Multi-criteria Decision Making Method Based on Cross-entropy with Interval Neutrosophic Sets

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Abstract: The main purpose of this paper is to provide a method of multi-criteria decision making that combines interval neutrosophic sets and TOPSIS involving the relative likelihood-based comparison relations of the performances of alternatives that are aggregated into interval numbers. A fuzzy cross-entropy approach is proposed to state the discrimination measure between alternatives and the absolute ideal solutions after a transformation operator has been developed to convert interval neutrosophic numbers to simplified neutrosophic numbers. Finally, an illustrative example is given and a comparison analysis is conducted between the proposed approach and other existing methods to verify the feasibility and effectiveness of the developed approach.

Keywords: multi-criteria decision making; interval neutrosophic sets; cross-entropy; TOPSIS

1. Introduction

In practice, multi-criteria decision making (MCDM) methods are widely used to rank alternatives or select the optimal one with respect to several concerned criteria. However, in some cases, it is difficult for decision-makers to explicitly express preference in solving MCDM problems with uncertain or incomplete information. Under these circumstances, fuzzy sets (FSs), proposed by Zadeh [1], where each element has a membership degree represented by a real number in the standard interval $[0,1]$, are regarded as a significant tool for solving MCDM problems [2,3], fuzzy logic and approximate reasoning [4]. Sometimes, FSs cannot handle the cases where the membership degree is uncertain and hard to be defined by a crisp value. therefore, the concept of interval-valued fuzzy sets (IVFSs) was proposed [5] to capture the uncertainty of membership degree. Generally, if the membership degree is defined, then the non-membership degree can be calculated by default. In order to deal with the uncertainty of non-membership degree, Atanassov [6] introduced intuitionistic fuzzy sets (IFSs) which is an extension of Zadeh's FSs, and the corresponding intuitionistic fuzzy logic [7] was proposed. IFSs consider both the membership degree and the non-membership degree simultaneously. So IFSs and intuitionistic fuzzy logic are more flexible in handling information containing uncertainty and incompleteness than traditional FSs. Currently, IFSs have been widely applied in solving MCDM problems [8- 16]. Moreover, in reality, the degree of membership and non-membership in IFSs may be expressed as interval numbers instead of specific numbers. Hence, interval-valued intuitionistic fuzzy sets (IVIFSs) [17] were proposed, which is an extension of FSs and IVFSs. In recent years, MCDM problems with evaluation information derived from IVFSs have attracted much attention of researchers [18- 24], in which averaging operators, aggregation operators, prospect score function and possibility degree method are involved. Furthermore, TOPSIS, proposed by Hwang and Yoon [25], was also used for MCDM problems under IVIFSs environment in [26,27].

Although FSs and IFSs theory have been developed and generalized, they cannot deal with all sorts of uncertainties in real problems. Some types of uncertainties, such as the indeterminate and inconsistent information, cannot be handled. For example, a paper is sent to a reviewer, he or she says it is 70% acceptable, 60% unacceptable and the statement is 20% uncertain. This issue cannot be handled effectively with FSs and IFSs. Therefore, some new theories are required.

Smarandache [28,29] coined the concept of neutrosophic logic and neutrosophic sets (NSs). Rivieccio [30] pointed out that a NS is a set where each element of the universe has a truth-membership, indeterminacy-membership and falsity-membership, respectively, and it lies in the non-standard unit interval $]0^{-}, 1^{+}[$. The uncertainty presented here, i.e. indeterminacy-membership, is quantified and independent of truth-membership and falsity-membership, which is different from the argument that the incorporated uncertainty is dependent on the degree of membership and non-membership of IFSs [31]. Thus, it is practical and flexible to address problems containing uncertain, incomplete and inconsistent information with NSs. For the aforementioned example, it can be presented as $x(0.7, 0.2, 0.6)$ by means of NSs. In recent years, the NS theory has found practical applications in various fields, such as relational database systems, semantic web services [32], mineral prospectivity prediction [33], image processing [34-36], granular computing [37], medical diagnosis [38] and information fusion [39]. It is clear that all of those proposals, promising as they are, still need to be refined from a formal point of view. Without specific description, it is hard to apply NSs in real scientific and engineering situations. Hence, the notion of single-valued neutrosophic sets (SVNSs) was introduced by Wang et al. [40], which is an instance of NSs. Majumdar et al. [31] introduced similarity and entropy measures of SVNSs. Furthermore, the correlation and correlation coefficient of SVNSs [41] as well as the cross-entropy of SVNSs [42] were presented. In addition, Ye [43] also introduced the concept of simplified neutrosophic sets (SNSs), which can be described by three real numbers in the real unit interval [0,1]. Peng et al. [44] defined the novel operations and aggregation operators of SNSs and applied them in solving multi-criteria group decision making problems.

Actually, the degree of truth, falsity and indeterminacy about a certain statement are denoted by several possible interval numbers instead of three real numbers in SNSs in some real situations. For example, an expert is asked to give the opinion about a certain statement, he or she may provide his or her own evaluations based on surveys as well as his or her knowledge and experience, and the evaluations are gathered in the form of interval numbers. The possibility that the statement is true is between 0.5 and 0.7 and that the statement is false is between 0.2 and 0.4 and the degree that he or she is not sure is between 0.1 and 0.3. That is beyond the scope of SNSs. So the concept of interval neutrosophic sets (INSs) was proposed by Wang et al. [32], which is a particular instance of NSs. For the aforementioned example, it can be presented as $x([0.5,0.7],[0.1,0.3],[0.2,0.4])$ by means of INSs. Subsequently, the studies of INSs had been conducted in various aspects, which concentrated mainly on defining and amending operational laws [45,46], correlation coefficients [47], similarity measures [48] and distances [49] to aggregate opinions of experts or decision-makers in MCDM problems, where the opinions of experts are expressed by INSs. Nevertheless, it is complex to directly process the assessment information with INSs. Furthermore, most of current studies coping with MCDM problems with INSs by means of aggregating interval numbers into real numbers, which may bring some drawbacks in operations and lead to great information loss [27,50].

