Undefined (2015) 1–9
IOS Press

On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets

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Abstract. A notion of Quadripartitioned Single Valued Neutrosophic Sets (QSVNS) is introduced and a theoretical study on various set-theoretic operations on them has been carried out. The definitions of distance, similarity measure and entropy have been proposed. Finally an application of the proposed similarity measure in a problem pertaining to pattern recognition has been shown.

Keywords: Single valued neutrosophic sets, quadripartitioned single valued neutrosophic sets, similarity measure, entropy

1. Introduction

The study of logic stretches from the fundamental classical 2-valued or Boolean logic to the study of the most general multi-valued logic. In case of classical logic, the values attributed to truth T and falsity F are 1 and 0 respectively. Later, the development of fuzzy logic was proposed as a generalization of the Boolean Logic [12] where T and F could assume any values from $[0, 1]$. Although the theory of fuzzy sets, which was proposed by L. A. Zadeh [15] in 1965, revolutionized the approach of dealing with uncertainties, yet it had its own limitations. Hence, in due course of time, several other improvizations of the fuzzy theory came into existance. Some of these include the theory of L-Fuzzy sets by Goguen [7], the theory of rough sets by Pawlak [9], the theory of intuitionistic fuzzy sets by K. T. Atanassov [1,2] etc. Unlike the theory of fuzzy sets which associates a certain degree of membership, $\mu \,\epsilon\, [0, 1]$ to each element of the universe of discourse, intuitionistic fuzzy sets associate a degree of non-membership $\nu \,\epsilon\, [0, 1]$ as well, to each element where $0 \leq \mu + \nu \leq 1$. However, the notion of indeterminacy, generally referred to as the hesitation margin, π, defined as, $\pi = 1 - \mu - \nu$ in case of intuitionistic fuzzy sets was somewhat specific and completely dependent on the values of membership and non-membership of an element. This particular shortcoming of the theory of intuitionistic fuzzy sets was compensated by the introduction of the theory of neutrosophic sets by Florentin Samarandache in 1995 [11,10]. Neutrosophic sets were proposed as a generalization of the intuitionistic fuzzy sets and neutrosophic logic sprouted from the branch of philosophy '$neutrosophy$' which means the study of neutralities. In case of neutrosophic sets, indeterminacy is taken care of separately and each element x is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, each of which belongs to the non-standard unit interval $]0^-, 1^+[$.

Although the neutrosophic indeterminacy is independent of the truth and falsity-membership values and

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is more general than the hesitation margin of intuitionistic fuzzy sets yet, it is not very clear whether the indeterminacy associated to a particular element refers to the hesitation regarding its belongingness or non-belongingness. Expounding it more clearly, it might be stated that if for a particular event x, a person associates an indeterminacy membership $I_A(x)$, it becomes difficult to comprehend whether the degree of uncertainty of the person regarding the occurance of the event is $I_A(x)$ or the degree of uncertainty of the person regarding the non-occurance of that event is $I_A(x)$. Thus, while some authors prefer to model the behaviour of indeterminacy in a way similar to that of the truth-membership, others may prefer to model its behaviour in a way similar to that of the falsity-membership. Quite naturally, this often leads to diverse approaches in dealing with uncertainty while executing various operations over neutrosophic sets as can be seen from the works of [13,14] etc.

At this juncture, it became necessary to look for means to find a solution to this conflict of interests. In this regard, Belnap's four valued logic [3], which involves the study of truth T, falsity F, unknown U and contradictiton C proves to be a more general approach. Based on this, Smarandache proposed the notion of Four Numerical-valued neutrosophic logic [12] where the indeterminacy is split into two parts namely, 'unknown' viz. neither true nor false and 'contradiction' viz. both true and false, thereby providing a solution to the difficulties encountered in dealing with usual neutrosophic indeterminacy. This four-valued neutrosophic logic being of special interest to us, a notion of Quadripartitioned Single Valued Neutrosophic Sets (QSVNS, in short) is introduced in this paper whereby some of their properties have been studied and an application to an example of a pattern recognition problem has been shown.

The organization of the paper is as follows:
Section 1 provides a brief introduction; Section 2 is dedicated to recalling some preliminary results; Section 3 introduces the concept of a quadripartitioned neutrosophic set and deals with some basic set-theoretic operations over quadripartitioned neutrosophic sets; Section 4 introduces the definition of similarity and distance measure; Section 5 deals with the concept of entropy over QSVNS; Section 6 consists of a comparative study of the proposed similarity measures in the context of classification of patterns; Section 7 concludes the paper.

2. Preliminaries

In this section we discuss some preliminary results that would prove to be useful in the following sections.

2.1. An overview of four valued logic

Belnap [3], with a view to device a practical tool for inference, introduced the concept of a four valued logic. In his work, corresponding to a certain information he considered four possibilities namely
T: just True
F: just false
$None$: neither True nor False and,
$Both$: both True and False.
He symbolized these four truth values as
$4 = \{T, F, Both, None\}$
such that the possible values satisfied the conditions as shown in Table 1.

