 1$ . If for any integer  $i, i \geq 1$ ,*

$$\frac{\rho(u_i)}{\rho(v_i)} \leq \frac{\mu(v_i)}{\mu(u_i)},$$

then under the linear assumption, a family  $L$  of parallel lines passing through  $C$  is a parallel bundle.

*Proof* Notice that under the linear assumption, for any integer  $i, i \geq 1$ , we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_iv_i)}$$

by the Proposition 3.1. Whence,  $f'_{i+}(x) \geq 0$  for  $i = 1, 2, \dots, l$ . Therefore, we get that

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ &\dots\dots\dots \\ f'_{1+}(x) + f'_{2+}(x) + \dots + f'_{l+}(x) &\geq 0. \end{aligned}$$

By the Proposition 2.2, we know that a family  $L$  of parallel lines passing through  $C$  is a parallel bundle.  $\square$

#### 4. Classification of parallel bundles

For a cut  $C$  in a planar map geometry and  $e \in C$ , denote by  $f_e(x)$  the angle function on the edge  $e$ ,  $f(C, x) = \sum_{e \in C} f_e(x)$ . If  $f(C, x)$  is independent on  $x$ , then we abbreviate it to  $f(C)$ . According to the results in the Section 2 and 3, we can classify the parallel bundles with a given orientation  $\vec{O}$  in planar map geometries into the following 15 classes, where, each class is labelled by a 4-tuple 0, 1 code.

##### Classification of parallel bundles

- (1)  $\mathcal{C}_{1000}$ : for any cut  $C$  along  $\vec{O}$ ,  $f(C) = |C|\pi$ ;
- (2)  $\mathcal{C}_{0100}$ : for any cut  $C$  along  $\vec{O}$ ,  $f(C) < |C|\pi$ ;
- (3)  $\mathcal{C}_{0010}$ : for any cut  $C$  along  $\vec{O}$ ,  $f(C) > |C|\pi$ ;
- (4)  $\mathcal{C}_{0001}$ : for any cut  $C$  along  $\vec{O}$ ,  $f'_+(C, x) > 0$  for  $\forall x, x \geq 0$ ;
- (5)  $\mathcal{C}_{1100}$ : There exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$  and  $f(C_2) = c < |C_2|\pi$ ;
- (6)  $\mathcal{C}_{1010}$ : there exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$  and  $f(C_2) > |C_2|\pi$ ;
- (7)  $\mathcal{C}_{1001}$ : there exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$  and  $f'_+(C_2, x) > 0$  for  $\forall x, x \geq 0$ ;
- (8)  $\mathcal{C}_{0110}$ : there exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) < |C_1|\pi$  and  $f(C_2) > |C_2|\pi$ ;

- (9)  $\mathcal{C}_{0101}$ : there exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) < |C_1|\pi$  and  $f'_+(C_2, x) > 0$  for  $\forall x, x \geq 0$ ;
- (10)  $\mathcal{C}_{0011}$ : there exist cuts  $C_1, C_2$  along  $\vec{O}$ , such that  $f(C_1) > |C_1|\pi$  and  $f'_+(C_2, x) > 0$  for  $\forall x, x \geq 0$ ;
- (11)  $\mathcal{C}_{1110}$ : there exist cuts  $C_1, C_2$  and  $C_3$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$ ,  $f(C_2) < |C_2|\pi$  and  $f(C_3) > |C_3|\pi$ ;
- (12)  $\mathcal{C}_{1101}$ : there exist cuts  $C_1, C_2$  and  $C_3$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$ ,  $f(C_2) < |C_2|\pi$  and  $f'_+(C_3, x) > 0$  for  $\forall x, x \geq 0$ ;
- (13)  $\mathcal{C}_{1011}$ : there exist cuts  $C_1, C_2$  and  $C_3$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$ ,  $f(C_2) > |C_2|\pi$  and  $f'_+(C_1, x) > 0$  for  $\forall x, x \geq 0$ ;
- (14)  $\mathcal{C}_{0111}$ : there exist cuts  $C_1, C_2$  and  $C_3$  along  $\vec{O}$ , such that  $f(C_1) < |C_1|\pi$ ,  $f(C_2) > |C_2|\pi$  and  $f'_+(C_1, x) > 0$  for  $\forall x, x \geq 0$ ;
- (15)  $\mathcal{C}_{1111}$ : there exist cuts  $C_1, C_2, C_3$  and  $C_4$  along  $\vec{O}$ , such that  $f(C_1) = |C_1|\pi$ ,  $f(C_2) < |C_2|\pi$ ,  $f(C_3) > |C_3|\pi$  and  $f'_+(C_4, x) > 0$  for  $\forall x, x \geq 0$ .