As an important topic in the theory of FSs, entropy measure of FSs mentioned by Zadeh [1], is an useful tool to measure uncertain information. Burillo and Bustince [51] introduced the notion that entropy of IFSs and IVIFSs can be used to assess the degree of intuition of an IFS or IVIFS. In the course of determining criterion weights under intuitionistic fuzzy environment, Chen and Li [52] utilized the intuitionistic fuzzy entropy measure to estimate the objective criteria weights. Despite the existing research effort, dealing with MCDM problems with completely unknown criteria weights in the framework of IVIFS remains an open problem [53,54]. The starting point for the cross-entropy method is information theory as developed by Shannon [55]. Cross-entropy is applied to measure the discrimination information, according to Shannon's inequality [56]. Then Shang and Jiang [57] proposed a fuzzy cross-entropy and a symmetric discrimination information measure between two FSs. Furthermore, the concepts of intuitionistic fuzzy cross-entropy and discrimination for IFSs were introduced by Vlachos and Sergiadis [58]. A fuzzy cross-entropy on interval-valued intuitionistic fuzzy numbers was proposed by analogy with the intuitionistic fuzzy cross-entropy [59]. A series of mathematical programming models based on cross-entropy were constructed to determine the weights of criteria under IVIFSs environment [27]. Other applications with the fuzzy cross-entropy include portfolio selection [60], the divergence of uncertain variables [61], MCDM problems [62-64] and so on.

However, compared with fuzzy cross-entropy that is widely applied in FSs and IFSs, there exists few studies on the application of entropy theory under interval neutrosophic environment and attention paid to avoid information loss is not enough in the process of information aggregation. What's more, TOPSIS which is one of popular decision making methods has been applied effectively and availably in practice to address problems in various fields. For this purpose, a MCDM method on the basis of cross-entropy is proposed, in which a transformation operator is defined to convert each interval neutrosophic number (INN) to a simplified neutrosophic number (SNN). It can effectively avoid artificially setting parameter values of the transformation operator in IVIFSs and IFSs [54,65] and to some degree, lower the computational complexity in directly processing evaluation information with INSs at the same time. In addition, a TOPSIS method associated with possibility degree method is proposed to rank the performances of alternatives in the form of interval numbers, which can decrease the loss of assessment information and guarantee the reasonability of the ultimate ranking results.

The rest of the paper is organized as follows. In Section 2, interval numbers as well as the concepts and operations of NSs, SNSs and INSs are briefly reviewed. In Section 3, an operator is put forward, which can transform each INN into a SNN, together with a cross-entropy for SNNs and some useful properties. Subsequently, TOPSIS for MCDM problems with INSs is developed in Section 4. In Section 5, an illustrative example is given and a comparison analysis is conducted between the proposed approach and other existing methods to verify the feasibility and effectiveness of the developed approach. Some summary remarks are given in Section 6.

2. Preliminaries

In this section, some basic concepts and definitions related to INSs, including interval numbers, definitions and operational laws of NSs, SNSs and INSs are introduced, which will be utilized in the latter analysis.

2.1 Interval numbers

Interval numbers and their operations are of utmost significance to explore the operations for INSs. In the following, some definitions and operational laws of interval numbers are given.

Definition 1 [66,67]. Let $\tilde{a}=[a^L,a^U]=\{x\,|\,a^L\le x\le a^U\}$, then $\tilde{a}$ is called an interval number. Especially, $\tilde{a}=[a^L,a^U]$ will be degenerated to a real number if $a^L=a^U$.

Consider any two nonnegative interval numbers $\tilde{a}=[a^L,a^U]$ and $\tilde{b}=[b^L,b^U]$, where $0\le a^L\le x\le a^U$, $0\le b^L\le x\le b^U$. Then their operations are defined as follows [66]:

(1) $\tilde{a}=\tilde{b}\Leftrightarrow a^L=b^L, a^U=b^U$;

(2) $\tilde{a}+\tilde{b}=[a^L+b^L, a^U+b^U]$;

(3) $\tilde{a}-\tilde{b}=[a^L-b^U, a^U-b^L]$;

(4) $\lambda\tilde{a}=[\lambda a^L,\lambda a^U], \lambda>0.$

Definition 2 [68,69]**.** For any two interval numbers $\tilde{a}=[a^L,a^U]$ and $\tilde{b}=[b^L,b^U]$, the possibility of $\tilde{a}\ge\tilde{b}$ is formulated by $p(\tilde{a}\ge\tilde{b})=\max\left\{1-\max\left\{\frac{b^U-a^L}{L(\tilde{a})+L(\tilde{b})},0\right\},0\right\}$, where $L(\tilde{a})=a^U-a^L$ and $L(\tilde{b})=b^U-b^L$.

The possibility degree of $\tilde{a}\ge\tilde{b}$ has the following properties [68]:

(1) $0\le p(\tilde{a}\ge\tilde{b})\le 1$;

(2) $p(\tilde{a}\ge\tilde{b})=p(\tilde{b}\ge\tilde{a})=0.5$ if $p(\tilde{a}\ge\tilde{b})=p(\tilde{b}\ge\tilde{a})$;

(3) $p(\tilde{a}\ge\tilde{b})+p(\tilde{b}\ge\tilde{a})=1$;

(4) $p(\tilde{a}\ge\tilde{b})=0$ if $a^U\le b^L$, $p(\tilde{a}\ge\tilde{b})=1$ if $a^L\ge b^U$;

(5) For any interval numbers $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, $p(\tilde{a} \geq \tilde{c}) \geq 0.5$ if $p(\tilde{a} \geq \tilde{b}) \geq 0.5$ and $p(\tilde{b} \geq \tilde{c}) \geq 0.5$; $p(\tilde{a} \geq \tilde{c}) = 0.5$ if and only if $p(\tilde{a} \geq \tilde{b}) = p(\tilde{b} \geq \tilde{c}) = 0.5$.

2.2 NSs and SNSs

Definition 3 [28]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in X is characterized by a truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. $t_A(x)$, $i_A(x)$ and $f_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is, $t_A(x): X \to]0^-, 1^+[$, $i_A(x): X \to]0^-, 1^+[$, and $f_A(x): X \to]0^-, 1^+[$. There is no restriction on the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$, so $0^- \leq \sup t_A(x) + \sup i_A(x) + \sup f_A(x) \leq 3^+$.

Definition 4 [28]. A NS A is contained in the other NS B, denoted by $A \subseteq B$ if and only if $\inf t_A(x) \leq \inf t_B(x)$, $\sup t_A(x) \leq \sup t_B(x)$, $\inf i_A(x) \geq \inf i_B(x)$, $\sup i_A(x) \geq \sup i_B(x)$, $\inf f_A(x) \geq \inf f_B(x)$ and $\sup f_A(x) \geq \sup f_B(x)$, for any $x \in X$.

Since it is hard to apply NSs to practical problems, Ye [43] reduced NSs of nonstandard interval numbers into a kind of SNSs of standard interval numbers.

Definition 5 [30,43]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in X is characterized by $t_A(x)$, $i_A(x)$ and $f_A(x)$, which are single subintervals/subsets in the real standard $[0,1]$, that is, $t_A(x): X \to [0,1]$, $i_A(x): X \to [0,1]$, and $f_A(x): X \to [0,1]$. And the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$ satisfies the condition $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$. Then a simplification of A is denoted by $A = \{(x, t_A(x), i_A(x), f_A(x)) | x \in X\}$, which is called a SNS. It is a subclass of NSs. If $\|X\| = 1$, a SNS will be degenerated to a SNN, denoted by $A = (t_A, i_A, f_A)$.