Also, for a mapping s from any atomic information into 4, the semantics was induced as
$s(A\&B) = s(A)\&s(B)$
$s(A \vee B) = s(A) \vee s(B)$
$s(\sim A) = \sim s(A)$

And, for any formulae A, B, C the following results hold:
(i) $A \vee B \Leftrightarrow B \vee A$; $A\&B \Leftrightarrow B\&A$.
(ii) $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$;
$A\& (B\&C) \Leftrightarrow (A\&B) \&C$.
(iii) $A\& (B \vee C) \Leftrightarrow (A\&B) \vee (A\&C)$;
$A \vee (B\&C) \Leftrightarrow (A \vee B) \& (A \vee C)$.
(iv) $(B \vee C) \&A \Leftrightarrow (B\&A) \vee (C\&A)$;
$(B\&C) \vee A \Leftrightarrow (B \vee A) \& (C \vee A)$.
(v) $\sim\sim A \Leftrightarrow A$
(vi) $\sim (A\&B) \Leftrightarrow \sim A \vee \sim B$;
$\sim (A \vee B) \Leftrightarrow \sim A \& \sim B$.

In [12], Smarandache recast Belnap's concept of four valued logic as "Four-numerical valued neutrosophic logic" where the indeterminacy I is split as $U = unknown$ and $C = contradiction$. T, F, C, U are subsets of $[0, 1]$ instead of symbols.

2.2. Some results regarding neutrosophic sets

Definition 2.1[13]. Let X be a space of points with a generic element in X denoted by x. A single valued neutrosophic set A in X is an object of the form
$A = \sum_{i=1}^{n} \langle T_A(x_i), F_A(x_i), I_A(x_i)\rangle / x_i, x_i \in X$

Table 1

Table representing the properties of the four truth values.

		N	F	T	B
$\sim$		B	T	F	N
&	N	N	F	N	F
	F	F	F	F	F
	T	N	F	T	B
	B	F	F	B	B
$\vee$	N	N	N	T	T
	F	N	F	T	B
	T	T	T	T	T
	B	T	B	T	B

when the universe of discourse is discrete.
It is represented as
$A = \int_X \langle T_A(x), I_A(x), F_A(x)\rangle /x, x \,\epsilon\, X$
when the universe of discourse is continuous.
T_A, I_A, F_A respectively denote the truth-membership, indeterminacy membership and falsity-membership functions such that for each point x in X,
$T_A(x), I_A(x), F_A(x) \,\epsilon\, [0, 1]$.

The various operations are defined as,
Containment: $A \subseteq B$ iff $T_A(x_i) \leq T_B(x_i)$, $I_A(x_i) \leq I_B(x_i), F_A(x_i) \geq F_B(x_i)$
Complement: $c(A) = \sum_{i=1}^{n} < F_A(x_i), 1-I_A(x_i), T_A(x_i) > /x_i$
Union: $A \cup B = \sum_{i=1}^{n} < T_A(x_i) \vee T_B(x_i), I_A(x_i) \vee I_B(x_i), F_A(x_i) \wedge F_B(x_i) > /x_i$
Intersection: $A \cap B = \sum_{i=1}^{n} < T_A(x_i) \wedge T_B(x_i), I_A(x_i) \wedge I_B(x_i), F_A(x_i) \vee F_B(x_i) > /x_i$ for $x_i \,\epsilon\, X$

Remark 2.2 [14]. As stated before, an alternative approach where the behaviour of the indeterminacy is assumed to be similar to that of the falsity membership, exists and is widely in use. In such cases the operations are defined as,
Containment: $A \subseteq B$ iff $T_A(x_i) \leq T_B(x_i)$, $I_A(x_i) \geq I_B(x_i), F_A(x_i) \geq F_B(x_i)$
Union: $A \cup B = \sum_{i=1}^{n} < T_A(x_i) \vee T_B(x_i), I_A(x_i) \wedge I_B(x_i), F_A(x_i) \wedge F_B(x_i) > /x_i$
Intersection: $A \cap B = \sum_{i=1}^{n} < T_A(x_i) \wedge T_B(x_i), I_A(x_i) \vee I_B(x_i), F_A(x_i) \vee F_B(x_i) > /x_i$ for $x_i \,\epsilon\, X$.

Proposition 2.3 [13]. SVNS satisfy the following properties under set-theoretic operations:
(i) $A \cup B = B \cup A$; $A \cap B = B \cap A$.
(ii) $A \cup (B \cup C) = (A \cup B) \cup C$;
$A \cap (B \cap C) = (A \cap B) \cap C$.
(iii) $A \cup A = A$; $A \cap A = A$.
(iv) $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$.
(v) $c\,(c(A)) = A$.
(vi) De-Morgan's laws hold viz.
$c(A \cup B) = c(A) \cap c(B)$;
$c(A \cap B) = c(A) \cup c(B)$.

3. Quadripartitioned Single Valued Neutrosophic Sets

In this section we propose some set-theoretic operations on quadripartitioned neutrosophic sets over a common universe X and study some of their basic properties.

Definition 3.1. Consider two QSVNS A and B, over X. A is said to be contained in B, denoted by $A \subseteq B$ iff $T_A(x) \leq T_B(x), C_A(x) \leq C_A(x), U_A(x) \geq U_A(x)$ and $F_A(x) \geq F_A(x)$.