Notice that only the first three classes may be parallel lines after them passing through the cut  $C$ . All of the other classes are only parallel bundles, not parallel lines in the usual meaning.

**Proposition 4.1** *For an orientation  $\vec{O}$ , the 15 classes  $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$  are all the parallel bundles in planar map geometries.*

*Proof* Not loss of generality, we assume  $C_1, C_2, \dots, C_m, m \geq 1$ , are all the cuts along  $\vec{O}$  in a planar map geometry  $(M, \mu)$  from the upon side of  $\vec{O}$  to its down side. We find their structural characters for each case in the following discussion.

$\mathcal{C}_{1000}$ : By the Proposition 2.3, a family  $\mathcal{L}$  of parallel lines parallel their initial lines before meeting  $M$  after the passing through  $M$ .

$\mathcal{C}_{0100}$ : By the definition, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if

$$f(C_1) \leq f(C_2) \leq \dots \leq f(C_m) < \pi.$$

Otherwise, some lines in  $\mathcal{L}$  will intersect. According to the Corollary 2.1, they parallel each other after they passing through  $M$  only if

$$f(C_1) = f(C_2) = \dots = f(C_m) < \pi.$$

$\mathcal{C}_{0010}$ : Similar to the case  $\mathcal{C}_{0100}$ , a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if

$$\pi < f(C_1) \leq f(C_2) \leq \dots \leq f(C_m)$$

and parallel each other after they passing through  $M$  only if

$$\pi < f(C_1) = f(C_2) = \dots = f(C_m).$$

$\mathcal{C}_{0001}$ : Notice that by the proof of the Proposition 2.3, a line has angle  $f(C, x) - (|C| - 1)\pi$  after it passing through  $C$  with the initial line before meeting  $C$ . In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if for  $\forall x_i, x_i \geq 0, 1 \leq i \leq m$ ,

$$f(C_1, x_1) \leq f(C_2, x_2) \leq \cdots \leq f(C_m, x_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{1100}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, 2 \leq k \leq m$ , such that

$$f(C_1) \leq f(C_2) \leq \cdots \leq f(C_{k-1}) < f(C_k) = f(C_{k+1}) = \cdots = f(C_m) = \pi.$$

Otherwise, they will intersect.

$\mathcal{C}_{1010}$ : Similar to the case  $\mathcal{C}_{1100}$ , in this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, 2 \leq k \leq m$ , such that

$$\pi = f(C_1) = f(C_2) = \cdots = f(C_k) < f(C_{k+1}) \leq \cdots \leq f(C_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{1001}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, l, 1 \leq k < l \leq m$ , such that for  $\forall x_i, x_i \geq 0, 1 \leq i \leq k$  or  $l \leq i \leq m$ ,

$$\begin{aligned} f(C_1, x_1) &\leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < f(C_{k+1}) \\ &= f(C_{k+2}) = \cdots = f(C_{l-1}) = \pi < f(C_l, x_l) \leq \cdots \leq f(C_m, x_m). \end{aligned}$$

Otherwise, they will intersect.

$\mathcal{C}_{0110}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is integers  $k, 1 \leq k < m$ , such that

$$f(C_1) \leq f(C_2) \leq \cdots \leq f(C_k) < \pi < f(C_{k+1}) \leq \cdots \leq f(C_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{0101}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is integers  $k, 1 \leq k \leq m$ , such that for  $\forall x_i, x_i \geq 0, 1 \leq i \leq m$ ,

$$f(C_1, x_1) \leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < \pi \leq f(C_{k+1}, x_{k+1}) \leq \cdots \leq f(C_m, x_m),$$

and there must be a constant in  $f(C_1, x_1), f(C_2, x_2), \cdots, f(C_k, x_k)$ .

$\mathcal{C}_{0011}$ : In this case, the situation is similar to the case  $\mathcal{C}_{0101}$  and there must be a constant in  $f(C_{k+1}, x_{k+1}), f(C_{k+2}, x_{k+2}), \cdots, f(C_m, x_m)$ .