Definition 6 [30,43]. A SNS A is contained in the other SNS B, denoted by $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$, $i_A(x) \geq i_B(x)$ and $f_A(x) \geq f_B(x)$, for any $x \in X$. Especially, $A = B$ if $A \subseteq B$ and $B \subseteq A$.

The complement set of A, denoted by A^c is defined as $A^c = \{(x, f_A(x), i_A(x), t_A(x)) | x \in X\}$. If A is a SNN, then $A^c = \{(f_A, i_A, t_A)\}$.

2.3 INSs

In the actual applications, sometimes, it is not easy to express the truth-membership, indeterminacy-membership and falsity-membership by crisp values, and they may be easily described by interval numbers. Wang et al. [32] further defined INSs.

Definition 7 [30,32]. Let X be a space of points (objects) with generic elements in X, denoted by x. An INS $\tilde{A}$ in X is characterized by a truth-membership function $t_{\tilde{A}}(x)$, an indeterminacy-membership function $i_{\tilde{A}}(x)$ and a falsity-membership function $f_{\tilde{A}}(x)$. For each point x in X, we have that $t_{\tilde{A}}(x) = [t_{\tilde{A}}^L(x), t_{\tilde{A}}^U(x)]$, $i_{\tilde{A}}(x) = [i_{\tilde{A}}^L(x), i_{\tilde{A}}^U(x)]$, $f_{\tilde{A}}(x) = [f_{\tilde{A}}^L(x), f_{\tilde{A}}^U(x)] \subseteq [0,1]$, and $0 \le t_{\tilde{A}}^U(x) + i_{\tilde{A}}^U(x) + f_{\tilde{A}}^U(x) \le 3$. We only consider the subunitary interval of $[0,1]$. An INS is the subclass of a NS. Therefore, all INSs are clearly NSs. Especially, an INS will be reduced to a SNS if $t_{\tilde{A}}^L(x) = t_{\tilde{A}}^U(x)$, $i_{\tilde{A}}^L(x) = i_{\tilde{A}}^U(x)$ and $f_{\tilde{A}}^L(x) = f_{\tilde{A}}^U(x)$. In addition, if $\|X\|=1$, an INS will be degenerated to an INN, denoted by $\tilde{A} = \left([t_{\tilde{A}}^L, t_{\tilde{A}}^U], [i_{\tilde{A}}^L, i_{\tilde{A}}^U], [f_{\tilde{A}}^L, f_{\tilde{A}}^U]\right)$.

Definition 8 [30,32]. An INS $\tilde{A}$ is contained in the other INS $\tilde{B}$, denoted by $\tilde{A} \subseteq \tilde{B}$ if and only if $t_{\tilde{A}}^L(x) \le t_{\tilde{B}}^L(x)$, $t_{\tilde{A}}^U(x) \le t_{\tilde{B}}^U(x)$, $i_{\tilde{A}}^L(x) \ge i_{\tilde{B}}^L(x)$, $i_{\tilde{A}}^U(x) \ge i_{\tilde{B}}^U(x)$, $f_{\tilde{A}}^L(x) \ge f_{\tilde{B}}^L(x)$ and $f_{\tilde{A}}^U(x) \ge f_{\tilde{B}}^U(x)$, for any $x \in X$. Especially, $\tilde{A} = \tilde{B}$ if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$. The complement set of $\tilde{A}$, denoted by $\tilde{A}^c$ is defined as $\tilde{A}^c = \left\{\left(x, [f_{\tilde{A}}^L(x), f_{\tilde{A}}^U(x)], [i_{\tilde{A}}^L(x), i_{\tilde{A}}^U(x)], [t_{\tilde{A}}^L(x), t_{\tilde{A}}^U(x)]\right) | x \in X\right\}$. If $\tilde{A}$ is an INN, then $\tilde{A}^c = \left\{\left([f_{\tilde{A}}^L, f_{\tilde{A}}^U], [i_{\tilde{A}}^L, i_{\tilde{A}}^U], [t_{\tilde{A}}^L, t_{\tilde{A}}^U]\right)\right\}$.

Definition 9 [46]. Let $\tilde{A} = \left([t_{\tilde{A}}^L, t_{\tilde{A}}^U], [i_{\tilde{A}}^L, i_{\tilde{A}}^U], [f_{\tilde{A}}^L, f_{\tilde{A}}^U]\right)$ and $\tilde{B} = \left([t_{\tilde{B}}^L, t_{\tilde{B}}^U], [i_{\tilde{B}}^L, i_{\tilde{B}}^U], [f_{\tilde{B}}^L, f_{\tilde{B}}^U]\right)$ be two INNs, where $\lambda > 0$. The operations for INNs are defined as below.

(1) $\lambda \cdot \tilde{A} = \left([1-(1-t_{\tilde{A}}^{L})^{\lambda}, 1-(1-t_{\tilde{A}}^{U})^{\lambda}], [(i_{\tilde{A}}^{L})^{\lambda}, (i_{\tilde{A}}^{U})^{\lambda}], [(f_{\tilde{A}}^{L})^{\lambda}, (f_{\tilde{A}}^{U})^{\lambda}]\right)$;

(2) $\tilde{A}^{\lambda} = \left([(t_{\tilde{A}}^{L})^{\lambda}, (t_{\tilde{A}}^{U})^{\lambda}], [1-(1-i_{\tilde{A}}^{L})^{\lambda}, 1-(1-i_{\tilde{A}}^{U})^{\lambda}], [1-(1-f_{\tilde{A}}^{L})^{\lambda}, 1-(1-f_{\tilde{A}}^{U})^{\lambda}]\right)$;

(3) $\tilde{A} + \tilde{B} = \left([t_{\tilde{A}}^{L} + t_{\tilde{B}}^{L} - t_{\tilde{A}}^{L} \cdot t_{\tilde{B}}^{L}, t_{\tilde{A}}^{U} + t_{\tilde{B}}^{U} - t_{\tilde{A}}^{U} \cdot t_{\tilde{B}}^{U}], [i_{\tilde{A}}^{L} \cdot i_{\tilde{B}}^{L}, i_{\tilde{A}}^{U} \cdot i_{\tilde{B}}^{U}], [f_{\tilde{A}}^{L} \cdot f_{\tilde{B}}^{L}, f_{\tilde{A}}^{U} \cdot f_{\tilde{B}}^{U}]\right)$;

(4) $\tilde{A} \cdot \tilde{B} = \left([t_{\tilde{A}}^{L} \cdot t_{\tilde{B}}^{L}, t_{\tilde{A}}^{U} \cdot t_{\tilde{B}}^{U}], [i_{\tilde{A}}^{L} + i_{\tilde{B}}^{L} - i_{\tilde{A}}^{L} \cdot i_{\tilde{B}}^{L}, i_{\tilde{A}}^{U} + i_{\tilde{B}}^{U} - i_{\tilde{A}}^{U} \cdot i_{\tilde{B}}^{U}], [f_{\tilde{A}}^{L} + f_{\tilde{B}}^{L} - f_{\tilde{A}}^{L} \cdot f_{\tilde{B}}^{L}, f_{\tilde{A}}^{U} + f_{\tilde{B}}^{U} - f_{\tilde{A}}^{U} \cdot f_{\tilde{B}}^{U}]\right)$.