Definition 3.2. The complement of a QSVNS A is denoted by A^c and is defined as
$A^c = \sum_{i=1}^{n} \langle F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i)\rangle /x_i$, $x_i \,\epsilon\, X$
i.e. $T_{A^c}(x_i) = F_A(x_i)$, $C_{A^c}(x_i) = U_A(x_i)$, $U_{A^c}(x_i) = C_A(x_i)$ and $F_{A^c}(x_i) = T_A(x_i), x_i \,\epsilon\, X$.

Definition 3.3. The union of two QSVNS A and B is denoted by $A \cup B$ and is defined as
$A \cup B = \sum_{i=1}^{n} < T_A(x_i) \vee T_B(x_i), C_A(x_i) \vee C_B(x_i), U_A(x_i) \wedge U_B(x_i), F_A(x_i) \wedge F_B(x_i) > /x_i$

Definition 3.4. The intersection of two QSVNS A and B is denoted by $A \cap B$ and is defined as
$A \cap B =< T_A(x_i) \wedge T_B(x_i), C_A(x_i) \wedge C_B(x_i), U_A(x_i) \vee$

$U_B(x_i), F_A(x_i) \vee F_B(x_i) > /x_i$

Example 3.5. Consider two QSVNS defined over X, given by
$A = \langle 0.7, 0.5, 0.2, 0.1\rangle /x_1 + \langle 0.2, 0.6, 0.1, 0.5\rangle /x_2 +$
$\langle 0.5, 0.2, 0.7, 0.1\rangle /x_3 + \langle 0.4, 0.2, 0.5, 0.8\rangle /x_4$
$B = \langle 0.0, 0.25, 0.7, 0.6\rangle /x_1 + \langle 0.5, 0.5, 0.3, 0.1\rangle /x_2 +$
$\langle 0.9, 0.2, 0.0, 0.0\rangle /x_3 + \langle 0.01, 0.1, 0.2, 1.0\rangle /x_4$
Then we have
$A^c = \langle 0.1, 0.2, 0.5, 0.7\rangle /x_1 + \langle 0.5, 0.1, 0.6, 0.2\rangle /x_2 +$
$\langle 0.1, 0.7, 0.2, 0.5\rangle /x_3 + \langle 0.8, 0.5, 0.2, 0.4\rangle /x_4$
$A \cup B = \langle 0.7, 0.5, 0.2, 0.1\rangle /x_1 + \langle 0.5, 0.6, 0.1, 0.1\rangle /x_2 +$
$\langle 0.9, 0.2, 0.0, 0.0\rangle /x_3 + \langle 0.4, 0.2, 0.2, 0.8\rangle /x_4$
$A \cap B = \langle 0.0, 0.25, 0.7, 0.6\rangle /x_1 + \langle 0.2, 0.5, 0.3, 0.5\rangle /x_2 +$
$\langle 0.5, 0.2, 0.7, 0.1\rangle /x_3 + \langle 0.01, 0.1, 0.5, 1.0\rangle /x_4$

Proposition 3.6. Quadripartitioned single valued neutrosophic sets satisfy the following properties under the aforementioned set-theoretic operations:
1. $A \cup B = B \cup A; A \cap B = B \cap A$
2. $A \cup (B \cup C) = (A \cup B) \cup C; A \cap (B \cap C) = (A \cap B) \cap C$
3. $A \cup (A \cap B) = A; A \cap (A \cup B) = A$
4.(i) $(A^c)^c = A$
(ii) $\mathbf{A^c} = \mathbf{\Theta}$
(iii) $\Theta^c = \mathbf{A}$
(iv) De-Morgan's laws hold viz.
$(A \cup B)^c = A^c \cap B^c; (A \cap B)^c = A^c \cup B$
5.(i) $A \cup \mathbf{A} = \mathbf{A}$
(ii) $A \cap \mathbf{A} = \mathbf{A}$
(iii) $A \cup \Theta = A$
(iv) $A \cap \Theta = \Theta$

Proof: The proofs are straight-forward.

4. Various similarity measures on quadripartitioned neutrosophic sets

Definition 4.1. Let $QSVNS(X)$ denote the set of all quadripartitioned neutrosophic sets, over the non-empty universe of discourse X. Then a mapping $s : QSVNS(X) \times QSVNS(X) \to [0,1]$ is said to be a similarity measure iff for $A, B \,\epsilon\, QSVNS(X)$ it satisfies the following properties viz.
$(S1)$ $s(A,B) = s(B,A)$
$(S2)$ $0 \leq s(A,B) < 1$ and $s(A,B) = 1$ iff $A = B$
$(S3)$ for any $A, B, C \,\epsilon\, QSVNS(X)$, such that, $A \subset B \subset C$, $s(A,C) \leq s(A,B) \wedge s(B,C)$.