$\mathcal{C}_{1110}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, l, 1 \leq k < l \leq m$ , such that

$$\begin{aligned} f(C_1) &\leq f(C_2) \leq \cdots \leq f(C_k) < f(C_{k+1}) \\ &= \cdots = f(C_{l-1}) = \pi < f(C_l) \leq \cdots \leq f(C_m). \end{aligned}$$

Otherwise, they will intersect.

$\mathcal{C}_{1101}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, l, 1 \leq k < l \leq m$ , such that for  $\forall x_i, x_i \geq 0, 1 \leq i \leq k$  or  $l \leq i \leq m$ ,

$$\begin{aligned} f(C_1, x_1) &\leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < f(C_{k+1}) \\ &= \cdots = f(C_{l-1}) = \pi < f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in  $f(C_1, x_1), f(C_2, x_2), \dots, f(C_k, x_k)$ . Otherwise, they will intersect.

$\mathcal{C}_{1011}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, l, 1 \leq k < l \leq m$ , such that for  $\forall x_i, x_i \geq 0, 1 \leq i \leq k$  or  $l \leq i \leq m$ ,

$$\begin{aligned} f(C_1, x_1) &\leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < f(C_{k+1}) \\ &= \cdots = f(C_{l-1}) = \pi < f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in  $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \dots, f(C_m, x_m)$ . Otherwise, they will intersect.

$\mathcal{C}_{0111}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, 1 \leq k \leq m$ , such that for  $\forall x_i, x_i \geq 0$ ,

$$f(C_1, x_1) \leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < \pi < f(C_l, x_l) \leq \cdots \leq f(C_m, x_m)$$

and there must be a constant in  $f(C_1, x_1), f(C_2, x_2), \dots, f(C_k, x_k)$  and a constant in  $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \dots, f(C_m, x_m)$ . Otherwise, they will intersect.

$\mathcal{C}_{1111}$ : In this case, a family  $\mathcal{L}$  of parallel lines is a parallel bundle along  $\vec{O}$  only if there is an integer  $k, l, 1 \leq k < l \leq m$ , such that for  $\forall x_i, x_i \geq 0, 1 \leq i \leq k$  or  $l \leq i \leq m$ ,

$$\begin{aligned} f(C_1, x_1) &\leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) < f(C_{k+1}) \\ &= \cdots = f(C_{l-1}) = \pi < f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in  $f(C_1, x_1), f(C_2, x_2), \dots, f(C_k, x_k)$  and a constant in  $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \dots, f(C_m, x_m)$ . Otherwise, they will intersect.

Following the structural characters of the classes  $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$ , by the Proposition 2.2, 2.3 and Proposition 3.1, we know that any parallel bundle is in one of the classes  $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$  and each class in  $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$  is non-empty. This completes the proof.  $\square$

A example of parallel bundle in a planar map geometry is shown in the Fig.5, in where the number on a vertex  $u$  denotes the number  $\rho(u)\mu(u)$ .

**Fig.5**

## 5. Generalization

All the planar map geometries considered in this paper are without boundary. For planar map geometries with boundary, i.e., some faces are deleted ([10]), which are correspondence with the maps with boundary ([2]). We know that they are the Smarandache *non-geometries*, satisfying one or more of the following conditions:

(A1<sup>-</sup>) *It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

(A2<sup>-</sup>) *It is not always possible to extend by continuity a finite line to an infinite line.*

(A3<sup>-</sup>) *It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

(A4<sup>-</sup>) *not all the right angles are congruent.*

(A5<sup>-</sup>) *if a line, cutting two other lines, forms the interior angles of the same side*

of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.

Notice that for an one face planar map geometry  $(M, \mu)^{-1}$  with boundary, if we choose all points being euclidean, then  $(M, \mu)^{-1}$  is just the Poincaré's model for the hyperbolic geometry.

Using the neutrosophic logic idea, we can also define the conception of *neutrosophic surface* as follow, comparing also with the surfaces in [8] and [14].

**Definition 5.1** *A neutrosophic surface is a Hausdorff, connected, topological space  $S$  such that every point  $v$  is elleptic, euclidean, or hyperbolic.*

For this kind of surface, we present the following problem for the further researching.

**Problem 5.1** *To determine the behaviors of elements, such as, the line, angle, area,  $\dots$ , in neutrosophic surfaces.*

Notice that results in this paper are just the behaviors of line bundles in a neutrosophic plane.

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