Example 1. Assume two INNs $\tilde{A} = ([0.7, 0.8], [0, 0.1], [0.1, 0.2])$ and $\tilde{B} = ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])$, and $\lambda = 2$. Then the following results can be calculated.

(1) $2 \cdot \tilde{A} = ([0.91, 0.96], [0, 0.01], [0.01, 0.04])$;

(2) $\tilde{A}^{2} = ([0.49, 0.64], [0, 0.19], [0.19, 0.36])$;

(3) $\tilde{A} + \tilde{B} = ([0.82, 0.90], [0, 0.05], [0.03, 0.08])$;

(4) $\tilde{A} \cdot \tilde{B} = ([0.28, 0.40], [0.20, 0.37], [0.37, 0.52])$.

Bustince and Burillo [70] put forward an operator $H_{p,q}$, which can transform each IVFS into an IFS. As an improved extension of $H_{p,q}$, an operator $H_{p,q,r}$ is defined to convert each INN to a SNN.

Definition 10. Let $p, q, r \in [0,1]$ be three fixed numbers, any INN can be transformed into a SNN through the operator $H_{p,q,r}$.

$$H_{p,q,r}(\tilde{A}) = \left(t_{\tilde{A}}^{L} + p \cdot W_{t_{\tilde{A}}}, i_{\tilde{A}}^{L} + q \cdot W_{i_{\tilde{A}}}, f_{\tilde{A}}^{L} + r \cdot W_{f_{\tilde{A}}}\right),$$

where $W_{t_{\tilde{A}}} = t_{\tilde{A}}^{U} - t_{\tilde{A}}^{L}$, $W_{i_{\tilde{A}}} = i_{\tilde{A}}^{U} - i_{\tilde{A}}^{L}$ and $W_{f_{\tilde{A}}} = f_{\tilde{A}}^{U} - f_{\tilde{A}}^{L}$. Obviously, $H_{p,q,r}(\tilde{A})$ is a SNN and can be determined with respect to p, q and r. That is, $H_{p,q,r}(\tilde{A})$ is well defined in all value ranges of p, q and r.

Example 2. Use the data of Example 1. Then the following transformation results can be obtained utilizing the operator $H_{p,q,r}$.

(1) $H_{p,q,r}(\tilde{A}) = (0.7 + p \cdot 0.1, 0 + q \cdot 0.1, 0.1 + r \cdot 0.1)$;

(2) $H_{p,q,r}(\tilde{B}) = (0.4 + p \cdot 0.1, 0.2 + q \cdot 0.1, 0.3 + r \cdot 0.1)$.

It is shown that $H_{p,q,r}(\tilde{A})$ and $H_{p,q,r}(\tilde{B})$ will be two specific SNNs if the values of p, q and r are given. And the method to determine the parameter values will be discussed in detail in Section 4.

3. Cross-entropy for SNNs

In an analogous manner to the proposals of Ye [42] and Vlachos and Sergiadis [58], the following definition of fuzzy cross-entropy for SNNs is proposed.

Definition 11. Let A and B be two SNNs. Then the cross-entropy between A and B can be defined as:

$$I_{NS}(A,B) = t_A \ln\frac{2\cdot t_A}{t_A + t_B} + i_A \ln\frac{2\cdot i_A}{i_A + i_B} + f_A \ln\frac{2\cdot f_A}{f_A + f_B}. \tag{1}$$

Eq. (1) can indicate the degree of discrimination of A from B. It is obvious that $I_{NS}(A,B)$ is not symmetric with respect to its arguments. Therefore, a modified symmetric discrimination information measure based on $I_{NS}(A,B)$ can be defined as:

$$D_{NS}(A,B) = I_{NS}(A,B) + I_{NS}(B,A). \tag{2}$$

The larger $D_{NS}(A,B)$ is, the larger the difference between A and B will be, and vice versa.

Property 1. Let A and B be two SNNs. Define the degree of discrimination of A from B as $D_{NS}(A,B)$. Then the following properties hold.

(1) $D_{NS}(A,B) = D_{NS}(B,A)$,

(2) $D_{NS}(A,B) = D_{NS}(A^c,B^c)$, where A^c and B^c are the complement sets of A and B, respectively, defined in Definition 6.

(3) $D_{NS}(A,B) \ge 0$ and $D_{NS}(A,B) = 0$ if and only if $A = B$.

Proof. Obviously, it can be easily verified that (1) and (2) hold. In the following, the proof of (3) is shown as below.

Now consider the function $f(x) = x\ln x$, where $x \in (0,1]$. Then $f'(x) = 1 + \ln x$ and $f''(x) = \frac{1}{x} > 0$, where $x \in (0,1]$. Accordingly, $f(x) = x\ln x$ is a convex function. Therefore, for any two points $x_1, x_2 \in (0,1]$, the inequality $\frac{f(x_1) + f(x_2)}{2} \ge f(\frac{x_1 + x_2}{2})$ holds.

Utilize $f(x)=x\ln x$ in the above inequality and $x_1\ln x_1+x_2\ln x_2-\left(x_1+x_2\right)\ln\frac{x_1+x_2}{2}\geq 0$ can be got, where the equality holds only if $x_1=x_2$. Similarly, the following equation can be obtained.

$$\begin{aligned}D_{NS}(A,B)=I_{NS}(A,B)+I_{NS}(B,A)=&\underbrace{\left(t_A\ln t_A+t_B\ln t_B\right)-\left(t_A+t_B\right)\ln\frac{t_A+t_B}{2}}_{T}+\\&\underbrace{\left(i_A\ln i_A+i_B\ln i_B\right)-\left(i_A+i_B\right)\ln\frac{i_A+i_B}{2}}_{I}+\\&\underbrace{\left(f_A\ln f_A+f_B\ln f_B\right)-\left(f_A+f_B\right)\ln\frac{f_A+f_B}{2}}_{F}.\end{aligned}$$

Because $T\geq 0$, $I\geq 0$ and $F\geq 0$, $D_{NS}(A,B)\geq 0$ holds. $D_{NS}(A,B)=0$ holds only if $t_A=t_B$, $i_A=i_B$ and $f_A=f_B$, namely $A=B$.