A recent study shows that several measures of similarity exist in the literature which do not satisfy the triangle inequality $(S3)$. Some example of such similarity measures are,

Weighted similarity measure for SVNS based on matching function [8]:

$$s^w(A,B) = \frac{\sum_{i=1}^n \omega_i (T_A(x_i).T_B(x_i) + I_A(x_i).I_B(x_i) + F_A(x_i).F_B(x_i))^2}{\sum_{i=1}^n \omega_i \{(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)).(T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i))\}}$$

where $\omega_i \,\epsilon\, [0,1]$ is the weight associated to each $x_i \,\epsilon\, X$.

Cosine similarity measure for interval valued neutrosophic sets [4]:

$$C_N(A,B) = \frac{1}{n}\sum_{i=1}^n [(\triangle T_A(x_i).\triangle T_B(x_i) + \triangle I_A(x_i).\triangle I_B(x_i) + \triangle F_A(x_i).\triangle F_B(x_i))/(\triangle T_A^2(x_i) + \triangle I_A^2(x_i) + \triangle F_A^2(x_i))^{\frac{1}{2}}. (\triangle T_B^2(x_i) + \triangle I_B^2(x_i) + \triangle F_B^2(x_i))^{\frac{1}{2}}]$$

Dice similarity measure for single valued neutrosophic multisets [14]:

$$S_D(A,B) = \frac{1}{n}\sum_{j=1}^n \sum_{i=1}^{l_j} [2\{(T_A^i(x_j).T_B^i(x_j) + I_A^i(x_j)).(I_B^i(x_j) + F_A^i(x_j).F_B^i(x_j))\}/\{(T_A^i(x_j)^2 + I_A^i(x_j)^2 + F_A^i(x_j)^2) + (T_B^i(x_j)^2 + I_B^i(x_j)^2 + F_B^i(x_j)^2)\}]$$

where $l_j = L(x_j : A, B) = max\{L(x_j : A), L(x_j : B)\}$ is the maximum length of an element , $j = 1, 2, ..., n$.

These measures have found extensive applicability in various spheres pertaining to decision making problems and yet they do not satisfy $(S3)$. We thus introduce the definition of a different kind of similarity measure, which we term as quasi-similarity, a term which was first mentioned in [6] as follows:

Definition 4.2. A mapping $s : QSVNS(X) \times QSVNS(X) \to [0,1]$ is said to be a quasi-similarity measure if it satisfies $(S1)$and $(S2)$.

4.1. Distance based similarity measure

Before proceeding to define the distance based similarity measure, the notion of distance between QSVNS is introduced first.

Definition 4.3. A mapping $d : QSVNS(X) \times QSVNS(X) \to R^+$, where R^+ is the set of all positive real numbers, is said to be a distance measure iff for $A, B, C \,\epsilon\, QSVNS(X)$, it satisfies the following properties
$(D1)$ $d(A,B) \geq 0$ and equality holds iff $A = B$.
$(D2)$ $d(A,B) = d(B,A)$

$(D3)$ $d(A,B) \leq d(A,B) + d(A,C)$

Let $A, B \,\epsilon\, QSVNS(X)$ then for all $x_i \,\epsilon\, X$, we define the following distance measures.

Definition 4.4. The Hamming distance between A and B is defined as
$h(A,B) = \sum_{i=1}^{n}(|T_A(x_i) - T_B(x_i)| + |C_A(x_i) - C_B(x_i)| + |U_A(x_i) - U_B(x_i)| + |F_A(x_i) - F_B(x_i)|)$.

Definition 4.5. The Normalized Hamming distance between A and B is defined as $h_N(A,B) = \frac{1}{4n}h(A,B)$.

Definition 4.6. The Euclidean distance between A and B is defined as
$e(A,B) = \sum_{i=1}^{n}(|T_A(x_i) - T_B(x_i)|^2 + |C_A(x_i) - C_B(x_i)|^2 + |U_A(x_i) - U_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2)^{\frac{1}{2}}$

Definition 4.7. The Normalized Euclidean distance between A and B is defined as $e_N(A,B) = \frac{1}{2\sqrt{n}}e(A,B)$.

Definition 4.8. The distance based similarity measure between $A, B \,\epsilon\, QSVNS(X)$ is defined as
$S_d(A,B) = \frac{1}{1+d(A,B)}$
where the distance between A and B can be evaluated using any of the afore-stated methods.

Example 4.9. Consider the QSVNS of example 3.5, then we have, $h(A,B) = 5.14, h_N(A,B) = 0.32, e(A,B) = 1.52$ and $e_N(A,B) = 0.38$. Also, the distance based similarity measure corresponding to the Hamming distance, Normalized Hamming distance, Euclidean distance and Normalized Euclidean distance are found to be $0.163, 0.758, 0.397$ and 0.725 respectively.