Example 3. Assume two SNNs $A=\left(0.8,0.1,0.2\right)$ and $B=\left(0.5,0.3,0.4\right)$. Then the following result can be obtained by applying Eqs. (1) and (2).

$$D_{NS}(A,B)=I_{NS}(A,B)+I_{NS}(B,A)=0.1212.$$

4. MCDM approach based on cross-entropy and TOPSIS

This section presents an approach for MCDM problems with INSs by means of the cross-entropy and TOPSIS.

For a MCDM problem, let $A=\left\{a_1,a_2,\cdots,a_m\right\}$ be a set consisted of m alternatives and let $C=\left\{c_1,c_2,\cdots,c_n\right\}$ be a set consisting of n criteria. Assume that the weight of the criterion c_j $(j=1,2,\cdots,n)$, given by the decision-maker, is w_j, where $w_j\in\left[0,1\right]$ and $\sum_{j=1}^{n}w_j=1$. And the weight vector of criteria can be expressed as $w=\left(w_1,w_2,\cdots,w_n\right)$. Let $\tilde{B}=\left[\tilde{b}_{ij}\right]_{m\times n}=\left[\left(\left[t_{ij}^L,t_{ij}^U\right],\left[i_{ij}^L,i_{ij}^U\right],\left[f_{ij}^L,f_{ij}^U\right]\right)\right]_{m\times n}$ be the decision matrix, where $\left[t_{ij}^L,t_{ij}^U\right]$, $\left[i_{ij}^L,i_{ij}^U\right]$ and $\left[f_{ij}^L,f_{ij}^U\right]$ are interval numbers and represent the degrees of truth-membership, indeterminacy-membership and falsity-membership of each alternative a_i $(i=1,2,\cdots,m)$ on the criterion c_j with respect to the concept “excellent”, respectively.

In general, there are two types of criterion called maximizing criteria and minimizing criteria. In order to uniform criterion types, minimizing criteria need to be transformed into maximizing criteria. Suppose the standardized matrix is expressed as $\tilde{R}=\left[\tilde{r}_{ij}\right]_{m\times n}$. The original decision matrix $\tilde{B}=\left[\tilde{b}_{ij}\right]_{m\times n}$ can be converted to $\tilde{R}=\left[\tilde{r}_{ij}\right]_{m\times n}$ based on the primary transformation principle of Ref. [71], where

$$\tilde{r}_{ij}=\begin{cases}\tilde{b}_{ij}=\left(\left[t_{ij}^{L},t_{ij}^{U}\right],\left[i_{ij}^{L},i_{ij}^{U}\right],\left[f_{ij}^{L},f_{ij}^{U}\right]\right), \text{for maximizing criterion } c_j\\ \tilde{b}_{ij}^{c}=\left(\left[f_{ij}^{L},f_{ij}^{U}\right],\left[i_{ij}^{L},i_{ij}^{U}\right],\left[t_{ij}^{L},t_{ij}^{U}\right]\right), \text{for minimizing criterion } c_j\end{cases}, \tag{3}$$

in which $\tilde{b}_{ij}^{c}$ is the complement set of $\tilde{b}_{ij}$, defined in Definition 8.

The absolute positive ideal solution (PIS) and the absolute negative ideal solution (NIS) of INSs respectively denoted by a^{+} and a^{-}, and can be expressed as follows [49]:

$$a^{+}=\left(\left[t_{ij}^{L},t_{ij}^{U}\right],\left[i_{ij}^{L},i_{ij}^{U}\right],\left[f_{ij}^{L},f_{ij}^{U}\right]\right)=\left([1,1],[0,0],[0,0]\right),$$

$$a^{-}=\left(\left[t_{ij}^{L},t_{ij}^{U}\right],\left[i_{ij}^{L},i_{ij}^{U}\right],\left[f_{ij}^{L},f_{ij}^{U}\right]\right)=\left([0,0],[1,1],[1,1]\right).$$

In order to get the cross-entropy or degree of discrimination between $a_i\ (i=1,2,\cdots,m)$ and the ideal solutions, each INN is transformed into a SNN based on the operator $H_{p,q,r}$. Let $p_{ij},q_{ij},r_{ij}\in[0,1]$ be three crisp numbers and any INN, denoted by $\left(\left[t_{ij}^{L},t_{ij}^{U}\right],\left[i_{ij}^{L},i_{ij}^{U}\right],\left[f_{ij}^{L},f_{ij}^{U}\right]\right)$ can be transformed into the following form:

$$H_{p,q,r}\left(\tilde{r}_{ij}\right)=\left(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}},i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}},f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}}\right), \tag{4}$$

where $W_{t_{ij}}=t_{ij}^{U}-t_{ij}^{L}$, $W_{i_{ij}}=i_{ij}^{U}-i_{ij}^{L}$ and $W_{f_{ij}}=f_{ij}^{U}-f_{ij}^{L}$.

Using Eqs. (1), (2) and (4), the degree of discrimination of $a_i\ (i=1,2,\cdots,m)$ from a^{+} or a^{-} on $c_j\ (j=1,2,\cdots,n)$ can be obtained as follows:

$$\begin{aligned}D_{ij}^{+}&=(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}})\ln\frac{2(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}})}{(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}}+1)}+\ln\frac{2}{(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}}+1)}+\\&(i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}})\ln 2+(f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}})\ln 2,\end{aligned} \tag{5}$$

$$D_{ij}^{-}=(t_{ij}^{L}+p_{ij}\cdot W_{t_{ij}})\ln 2+(i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}})\ln\frac{2(i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}})}{(i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}}+1)}+\ln\frac{2}{(i_{ij}^{L}+q_{ij}\cdot W_{i_{ij}}+1)}+$$
$$(f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}})\ln\frac{2(f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}})}{(f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}}+1)}+\ln\frac{2}{(f_{ij}^{L}+r_{ij}\cdot W_{f_{ij}}+1)}. \tag{6}$$

Using Eqs. (5) and (6), $D_i\ (i=1,2,\cdots,m)$ can be obtained, which expresses the performance of the alternative $a_i\ (i=1,2,\cdots,m)$. In other words, the larger D_i is, the better the alternative a_i will be.

$$D_i=\sum_{j=1}^{n}w_j D_{ij}=\sum_{j=1}^{n}w_j\frac{D_{ij}^{-}}{D_{ij}^{-}+D_{ij}^{+}}, \tag{7}$$

where $D_{ij}=\dfrac{D_{ij}^{-}}{D_{ij}^{-}+D_{ij}^{+}}$ denotes the performance of the alternative $a_i\ (i=1,2,\cdots,m)$ on $c_j\ (j=1,2,\cdots,n)$ and $w_j\ (j=1,2,\cdots,n)$ represents the weight of c_j.