4.2. Similarity measures based on membership values

Suppose $A, B \,\epsilon\, QSVNS(X)$. At first some functions are defined which would be useful in defining the similarity measure.
For each $x_i \,\epsilon\, X, i = 1, 2, ..., n$ and for $j = 1, 2, 3, 4$ define the functions $h_j^{A,B} : X \to [0,1]$ respectively as,
$h_1^{A,B}(x_i) = |T_B(x_i) - T_A(x_i)|$
$h_2^{A,B}(x_i) = |F_A(x_i) - F_B(x_i)|$
$h_3^{A,B}(x_i) = \frac{1}{3}\left(h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + |C_B(x_i) - C_A(x_i)|\right)$
$h_4^{A,B}(x_i) = |U_A(x_i) - U_B(x_i)|$

The functions defined above measure the difference between the various membership values corresponding to the two sets A and B w.r.t. each x_i. Define a mapping,

$S^{'}(A,B) = 1 - \frac{1}{4n}\sum_{i=1}^{n}\sum_{j=1}^{4} h_j^{A,B}(x_i)$

Theorem 4.10. $S^{'}(A,B)$ is a measure of similarity between the two quadripartitioned single valued neutrosophic sets A and B over X.

Proof: The proofs are straight-forward.

When an information is represented in terms of a QSVNS, the uncertainty associated with the information are characterized by four membership functions describing the aspects 'true', 'both true and false', 'neither true nor false' and 'false'. Naturally, it would be a better attempt if most of the available information could be put into best use while defining the measure of similarity between two QSVNS. Thus, we improvize the definition of the proposed similarity measure as follows:
Suppose,
$\tau_1^{A,B}(x_i)$
$= \frac{1}{2}|\,(T_A(x_i).F_A(x_i) - C_A(x_i)) - (T_B(x_i).F_B(x_i) - C_B(x_i))\,|$
$= \frac{1}{2}|(T_A(x_i).F_A(x_i) - T_B(x_i).F_B(x_i)) + (C_B(x_i) - C_A(x_i))|$
$\tau_2^{A,B}(x_i) = \frac{1}{3}\left(h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + |U_B(x_i) - U_A(x_i)|\right)$
Finally define,
$S^{''}(A,B)$
$= 1 - [\frac{1}{n}\sum_{i=1}^{n}\frac{1}{4}(h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + \tau_1^{A,B}(x_i) + \tau_2^{A,B}(x_i))^p]^{\frac{1}{p}}$
where p is any positive integer, and is defined to be the *"order of similarity"*.

Theorem 4.11. $S^{''}(A,B)$ is a similarity measure.

Proof:

(i) It is easy to prove that $S^{''}(A,B) = S^{''}(B,A)$.

(ii) We have, $T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) \,\epsilon\, [0,1]$. Thus, $h_1^{A,B}(x_i)$ attains its maximum value if either one of $T_A(x_i)$ or $T_B(x_i)$ is equal to 1 while the other is 0 and in that case the maximum value is 1. Similarly, $h_1^{A,B}(x_i)$ attains a minimum value 0 if $T_A(x_i)$ and $T_B(x_i)$ are equal. So, it follows that $0 \leq h_1^{A,B}(x_i) \leq 1$. Similarly it can be shown that $h_2^{A,B}(x_i)$, lies within $[0,1]$ for all $x_i \,\epsilon\, X$. Similar proofs apply for $\tau_1^{A,B}(x_i)$ and $\tau_2^{A,B}(x_i)$.
So, $0 \leq h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + \tau_1^{A,B}(x_i) + \tau_2^{A,B}(x_i) \leq 4$
$\Rightarrow 0 \leq \frac{1}{n}\sum_{i=1}^{n}[\frac{1}{4}(h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + \tau_1^{A,B}(x_i) + \tau_2^{A,B}(x_i))]^p \leq 1$
which implies $0 \leq S^{''}(A,B) \leq 1$.

Now $S^{''}(A,B) = 1$ iff
$h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + \tau_1^{A,B}(x_i) + \tau_2^{A,B}(x_i) = 0$ for each

$x_i \in X$
$\Leftrightarrow h_1^{A,B}(x_i) = 0, h_2^{A,B}(x_i) = 0, \tau_1^{A,B}(x_i) = 0 = \tau_2^{A,B}(x_i)$
$\Leftrightarrow T_A(x_i) = T_B(x_i), C_A(x_i) = C_B(x_i), U_A(x_i) = U_B(x_i),$
$F_A(x_i) = F_B(x_i)$
i.e. iff $A = B$.

(iii) Suppose $P \subset Q \subset R$. then, we have,
$T_P(x_i) \leq T_Q(x_i) \leq T_R(x_i), C_P(x_i) \leq C_Q(x_i) \leq C_R(x_i),$
$U_P(x_i) \geq U_Q(x_i) \geq U_R(x_i)$ and $F_P(x_i) \geq F_Q(x_i) \geq F_R(x_i)$ for all $x_i \in X$.

Consider $h_1^{P,Q}(x_i)$ and $h_1^{P,R}(x_i)$. Since $T_Q(x_i) \leq T_R(x_i)$, it follows that,
$|T_R(x_i) - T_P(x_i)| \geq |T_Q(x_i) - T_P(x_i)|$
$\Rightarrow h_1^{P,R}(x_i) \geq h_1^{P,Q}(x_i)$.
Similarly it can be shown that $h_2^{P,R}(x_i) \geq h_2^{P,Q}(x_i)$, for all $x_i \in X$.