In view of the fact that an INS is characterized by a truth-membership function, a indeterminacy-membership function and a falsity-membership function, whose values are intervals rather than specific numbers. It is unreasonable to designate a SNN for the given INN by artificially choose only a certain p_{ij}, q_{ij} and r_{ij} in Eq. (4) to indicate the evaluation information. Because it may lead to distortion or loss of original information [27]. Thus, D_{ij} may take different values as numerical variations p_{ij}, q_{ij} and r_{ij} change. In order to avoid information loss, an interval $\tilde{D}_{ij}=\left[D_{ij}^{L},D_{ij}^{U}\right]$ is applied to represent the performance of the alternative $a_i\ (i=1,2,\cdots,m)$ on $c_j\ (j=1,2,\cdots,n)$, where D_{ij}^{L} and D_{ij}^{U} are the lower and upper bounds, respectively.

The values of D_{ij}^{L} and D_{ij}^{U} are determined as below.

As it is shown in Eqs. (5) and (6), both D_{ij}^{+} and D_{ij}^{-} are multivariate continuous functions with respect to p_{ij}, q_{ij} and r_{ij}, respectively. They also can reach the maximum and minimum in the domain $p_{ij},q_{ij},r_{ij}\in\left[0,1\right]$. Calculate the partial derivative of D_{ij}^{+} and D_{ij}^{-} with respect to p_{ij}, q_{ij} and r_{ij}, respectively.

$$\frac{\partial D_{ij}^{+}}{\partial p_{ij}} = W_{t_{ij}} \ln \frac{2(t_{ij}^{L} + p_{ij} \cdot W_{t_{ij}})}{(t_{ij}^{L} + p_{ij} \cdot W_{t_{ij}} + 1)}.$$

Since $2(t_{ij}^{L} + p_{ij} \cdot W_{t_{ij}}) \le t_{ij}^{L} + p_{ij} \cdot W_{t_{ij}} + 1$, $\dfrac{\partial D_{ij}^{+}}{\partial p_{ij}} \le 0$.

$\dfrac{\partial D_{ij}^{+}}{\partial q_{ij}} = W_{i_{ij}} \ln 2 \ge 0$, and $\dfrac{\partial D_{ij}^{+}}{\partial r_{ij}} = W_{f_{ij}} \ln 2 \ge 0$.

Similarly, $\dfrac{\partial D_{ij}^{-}}{\partial p_{ij}} = W_{t_{ij}} \ln 2 \ge 0$, $\dfrac{\partial D_{ij}^{-}}{\partial q_{ij}} = W_{i_{ij}} \ln \dfrac{2(i_{ij}^{L} + q_{ij} \cdot W_{i_{ij}})}{(i_{ij}^{L} + q_{ij} \cdot W_{i_{ij}} + 1)} \le 0$,

and $\dfrac{\partial D_{ij}^{-}}{\partial r_{ij}} = W_{f_{ij}} \ln \dfrac{2(f_{ij}^{L} + r_{ij} \cdot W_{f_{ij}})}{(f_{ij}^{L} + r_{ij} \cdot W_{f_{ij}} + 1)} \le 0.$

The analysis above indicates that D_{ij}^{+} can reach its maximum and D_{ij}^{-} reaches the minimum if $p_{ij} = 0$ and $q_{ij} = r_{ij} = 1$. Likewise, D_{ij}^{+} reaches its minimum and D_{ij}^{-} reaches the maximum if $p_{ij} = 1$ and $q_{ij} = r_{ij} = 0$. As a result, it easy to understand that $D_{ij} = \dfrac{D_{ij}^{-}}{D_{ij}^{-} + D_{ij}^{+}}$ can have a minimum when $p_{ij} = 0$ and $q_{ij} = r_{ij} = 1$ and a maximum when $p_{ij} = 1$ and $q_{ij} = r_{ij} = 0$. Then the lower and upper bounds of $\tilde{D}_{ij}$ can be obtained, respectively.

$$D_{ij}^{L} = \frac{t_{ij}^{L} \ln 2 + i_{ij}^{U} \ln \frac{2 \cdot i_{ij}^{U}}{(i_{ij}^{U} + 1)} + f_{ij}^{U} \ln \frac{2 \cdot f_{ij}^{U}}{(f_{ij}^{U} + 1)} + \ln \frac{2}{(i_{ij}^{U} + 1)} + \ln \frac{2}{(f_{ij}^{U} + 1)}}{(t_{ij}^{L} + i_{ij}^{U} + f_{ij}^{U}) \ln 2 + t_{ij}^{L} \ln \frac{2 \cdot t_{ij}^{L}}{(t_{ij}^{L} + 1)} + i_{ij}^{U} \ln \frac{2 \cdot i_{ij}^{U}}{(i_{ij}^{U} + 1)} + f_{ij}^{U} \ln \frac{2 \cdot f_{ij}^{U}}{(f_{ij}^{U} + 1)} + \ln \frac{2}{(t_{ij}^{L} + 1)} + \ln \frac{2}{(i_{ij}^{U} + 1)} + \ln \frac{2}{(f_{ij}^{U} + 1)}}, \tag{8}$$

$$D_{ij}^{U} = \frac{t_{ij}^{U} \ln 2 + i_{ij}^{L} \ln \frac{2 \cdot i_{ij}^{L}}{(i_{ij}^{L} + 1)} + f_{ij}^{L} \ln \frac{2 \cdot f_{ij}^{L}}{(f_{ij}^{L} + 1)} + \ln \frac{2}{(i_{ij}^{L} + 1)} + \ln \frac{2}{(f_{ij}^{L} + 1)}}{(t_{ij}^{U} + i_{ij}^{L} + f_{ij}^{L}) \ln 2 + t_{ij}^{U} \ln \frac{2 \cdot t_{ij}^{U}}{(t_{ij}^{U} + 1)} + i_{ij}^{L} \ln \frac{2 \cdot i_{ij}^{L}}{(i_{ij}^{L} + 1)} + f_{ij}^{L} \ln \frac{2 \cdot f_{ij}^{L}}{(f_{ij}^{L} + 1)} + \ln \frac{2}{(t_{ij}^{U} + 1)} + \ln \frac{2}{(i_{ij}^{L} + 1)} + \ln \frac{2}{(f_{ij}^{L} + 1)}}. \tag{9}$$

Based on the analysis above, Eq. (7) can be rewritten using operational laws of interval numbers given in Definition 1.