Next, consider $\tau_1^{P,Q}(x_i)$ and $\tau_1^{P,R}(x_i)$.
Since $T_P(x_i) \leq T_Q(x_i) \leq T_R(x_i), F_P(x_i) \geq F_Q(x_i) \geq F_R(x_i)$, it follows that,
$0 \leq T_P(x_i).F_P(x_i) - T_Q(x_i).F_Q(x_i) \leq T_P(x_i).F_P(x_i) - T_R(x_i).F_R(x_i)$. Again, $C_P(x_i) \leq C_Q(x_i) \leq C_R(x_i)$, we have $0 \leq C_Q(x_i) - C_P(x_i) \leq C_R(x_i) - C_P(x_i)$. Thus,
$\left|(T_P(x_i).F_P(x_i) - T_Q(x_i).F_Q(x_i)) + (C_Q(x_i) - C_P(x_i))\right|$
$\leq |(T_P(x_i).F_P(x_i) - T_R(x_i).F_R(x_i)) + (C_R(x_i) - C_P(x_i))|$
$\Rightarrow \tau_1^{P,Q}(x_i) \leq \tau_1^{P,R}(x_i)$

Similar proof can be constructed for $\tau_2^{P,Q}(x_i)$ respectively for each x_i. Thus, one can safely say that,
$\frac{1}{4}\left(h_1^{P,R}(x_i) + h_2^{P,R}(x_i) + \tau_1^{P,R}(x_i) + \tau_2^{P,R}(x_i)\right)$
$\geq \frac{1}{4}\left(h_1^{P,Q}(x_i) + h_2^{P,Q}(x_i) + \tau_1^{P,Q}(x_i) + \tau_2^{P,Q}(x_i)\right)$
which means
$(\frac{1}{n}\sum_{i=1}^{n}[\frac{1}{4}(h_1^{P,R}(x_i)+h_2^{P,R}(x_i)+\tau_1^{P,R}(x_i)+\tau_2^{P,R}(x_i))]^p)^{\frac{1}{p}} \geq$
$(\frac{1}{n}\sum_{i=1}^{n}[\frac{1}{4}(h_1^{P,Q}(x_i)+h_2^{P,Q}(x_i)+\tau_1^{P,Q}(x_i)+\tau_2^{P,Q}(x_i))]^p)^{\frac{1}{p}}$,
for any positive integer p.
Thus, it automatically follows that, $S''(P,R) \leq S''(P,Q)$.
The proof of $S''(P,R) \leq S''(Q,R)$ follows in a similar manner. Hence, it can be concluded that
$S''(P,R) \leq S''(P,Q) \wedge S''(Q,R)$ which completes the proof.

Definition 4.12. The weighted similarity measure can be defined in a similar way as,
$S_w''(A,B)$
$= 1 - [\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3}w_i|h_1^{A,B}(x_i) + h_2^{A,B}(x_i) + \tau_1^{A,B}(x_i) + \tau_2^{A,B}(x_i)|)^p]^{\frac{1}{p}}$
where w_i are the weights associated to the elements x_i of the universe, $i = 1, 2, ..., n$ such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$.

It can be easily shown that $S_w''(A,B)$ is a measure of similarity.

4.3. Similarity measure based on correlation coefficient

Definition 4.13. The correlation coefficient between two QSVNS A and B is defined as ,

$\beta(A,B) = [\sum_{i=1}^{n}(T_A(x_i).T_B(x_i)+C_A(x_i).C_B(x_i)+U_A(x_i).U_B(x_i)$
$+F_A(x_i).F_B(x_i))]/[(\sum_{i=1}^{n}(T_A^2(x_i)+C_A^2(x_i)+U_A^2(x_i)+F_A^2(x_i))).$
$(\sum_{i=1}^{n}(T_B^2(x_i) + C_B^2(x_i) + U_B^2(x_i) + F_B^2(x_i)))]^{\frac{1}{2}}$

Remark 4.14. $\beta(A,B)$ is a quasi-similarity between the sets A and B.

4.4. Quadripartitioned similarity measure

Definition 4.15. For any two QSVNS A and B over a universe $X = \{x_1, x_2, ..., x_n\}$, define the quadripartitioned similarity measure $S_4 : QSVNS(X) \times QSVNS(X) \rightarrow [0,1]^4$ between two QSVNS A and B in the sense of Broumi and Smarandache [5] as
$S_4(A,B) = \langle S_T(A,B), S_C(A,B), S_U(A,B), S_F(A,B)\rangle$
where

$S_T(A,B) = \frac{\sum_{i=1}^{n} min(T_A(x_i),T_B(x_i))}{\sum_{i=1}^{n} max(T_A(x_i),T_B(x_i))}$
$S_C(A,B) = \frac{\sum_{i=1}^{n} min(C_A(x_i),C_B(x_i))}{\sum_{i=1}^{n} max(C_A(x_i),C_B(x_i))}$
$S_U(A,B) = \frac{\sum_{i=1}^{n} min(U_A(x_i),U_B(x_i))}{\sum_{i=1}^{n} max(U_A(x_i),U_B(x_i))}$
$S_F(A,B) = \frac{\sum_{i=1}^{n} min(F_A(x_i),F_B(x_i))}{\sum_{i=1}^{n} max(F_A(x_i),F_B(x_i))}$

A quadripartitioned similarity measure is in fact a quadruple comprising four different similarity measures in terms of the four membership values of a QSVNS. At times, for the sake of convenience $S_4(A,B)$ is also represented in the form of a matrix:

$$S_4(A,B) = \left\langle \begin{matrix} S_T(A,B) & S_F(A,B) \\ S_C(A,B) & S_U(A,B) \end{matrix} \right\rangle$$

Such a similarity measure takes into account, separately, the degrees of similarity between the various membership values. It is easy to show that $S_4(A,B)$ satisfies all the axiomatic properties of a similarity measure. We only give an outline of the proof of the triangle inequality as follows:

When $A \subset B \subset C$, for each $x_i \in X$,
$min\,(T_A(x_i), T_B(x_i)) = T_A(x_i)$
$max\,(T_A(x_i), T_B(x_i)) = T_B(x_i)$
$min\,(T_A(x_i), T_C(x_i)) = T_A(x_i)$ and
$max\,(T_A(x_i), T_C(x_i)) = T_C(x_i)$. Thus, in such a case,
$S_T(A,B) = \frac{\sum_{i=1}^{n} T_A(x_i)}{\sum_{i=1}^{n} T_B(x_i)}$.
Similarly, $S_T(A,C) = \frac{\sum_{i=1}^{n} T_A(x_i)}{\sum_{i=1}^{n} T_C(x_i)}$.

Since, $T_B(x_i) \leq T_C(x_i)$, it follows that $S_T(A,C) \leq S_T(A,B)$.
Similar proofs follow for $S_C(A,B), S_U(A,B)$ and $S_F(A,B)$. Thus, all the components of $S_4(A,B)$ individually satisfy the properties $(S1)-(S3)$.

If, further, we introduce the notations $\bar{0} = \langle 0,0,0,0\rangle$, $\bar{1} = \langle 1,1,1,1\rangle$ and an ordering on $[0,1]^4$ of the form, $\langle a_1,a_2,a_3,a_4\rangle \preceq \langle b_1,b_2,b_3,b_4\rangle$ iff $a_i \leq b_i$, for $a_i, b_i \,\epsilon\, [0,1]$, $i = 1,2,3,4$; then for any $A, B \,\epsilon\, QSVNS(X)$
(i) $\bar{0} \preceq S_4(A,B) \preceq \bar{1}$
(ii) $S_4(A,B) = S_4(B,A)$
(iii) for $A, B, C \,\epsilon\, QSVNS(X)$ such that $A \subseteq B \subseteq C$, we have $S_4(A,C) \preceq S_4(A,B) \wedge S_4(B,C)$.

5. Entropy measure for QSVNS

Definition 5.1. Let X be a non-empty universe of discourse. A mapping $e : QSVNS(X) \rightarrow [0,1]$ is said to be an entropy on $QSVNS(X)$ if e satisfies the following properties:
(i) $e(A) = 0$ iff $A \,\epsilon\, \mathcal{P}(X)$
(ii) $e(A) = 1$ for $A \,\epsilon\, QSVNS(X)$ if $T_A(x) = C_A(x) = U_A(x) = F_A(x) = 0.5$ for all $x \,\epsilon\, X$.
(iii) $e(A) \geq e(B)$ iff $T_A(x) + F_A(x) \leq T_B(x) + F_B(x)$ and $|C_A(x) - U_A(x)| \leq |C_B(x) - U_B(x)|$, for all $x \,\epsilon\, X$.
(iv) $e(A) = e(A^c)$, for all $A \,\epsilon\, QSVNS(X)$.

Proposition 5.2.
$e_m(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}(T_A(x_i) + F_A(x_i)) \,.\, |C_A(x_i) - U_A(x_i)|$, for all $x_i \,\epsilon\, X$ is an entropy measure.

Proof: Proofs are straight-forward.

Remark 5.3. e_m is referred to as the entropy based on membership degrees.

Example 5.4. Consider the QSVNS A of example 3.5 then we have
$e_m(A) = 0.688$. Also, $e_m(A^c) = 0.688 = e_m(A)$.

6. A comparative study of the proposed similarity measures in the context of an example pertaining to pattern recognition

Suppose x_1, x_2, x_3 respectively denote the saturation, sharpness and contrast of three similar hued patterns A, B and C which are represented in terms of three QSVNS as

$A = \langle 0.5, 0.4, 0.2, 0.01\rangle/x_1 + \langle 0.2, 0.1, 0.3, 0.5\rangle/x_2 + \langle 0.45, 0.2, 0.1, 0.3\rangle/x_3$
$B = \langle 0.4, 0.4, 0.01, 0.0\rangle/x_1 + \langle 0.5, 0.3, 0.4, 0.4\rangle/x_2 + \langle 0.5, 0.4, 0.3, 0.1\rangle/x_3$
$C = \langle 0.56, 0.8, 0.0, 0.0\rangle/x_1 + \langle 0.6, 0.5, 0.3, 0.2\rangle/x_2 + \langle 0.4, 0.4, 0.5, 0.6\rangle/x_3$