$$\tilde{D}_{i} = \sum_{j=1}^{n} w_{j} \tilde{D}_{ij} = \sum_{j=1}^{n} w_{j} \left[D_{ij}^{L}, D_{ij}^{U} \right] = \left[\sum_{j=1}^{n} w_{j} D_{ij}^{L}, \sum_{j=1}^{n} w_{j} D_{ij}^{U} \right]. \tag{10}$$

$\tilde{D}_{i} \ge \tilde{D}_{j}$ means that a_{i} $(i = 1, 2, \cdots, m)$ is not inferior to a_{j} $(j = 1, 2, \cdots, m)$. Then the likelihood matrix P can constructed as below.

$$P=\left(p_{ij}\right)_{m\times m}=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}, \tag{11}$$

where $p_{ij}=p(a_i\geq a_j)=p(\tilde{D}_i\geq\tilde{D}_j)=\max\left\{1-\max\left\{\dfrac{D_j^U-D_i^L}{L(\tilde{D}_i)+L(\tilde{D}_j)},0\right\},0\right\}$, and $L(\tilde{D}_i)=D_i^U-D_i^L$.

According to [69], the ranking vector of the likelihood matrix can be defined as follows:

$$\omega_i=\frac{\sum_{j=1}^{m}p_{ij}+\frac{m}{2}-1}{m\left(m-1\right)},\ i=1,2,\cdots,m. \tag{12}$$

Consequently, the ranking of all alternatives is determined according to the descending order of $\omega_i\ (i=1,2,\cdots,m)$. That is, the larger ω_i is, the better the alternative a_i will be.

In a word, the main procedure of the above MCDM approach is listed as below.

Step 1. Establish the original decision matrix $\tilde{B}=\left[\tilde{b}_{ij}\right]_{m\times n}$ according to the evaluation information given in the form of INNs. Then use Eq. (3) to transform $\tilde{B}=\left[\tilde{b}_{ij}\right]_{m\times n}$ into $\tilde{R}=\left[\tilde{r}_{ij}\right]_{m\times n}$.

Step 2. Use Eq. (8) and (9) to derive D_{ij}^L and D_{ij}^U and establish the matrix $\left(\tilde{D}_{ij}\right)_{m\times n}=\left[D_{ij}^L,D_{ij}^U\right]_{m\times n}$.

Step 3. Use Eq. (10) to calculate the performance of each alternative $\tilde{D}_i\ (i=1,2,\cdots,m)$.

Step 4. Construct the likelihood matrix P by using Eq. (11) and obtain the ranking vector $\omega=\left(\omega_1,\omega_2,\cdots,\omega_m\right)$ based on Eq. (12).

Step 5. Determine the ranking of all alternatives according to the descending order of $\omega_i\ (i=1,2,\cdots,m)$.

5. Illustrative example

In this section, an investment appraisal project is used to demonstrate the application of proposed decision making approach, as well as the validity and effectiveness of the proposed approach.

The example on the investment appraisal project of a company adapted from Ref. [48] is employed here. ABC company intends to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1) a_1 is a car company; (2) a_2 is a food company; (3) a_3 is a computer company; (4) a_4 is an arms company. This company must make a decision according to the three criteria: (1) c_1 is the risk; (2) c_2 is the growth prospects; (3) c_3 is the environmental impact, where c_1 and c_2 are maximizing criteria, and c_3 is a minimizing criterion. The weight vector of the criteria is $w=\left(0.35,0.25,0.4\right)$. The four possible alternatives are to be assessed under three criteria using INSs, and the evaluation information is shown in the following interval neutrosophic decision matrix $\tilde{B}=\left[\tilde{b}_{ij}\right]_{m\times n}$.

$$\tilde{B}=\begin{bmatrix} \left([0.4,0.5],[0.2,0.3],[0.3,0.4]\right) & \left([0.4,0.6],[0.1,0.3],[0.2,0.4]\right) & \left([0.7,0.9],[0.2,0.3],[0.4,0.5]\right) \\ \left([0.6,0.7],[0.1,0.2],[0.2,0.3]\right) & \left([0.6,0.7],[0.1,0.2],[0.2,0.3]\right) & \left([0.3,0.6],[0.3,0.5],[0.8,0.9]\right) \\ \left([0.3,0.6],[0.2,0.3],[0.3,0.4]\right) & \left([0.5,0.6],[0.2,0.3],[0.3,0.4]\right) & \left([0.4,0.5],[0.2,0.4],[0.7,0.9]\right) \\ \left([0.7,0.8],[0.0,0.1],[0.1,0.2]\right) & \left([0.6,0.7],[0.1,0.2],[0.1,0.3]\right) & \left([0.6,0.7],[0.3,0.4],[0.8,0.9]\right) \end{bmatrix}.$$

5.1 Procedures of decision making based on INSs

In the following, the main procedures of obtaining the optimal ranking of alternatives are presented.

Step 1. Use Eq. (3) to transform minimizing the criterion c_3 into a maximizing criterion and obtain $\tilde{R}=\left[\tilde{r}_{ij}\right]_{m\times n}$.

$$\tilde{R}=\begin{bmatrix} \left([0.4,0.5],[0.2,0.3],[0.3,0.4]\right) & \left([0.4,0.6],[0.1,0.3],[0.2,0.4]\right) & \left([0.4,0.5],[0.2,0.3],[0.7,0.9]\right) \\ \left([0.6,0.7],[0.1,0.2],[0.2,0.3]\right) & \left([0.6,0.7],[0.1,0.2],[0.2,0.3]\right) & \left([0.8,0.9],[0.3,0.5],[0.3,0.6]\right) \\ \left([0.3,0.6],[0.2,0.3],[0.3,0.4]\right) & \left([0.5,0.6],[0.2,0.3],[0.3,0.4]\right) & \left([0.7,0.9],[0.2,0.4],[0.4,0.5]\right) \\ \left([0.7,0.8],[0.0,0.1],[0.1,0.2]\right) & \left([0.6,0.7],[0.1,0.2],[0.1,0.3]\right) & \left([0.8,0.9],[0.3,0.4],[0.6,0.7]\right) \end{bmatrix}.$$

Step 2. Use Eqs. (8) and (9) to derive D_{ij}^{L} and D_{ij}^{U} and establish the matrix $\left(\tilde{D}_{ij}\right)_{4\times 3}=\left[D_{ij}^{L},D_{ij}^{U}\right]_{4\times 3}$.