Further suppose that there are two unidentified patterns P_1 and P_2 given by
$P_1 = \langle 0.45, 0.4, 0.11, 0.05\rangle/x_1 + \langle 0.35, 0.2, 0.35, 0.45\rangle/x_2 + \langle 0.45, 0.3, 0.2, 0.2\rangle/x_3$
$P_2 = \langle 0.6, 0.7, 0.01, 0.0\rangle/x_1 + \langle 0.5, 0.5, 0.4, 0.2\rangle/x_2 + \langle 0.3, 0.2, 0.5, 0.6\rangle/x_3$

In order to determine which unidentified pattern belongs to which one of the specified patterns at hand, the similarity measures between the given patterns A and B and the unknown patterns P_1 and P_2 are calculated. Finally, the unidentified pattern bearing the highest similarity to the given set of patterns is concluded to belong to that particular set of patterns. In this respect, it needs to be stated that the similarity measure S'' has been calculated taking 3 values of the order of similarity p viz. $p = 1, p = 2$ and $p = 3$. The obtained results are represented in a tabular form (ref. Table2).

Discussion:

From the given set of data, it is seen that, the calculated values of the similarity measures S', S'' and S_4 between P_1 and A, B, C indicate that the pattern P_1 belongs to the class of patterns, of which A is a member. However, the quasi similarity measure β throws close values for $\beta(P_1, A)$and $\beta(P_2, B)$ which creates confusion in classifying the pattern P_1. Thus, S', S'' and S_4 are better measures of similarity compared to β. Moreover, from the similarity measure $S_4(P_1, A)$, it is seen that membership values T, C, U and F of the patterns P_1 and A are similar upto $84.6\%, 77.8\%, 68\%$ and 77.6% respectively. Similarly, all the measures S', S'', β and S_4 show that the pattern P_2 belongs to the class of patterns C.

7. Conclusion

In this paper, Belnap's four valued logic has been used as a framework for proposing a set-theoretic structure which involves partitioned indeterminacies. Although apparently it might so appear that the inde-

Table 2
Table representing the similarity measures among the different patterns.

Unidentified Patterns ↓	Similarity Measures	Given Patterns: A	B	C
P_1	$S^{'}$	0.931	0.924	0.816
	$S^{''}$	$0.941_{p=1}$	$0.934_{p=1}$	$0.830_{p=1}$
		$0.940_{p=2}$	$0.932_{p=2}$	$0.825_{p=3}$
		$0.938_{p=3}$	$0.931_{p=2}$	$0.821_{p=3}$
	β	0.971	0.973	0.886
	S_4	$\left\langle \begin{matrix} 0.846\ 0.776 \\ 0.778\ 0.680 \end{matrix} \right\rangle$	$\left\langle \begin{matrix} 0.828\ 0.714 \\ 0.818\ 0.691 \end{matrix} \right\rangle$	$\left\langle \begin{matrix} 0.745\ 0.364 \\ 0.529\ 0.521 \end{matrix} \right\rangle$
P_2	$S^{'}$	0.794	0.842	0.956
	$S^{''}$	$0.821_{p=1}$	$0.834_{p=1}$	$0.959_{p=1}$
		$0.808_{p=2}$	$0.812_{p=2}$	$0.958_{p=2}$
		$0.798_{p=3}$	$0.792_{p=3}$	$0.957_{p=3}$
	β	0.825	0.866	0.986
	S_4	$\left\langle \begin{matrix} 0.645\ 0.450 \\ 0.500\ 0.373 \end{matrix} \right\rangle$	$\left\langle \begin{matrix} 0.750\ 0.300 \\ 0.625\ 0.780 \end{matrix} \right\rangle$	$\left\langle \begin{matrix} 0.850\ 1.000 \\ 0.824\ 0.879 \end{matrix} \right\rangle$

terminacy membership functions C and U are interdependent, often, in reality, while dealing with linguistic approaches, it is quite natural that the values corresponding to the functions are actually independent and the mutual dependence of these functional values boils down to a particular case under speculation.As for example, concerning a particular sample of information, a particular person may be utterly clueless as to whether the piece of information is true, false, both true and false or neither of them. In such cases his judgement may instinctively, yet quite involuntarily hover between 'both true and false' and 'neither true nor false', being totally unaware of the fact that these two truth values are somewhat logically complementary. For another instance, given two patterns under consideration, it might so happen that at a particular moment the graphical representations of the two patterns are same while they differ in terms of the constituent hues. Thus, at such stances the information that the patterns are similar are both true and false and simultaneously neither true nor false. Putting into considerations such situations such as these, it is evident that a structure like QSVNS prove to be useful and at times essential in representing and tackling the available information. At present some basic set-theoretic operations and similarity measures have been stated. Future works may involve dealing with actual problems, rather than fictitious ones, and implementing them in decision making problems.

Acknowledgments

The research of the first author is supported by University JRF (Junior Research Fellowship).
The research of the second author is supported by UGC (ERO) Minor Research Project (Project no. F.PSW-19/12-13).
The research of the third author is partially supported by the Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant no. F 510/3/DRS-III/(SAP-I)].

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