$$\left(\tilde{D}_{ij}\right)_{4\times 3}=\left[D_{ij}^{L},D_{ij}^{U}\right]_{4\times 3}=\begin{bmatrix} \left([0.4963,0.6597]\right) & \left([0.4963,0.8144]\right) & \left([0.3317,0.4838]\right) \\ \left([0.6954,0.8369]\right) & \left([0.6952,0.8369]\right) & \left([0.4714,0.7094]\right) \\ \left([0.4410,0.6952]\right) & \left([0.5433,0.6952]\right) & \left([0.5194,0.7146]\right) \\ \left([0.8369,0.9542]\right) & \left([0.6952,0.8902]\right) & \left([0.4800,0.5823]\right) \end{bmatrix}.$$

Step 3. Use Eq. (10) to calculate the performance of each alternative $\tilde{D}_i$ $(i=1,2,3,4)$.

$$\tilde{D}_1=[0.4305,0.6280],\ \tilde{D}_2=[0.6057,0.7859],$$

$$\tilde{D}_3=[0.4979,0.7030],\ \tilde{D}_4=[0.6587,0.7894].$$

Step 4. Construct the likelihood matrix P using Eq. (11) and obtain the ranking vector $\omega=(\omega_1,\omega_2,\omega_3,\omega_4)$ based on Eq. (12).

$$P=\left(p_{ij}\right)_{4\times 4}=\begin{bmatrix} 0.5 & 0.0593 & 0.3231 & 0 \\ 0.9407 & 0.5 & 0.7474 & 0.4091 \\ 0.6769 & 0.2526 & 0.5 & 0.1317 \\ 1 & 0.5909 & 0.8683 & 0.5 \end{bmatrix},$$

$\omega_1=0.1569$, $\omega_2=0.2998$, $\omega_3=0.2134$, and $\omega_4=0.3299$.

Step 5. According to the descending order of $\omega_i\ (i=1,2,3,4)$, the ranking of all alternatives is $a_4\succ a_2\succ a_3\succ a_1$ and a_4 is the best one.

5.2 Comparison analysis and discussion

In order to further verify the feasibility and effectiveness of the proposed approach based on INSs, a comparison analysis is conducted using three existing methods and the analysis is based on the same illustrative example.

In Ref. [48] and Ref. [49], some aggregation operators were developed to aggregate interval neutrosophic formation. Two methods in Ref. [48] utilized the similarity measures based on the relationship with distances ,where the similarity measure of Method 1 is on the basis of the Euclidean distance and the similarity measure of Method 2 is in view of the Hamming distance. The ranking results were got though calculating the similarity measures between each alternative and the PIS. The method in Ref. [49] relied on the Hamming distance between INSs and provided a compromise solution to rank alternatives, which considered the distance to both the absolute PIS and NIS. The comparison results can be found in Table 1.

Table 1

Comparison with different methods

Methods	Ranking vectors				Ranking results
Method 1 of Ref. [48]	0.7370	0.9323	0.8344	0.9034	$a_2 \succ a_4 \succ a_3 \succ a_1$
Method 2 of Ref. [48]	0.7667	0.9542	0.8625	0.9600	$a_4 \succ a_2 \succ a_3 \succ a_1$
Method of Ref. [49]	0.941	0.319	0.666	0.114	$a_4 \succ a_2 \succ a_3 \succ a_1$
The proposed approach	0.1569	0.2998	0.2134	0.3299	$a_4 \succ a_2 \succ a_3 \succ a_1$

The difference among the ranking results shown in Table 1 is the sequence of a_2 and a_4. (1) The comparison methods in Ref. [48] were conducted using the similarity measures that only considered the relationship between each alternative and the PIS. There are some drawbacks. Firstly, it is not easy to choose the PIS in the assessment information given in the form of INSs. Secondly, the ideal solution is closely related to the number of alternatives as well as the evaluation values of alternatives. Thus it may vary as the original information changes. If only PIS is taken into account and NIS is ignored, the ranking of alternatives may be incorrectly reversed, and this may be amplified in the final results. There will be an identical ranking result with the proposed approach in case of replacing the PIS with the absolute one and allowing the usage of the absolute NIS at the same time, and utilizing the closeness coefficient (performance of each alternative) to determine the ranking of alternatives. The updated results are shown in Table 2. Therefore, the methods in Ref. [48] are not reliable enough, which can be demonstrated through the changes of the final results. (2) The result acquired by the fuzzy decision making method utilizing TOPSIS in Ref. [49] is consistent with the one obtained through Method 2 of Ref. [48] and the proposed approach, but the basis of these approaches is not similar. The method in Ref. [49] considered the distances to the absolute PIS and NIS based on the primary principle of TOPSIS. The performance of each alternative was aggregated in the form of a real number. However, unlike the method in Ref. [49], the proposed approach takes advantage of the merits of interval number and converts the performance of each alternative to an interval number, which can decrease information loss. Then these interval numbers are compared by establishing the relative likelihood-based comparison relations so as to rank all alternatives. What's more, all methods in Ref. [48] and Ref. [49] were conducted by directly calculating similarity or distance measures between INSs, which are complex in

Table 2

Ranking results by revising the methods in Ref. [48]

Methods	Ranking vectors				Ranking results
Method 1 of Ref. [48]	0.5025	0.6908	0.5987	0.6983	$a_4 \succ a_2 \succ a_3 \succ a_1$
Method 2 of Ref. [48]	0.5025	0.6900	0.5983	0.6959	$a_4 \succ a_2 \succ a_3 \succ a_1$

Furthermore, compared to the considered methods, the proposed approach makes good use of both interval numbers and TOPSIS method. It is well known that adopting interval numbers to represent performances in the TOPSIS method can give up various kinds of aggregation operators and score functions that may bring different results and effectively reduce the loss of decision information. Accordingly, the proposed approach is more flexible and reliable in handling MCDM problems than the compared methods in the interval neutrosophic environment.

6. Conclusion

INSs can flexibly express uncertain, imprecise, incomplete and inconsistent information that widely exist in scientific and engineering situations. So it is of great significance to study MCDM methods with INSs. In this paper, the basis definitions related to INSs are proposed and the useful operational laws and properties are discussed in detail. Then based on related research achievements in IFSs, a transformation operator and cross-entropy are defined. Thus, a MCDM method is established based on cross-entropy and TOPSIS, which computes the cross-entropy after transforming INNs into SNNs based on the transformation operator, and aggregates the performances of alternatives into interval numbers. Finally, the ranking result is obtained by comparing these interval numbers with a possibility degree method.

The advantages of this study are that the approach can be both simple and convenient in computing and effective to decrease the loss of evaluation information. The feasibility and validity of the proposed approach have been verified through the illustrative example and comparison analysis. The comparison results demonstrate that the proposed approach can provide more reliable and precise outcomes than other methods. Therefore, this approach has much application potential in dealing with MCDM problems in the interval neutrosophic environment, in which criterion values with respect to alternatives are evaluated by the form of INNs and the criterion weights are known information.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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